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Research paper

$g^{\star}bp$-Continuous Multifunction

Alias B. Khalaf[1]*, Suzan N. Dawod[2]

[1] *University of Duhok, Department of Mathematics*
[2] *University of Zakho, Department of Mathematics*
**Corresponding author E-mail: aliasbkhalaf@gmail.com*

Abstract

In this paper we introduce a new class of multifunction called Upper(lower) $g^{\star}bp$-continuous multifunction, Upper(lower) almost $g^{\star}bp$-continuous multifunction, Upper(lower) weakly $g^{\star}bp$-continuous multifunction and Upper(lower) contra$g^{\star}bp$-continuous multifunction in topological spaces,and study some of their basic properties and relations among them.

Keywords: *$g^{\star}b$-closed set, $g^{\star}bp$-continuous, almost $g^{\star}bp$-continuous, weakly $g^{\star}bp$-continuous.*

1. Introduction

Many mathematicians and they devote a great part of their research work on the study of generalised continuous multifunction. In 1999, Mahmoud introduced the concept of pre-irresolute multi-valued function while in 1996 Popa and Noiri and in 2001 Abd-El-Monsef and Nasef introduced other types of multifunctions.
Throughout this paper (X, τ) and (Y, σ) (or simply X and Y) represents the non-empty topological spaces on which no separation axiom are assumed, unless otherwise mentioned. For a subset A of X, $Cl(A)$ and $Int(A)$ represents the closure of A and Interior of A respectively. A subset A is said to be preopen [17] (resp., α-open [19], semi open [12], regular open [25]) set if $A \subseteq IntCl(A)$ (resp., $A \subseteq IntClInt(A)$, $A \subseteq ClInt(A)$, $A = IntCl(A)$). The complement of a preopen set is called preclosed.

2. Preliminaries

We recall the following definition.

Definition 2.1 *A subset A of a topological space (X, τ) is called*

1. *b-open set [3], if $A \subseteq Cl(Int(A)) \cup Int(Cl(A))$ and b-closed set if $Cl(Int(A)) \cup Int(Cl(A)) \subseteq A$.*
2. *generalized closed set (briefly g-closed) [11] ($g^{\star}$-closed [23]), if $Cl(A) \subseteq U$ whenever $A \subseteq U$ and U is open(g-open) in X.*
3. *gb-closed [20], and ($g^{\star}b$-closed [24]) if $bCl(A) \subseteq U$ whenever $A \subseteq U$ and U is open(g-open) in X.*
4. *$p\delta$-open set [9], if for each $x \in A$, there exists a preopen set U in X such that $x \in U \subseteq pIntpCl(U) \subseteq A$.*
5. *regular preopen(resp., regular preclosed) set [6], if $A = pIntpCl(A)$(resp. $A = pClpInt(A)$).*

Definition 2.2 *[4] A space X is said to be*

1. *Pre-T_0 if and only if to each pair of distinct points x, y in X, there exists a preopen set containing one of the points but not the other.*
2. *Pre-T_1 if and only if to each pair of distinct points x, y of X, there exists a pair of preopen sets one containing x but not y and other containing y but not x.*
3. *Pre-T_2 if and only if to each pair of distinct points x, y of X, there exists a pair of disjoint preopen sets one containing x and the other containing y.*

Definition 2.3 *A topological space (X, τ) is said to be:*

1. *$g^{\star}b$-T_0 if for each pair of distinct points x, y in X, there exists a $g^{\star}b$-open set U such that either $x \in U$ and $y \notin U$ or $x \notin U$ and $y \in U$.*
2. *$g^{\star}b$-T_1 if for each pair of distinct points x, y in X, there exist two $g^{\star}b$-open sets U and V such that $x \in U$ but $y \notin U$ and $y \in V$ but $x \notin V$.*
3. *$g^{\star}b$-T_2 if for each distinct points x, y in X, there exist two disjoint $g^{\star}b$-open sets U and V containing x and y respectively.*
4. *$g^{\star}b$-$T_{\frac{1}{2}}$ if every $g^{\star}b$-closed set is g-closed.*
5. *$g^{\star}b$-space if every $g^{\star}b$-open set of X is open in X.*

Definition 2.4 *A topological space (X, τ) is said to be:*

1. *submaximal [7], if the closure of every open set of X is X.*
2. *extremally disconnected [15], if the closure of every open set of X is open in X.*
3. *pre-$T_{\frac{1}{2}}$ [16], space if every pg-closed set is preclosed.*
4. *r-T_1 [8], if for each pair of distinct points x and y of X, there exists regular open sets U and V containing x and y respectively, such that $y \notin U$ and $x \notin V$.*

Theorem 2.5 *[7] A space X is submaximal if and only if every preopen set is open.*

Theorem 2.6 *[2] Let (Y, τ_Y) be subspace of a space (X, τ). If $A \in PO(X, \tau)$ and $A \subseteq Y$, then $A \in PO(Y, \tau_Y)$.*

Theorem 2.7 *[25] Let A be a subset of a topological space (X, τ), if $A \in \tau$, then $Cl_{\theta}(A) = Cl(A)$.*

Theorem 2.8 *[24] Let $A \subseteq Y \subseteq X$ and suppose that A is $g^{\star}b$-closed in X, then A is $g^{\star}b$-closed relative to Y.*

Definition 2.9 *[14] A multifunction $F : X \to Y$ is said to be;*

1. *Upper pre-irresolute at $x \in X$ if for each preopen set A of Y containing $F(x)$ $(F(x) \cap V \neq \phi)$, there exists a preopen set U of X containing x such that $F(U) \subseteq A$.*
2. *Lower pre-irresolute at $x \in X$ if for each preopen set A of Y such that $F(x) \cap A \neq \phi$, there exists a preopen set U of X containing x such that $F(u) \cap A \neq \phi$ for every $u \in U$.*
3. *Upper (Lower) pre-irresolute if it has this property at each point of X.*

Definition 2.10 *[1] For a multifunction $F : X \to Y$, we shall denote the upper and lower inverse of a set A of Y by $F^{+}(A)$ and $F^{-}(A)$, respectively, that is, $F^{+}(A) = \{x \in X : F(x) \subseteq A\}$ and $F^{-}(A) = \{x \in X : F(x) \cap A \neq \phi\}$.*

Definition 2.11 *A multifunction $F : X \to Y$ is said to be;*

1. *Upper α-continuous [21] at $x \in X$ if for each open set V of Y containing $F(x)$, there exists $U \in \alpha(X, x)$ such that $F(U) \subseteq V$.*
2. *Lower α-continuous [21] at $x \in X$ if for each open set A of Y such that $F(x) \cap A \neq \phi$, there exists $U \in \alpha(X, x)$ such that $F(u) \cap A \neq \phi$ for every $u \in U$.*

3. *Upper (Lower) α-continuous [18] if it has this property at each point of X.*

Definition 2.12 *[22] A multifunction $F : X \to Y$ is said to be;*

1. *Upper almost α-continuous at $x \in X$ if for each open set V of Y containing $F(x)$, there exists $U \in \alpha(X, x)$ such that $F(U) \subseteq IntCl(V)$.*

2. *Lower almost α-continuous at $x \in X$ if for each open set V of Y such that $F(x) \cap A \neq \phi$, there exists $U \in \alpha(X, x)$ such that $F(u) \cap IntCl(V) \neq \phi$ for every $u \in U$.*

3. *Upper (Lower) almost α-continuous if it has this property at each point of X.*

Definition 2.13 *[13] A multifunction $F : X \to Y$ is said to be;*

1. *Upper δ-continuous at $x \in X$ if for each regular open set V of Y containing $F(x)$, there exists a regular open set U of X such that $F(U) \subseteq V$.*

2. *Lower δ-continuous at $x \in X$ if for each regular open set V of Y such that $F(x) \cap A \neq \phi$, there exists a regular open set U of X such that $F(u) \cap V \neq \phi$ for every $u \in U$.*

Definition 2.14 *[1] A multifunction $F : X \to Y$ is said to be;*

1. *Upper b-continuous at $x \in X$ if for each open set V of Y containing $F(x)$, there exists a b-open set U of X such that $F(U) \subseteq V$.*

2. *Lower b-continuous at $x \in X$ if for each open set V of Y such that $F(x) \cap A \neq \phi$, there exists a b-open set U of X such that $F(u) \cap V \neq \phi$ for every $u \in U$.*

3. Upper and lower $g^\star bp$-continuous multifunction

In this section, we Introduce the concept of upper and lower $g^\star bp$-continuous multifunctions in topological spaces.

Definition 3.1 *A multifunction $F : X \to Y$ is said to be:*

1. *Upper $g^\star bp$-continuous (U.$g^\star bp$.c.) at $x \in X$ if for each preopen set A of Y containing $F(x)$, there exists a $g^\star b$-open set U of X containing x such that $F(U) \subseteq A$.*

2. *Lower $g^\star bp$-continuous (L.$g^\star bp$.c.) at $x \in X$ if for each preopen set A of Y such that $F(x) \cap A \neq \phi$, there exists a $g^\star b$-open set U of X containing x such that $F(u) \cap A \neq \phi$ for every $u \in U$.*

3. *Upper (Lower) $g^\star bp$-continuous if it has this property at each point of X.*

Proposition 3.2 *Let X and Y be topological spaces. For a multifunction $F : X \to Y$, the following statements are equivalents:*

1. *F is U.$g^\star bp$.c. (L.$g^\star bp$.c.),*

2. *For every preopen set A, $F^+(A)(F^-(A))$ is a $g^\star b$-open set in X,*

3. *For every preclosed set K, $F^-(K)(F^+(K))$ is a $g^\star b$-closed set in X.*

proof. (1) $\Rightarrow$ (2). If A is preopen set of Y, then for each $x \in F^+(A), F(x) \subseteq A$. By(1) there exists a $g^\star b$-open set U of x such that $F(U) \subseteq A$ which implies that $x \in U \subseteq F^+(A)$, therefore $F^+(A)$ is $g^\star b$-open in X.
(2) $\Rightarrow$ (3). Let K be preclosed set of Y. Then $Y \setminus K$ is preopen set of Y. By(2), $F^+(Y \setminus K) = X \setminus F^-(k)$ is $g^\star b$-open set in X and hence $F^-(K)$ is $g^\star b$-closed in X.
(3) $\Rightarrow$ (1). Let A be any preopen set of Y. Then $(Y \setminus A)$ is preclosed in Y. By(3), $F^-(Y \setminus A)$ is $g^\star b$-closed set in X. But $F^-(Y \setminus A) = X \setminus F^+(A)$. Thus $X \setminus F^+(A)$ is $g^\star b$-closed in X so $F^+(A)$ is $g^\star b$-open in X. Therefore, we obtain $F(F^+(A)) \subseteq A$, hence F is $g^\star bp$-continuous.
The proof for the case where F is L.$g^\star bp$.c. is similarly proved.

Theorem 3.3 *If a multifunction $F : (X, \tau) \to (Y, \sigma)$ is upper b-continuous and Y is submaximal, then F is upper $g^\star bp$-continuous.*

proof. Let A be preopen set in Y, since Y is submaximal then A is open set in Y. Since F is upper b-continuous, then $F^{+}(A)$ is b-open in X and by Theorem(3.4) [24], it is $g^{\star}b$-open in X. Hence F is upper $g^{\star}bp$-continuous.

Proposition 3.4 *Let $X = R_1 \cup R_2$, where R_1 and R_2 are $g\star b$-closed set in X. Let $F : R_1 \to Y$ and $G : R_2 \to Y$ be upper $g^{\star}bp$-continuous. If $F(x) = G(x)$ for each $x \in R_1 \cap R_2$. Then $H : R_1 \cup R_2 \to Y$ such that*

$$H(x) = \begin{cases} F(x) & \text{if } x \in R_1 \\ G(x) & \text{if } x \in R_2 \end{cases}$$

is upper $g^{\star}bp$-continuous.

proof. Let A be any preopen set in Y. Clearly $H^{+}(A) = F^{+}(A) \cup G^{+}(A)$. Since F is upper $g^{\star}bp$-continuous, then $F^{+}(A)$ is $g^{\star}b$-open in R_1. But R_1 is $g^{\star}b$-open in X. Then by Theorem (3.30) [24], $F^{+}(A)$ is $g^{\star}b$-open in X. Similarly, $G^{+}(A)$ is $g^{\star}b$-open in R_2 and hence a $g^{\star}b$-open in X. Since a union of two $g^{\star}b$-open sets is $g^{\star}b$-open. Therefore, $H^{+}(A) = F^{+}(A) \cup G^{+}(A)$ is $g^{\star}b$-open in X. Hence H is upper $g^{\star}bp$-continuous.

Theorem 3.5 *For a multifunction $F : (X, \tau) \to (Y, \sigma)$ the following are equivalent.*

1. *F is upper $g^{\star}bp$-continuous.*
2. *$F(g^{\star}bCl(B)) \subseteq pCl(F(B))$, for every subset B of X,*
3. *$g^{\star}bCl(F^{+}(A)) \subseteq F^{+}(pCl(A))$, for each subset A of Y,*
4. *$F^{-}(pInt(A)) \subseteq g^{\star}bInt(F^{-}(A))$, for each subset A of Y,*
5. *$pInt(F(B)) \subseteq F(g^{\star}bInt(B))$, for each subset B of X.*

proof. (1) $\Rightarrow$ (2). Let B be any subset of X .Then $F(B) \subseteq pCl(F(B))$ and $pClF(B)$ is preclosed in Y. Hence $B \subseteq F^{+}(pClF(B))$, since F is $g^{\star}bp$-continuous. By Proposition 3.2, $F^{+}(pClF(B))$ is $g^{\star}b$-closed set in X. Therefore, $g^{\star}bCl(B) \subseteq F^{+}(pCl(F(B)))$. Hence $F(g^{\star}bCl(B)) \subseteq (pCl(F(B)))$.
(2) $\Rightarrow$ (3). Let A be any subset of Y, then $F^{+}(A)$ is a subset of X. By (2) we have $F(g^{\star}bClF^{+}(A)) \subseteq pCl(F(F^{+}(A))) = pCl(A)$. It follow that $g^{\star}b(ClF^{+}(A)) \subseteq F^{+}(pCl(A))$.
(3) $\Rightarrow$ (4). Let A be any subset of Y. Then apply(3) to $(Y \setminus A)$ we obtain $g^{\star}bCl(F^{+}(Y \setminus A)) \subseteq F^{+}(pCl(Y \setminus A)) \Leftrightarrow g^{\star}bCl(X \setminus F^{-}(A)) \subseteq F^{+}(Y \setminus pInt(A)) \Leftrightarrow X \setminus g^{\star}bInt(F^{-}(A)) \subseteq X \setminus F^{-}(pInt(A)) \Leftrightarrow F^{-}(pInt(A) \subseteq g^{\star}bInt(F^{-}(A))$.
(4) $\Rightarrow$ (5). Let B be any subset of X, Then $F(B)$ is a subset of Y. By(4), we have $F^{-}(pInt(f(A))) \subseteq g^{\star}bInt(F^{-}(F(A))) = g^{\star}bInt(A)$. Therefore, $pInt(F(A)) \subseteq F(g^{\star}bInt(A))$.
(5) $\Rightarrow$ (1). let $x \in X$ and let A be any preopen set of Y containing $F(x)$. Then $x \in F^{+}(A)$ and $F^{+}(A)$ is a subset of X. By(5), we have $pInt(F(F^{+}(A))) \subseteq F(g^{\star}bInt(F^{+}(A)))$. Then $pInt(A) \subseteq F(g^{\star}bInt(F^{+}(A)))$, since A is preopen, then $A \subseteq F(g^{\star}bInt(F^{+}(A)))$ implies that $F^{+}(A) \subseteq g^{\star}bInt(F^{+}(A))$. Therefore $F^{+}(A)$ is $g^{\star}b$-open in X containing x and clearly $F(F^{+}(A)) \subseteq A$. Hence F is upper $g^{\star}bp$-continuous.

Proposition 3.6 *Let $F : X \to Y$ be upper $g^{\star}bp$-continuous and $Y \subseteq Z$. If Y is preclosed subset of a topological space Z then $F : X \to Z$ is upper $g^{\star}bp$-continuous.*

proof. Let K be any preclosed set in Z. Then $K \cap Y$ is preclosed in Z, by Theorem(2.22) [2], it is preclosed in Y. Since F is upper $g^{\star}bp$-continuous $F^{+}(K \cap Y)$ is $g^{\star}b$-closed in X but $F(x) \in Y$ for each $x \in X$, and thus $F^{+}(K) = F^{+}(K \cap Y)$ is $g^{\star}b$-closed subset of X. Therefore, by Proposition 3.2 $F : X \to Z$ is upper $g^{\star}bp$-continuous.

Theorem 3.7 *If $F : X \to Y$ is upper $g^{\star}bp$-continuous and A is $g^{\star}b$-closed set in X then $F|A : A \to Y$ is upper $g^{\star}bp$-continuous.*

proof. Let B be preclosed set in Y, since F is upper $g^{\star}bp$-continuous, then $F^{+}(B)$ is $g^{\star}b$-closed in X. If $F^{+}(B) \cap A = A_1$ then A_1 is $g^{\star}b$-closed in X, since intersection of two $g^{\star}b$-closed is $g^{\star}b$-closed. Since $(F|A)^{+}(B) = A_1$ by Theorem 2.8, A_1 is $g^{\star}b$-closed set in A. Therefore $F|A$ is upper $g^{\star}bp$-continuous.

Theorem 3.8 *If $F : X \to Y$ and $G : Y \to Z$ be any two multifunctions, then $G \circ F : X \to Z$ is upper $g^{\star}bp$-continuous if G is preirresolute multifunction and F is upper $g^{\star}bp$-continuous.*

proof. Let A be any preclosed set in Z. Since G is preirresolute multifunction then $G^{+}(A)$ is preclosed in Y, since F is upper $g^{\star}bp$-continuous then $F^{+}(G^{+}(A))$ is $g^{\star}b$-closed in X. Hence $G \circ F$ is upper $g^{\star}bp$-continuous.

Theorem 3.9 *If $F : X \to Y$ is a upper $g^\star bp$-continuous injection and Y is pre-T_1, then X is $g^\star b$-T_1.*

proof. Assume that Y is pre-T_1. For any distinct points x and y in X, there exists preopen set A and W such that $F(x) \in A$, $F(y) \notin A$, $F(x) \notin W$ and $F(y) \in W$. Since F is upper $g^\star bp$-continuous, so there exists a $g^\star b$-open sets G and H such that $x \in G$, $y \in H$, $F(G) \subseteq A$ and $F(H) \subseteq W$. Thus we obtain $y \notin G$, $x \notin H$. this show that X is $g^\star b$-T_1.

Theorem 3.10 *If $F : X \to Y$ is upper $g^\star bp$-continuous injection and Y is pre-T_2 then X is $g^\star b$-T_2.*

proof. For any pair of distinct points x and y in X, there exists disjoint preopen sets U and V in Y such that $F(x) \in U$ and $F(y) \in V$. Since F is upper $g^\star bp$-continuous, there exists $g^\star b$-open sets G and H in X containing x and y, respectively, such that $F(G) \subseteq U$ and $F(H) \subseteq V$. Since U and V are disjoint, we have $U \cap V = \phi$, hence $G \cap H = \phi$. This shows that X is $g^\star b$-T_2.

Theorem 3.11 *An upper $g^\star bp$-continuous image of a $g^\star b$-connected space is $g^\star b$-connected for a multifunction F.*

proof. Let $F : X \to Y$ be an upper $g^\star bp$-continuous multifunction from a $g^\star b$-connected space X onto a space Y. Suppose Y is not connected and let $Y = A \cup B$ be a partition of Y. Then both A and B are preopen and preclosed subset of Y. Since F is upper $g^\star bp$-continuous, $F^+(A)$ and $F^+(B)$ are $g^\star b$-open subset of X. In view of the fact that $F^+(A)$ and $F^+(B)$ are disjoint, $X = F^+(A) \cup F^+(B)$ is a partition of X. This is contrary to the connectedness of X.

Definition 3.12 *A multifunction $F : X \to Y$ is said to be;*

1. *Upper almost $g^\star bp$-continuous at a point $x \in X$ if for each preopen set A of Y such that $F(x) \in A$, there exists a $g^\star b$-open set U containing x such that $F(U) \subseteq IntCl(A)$.*
2. *Lower almost $g^\star bp$-continuous at a point $x \in X$ if for each preopen set A of Y such that $F(x) \in A$, there exists a $g^\star b$-open set U of X containing X such that $F(U) \cap IntCl(A) \neq \phi$.*
3. *Upper (Lower) almost $g^\star bp$-continuous if it has this property at each point of X.*

Theorem 3.13 *A multifunction $F : X \to Y$ is upper almost $g^\star bp$-continuous if and only if for each $x \in X$ and each regular open set A containing $F(x)$, there exists a $g^\star b$-open set U in X containing x such that $F(U) \subseteq A$.*

proof. For every $x \in X$ and let A be any regular open set containing $F(x)$, then A is preopen set containing $F(x)$. Since F is upper almost $g^\star bp$-continuous, then there exists a $g^\star b$-open set U in X containing x such that $F(U) \subseteq IntCl(A) = A$. Conversely. Assume that for all regular open set A containing $F(x)$, there exists a $g^\star b$-open set U in X containing x with $F(U) \subseteq A = IntCl(A)$ then A is preopen set and hence F is upper almost $g^\star bp$-continuous.

Theorem 3.14 *For a multifunction $F : X \to Y$, the following statements are equivalent:*

1. *F upper almost $g^\star bp$-continuous,*
2. *$F^+(IntCl(A))$ is $g^\star b$-open set in X, for each preopen set A in Y,*
3. *$F^-(ClInt(B))$ is $g^\star b$-closed set in X, for each preclosed set B in Y,*
4. *$F^-(B)$ is $g^\star b$-closed set in X, for each regular closed set B in Y,*
5. *$F^+(A)$ is $g^\star b$-open set in X , for each regular open set A in Y.*

proof. (1) $\Rightarrow$ (2). Let A be any preopen set in Y. We have to show that $F^+(IntCl(A))$ is $g^\star b$-open set in X. Let $x \in F^+(IntCl(A))$. Then $F(x) \in IntCl(A)$ and $IntCl(A)$ is regular open set in Y. Since F is upper almost $g^\star bp$-continuous. By Theorem 3.13, there exists a $g^\star b$-open set U of X containing x such that $F(U) \subseteq IntCl(A)$. Which implies that $x \in U \subseteq F^+(IntCl(A))$. Therefore, $F^+(IntCl(A))$ is $g^\star b$-open set in X.
(2) $\Rightarrow$ (3). Let B be any preclosed set of Y. Then $Y \setminus B$ is preopen set of Y. By (2), $F^+(IntCl(Y \setminus B))$ is $g^\star b$-open set in X and $F^+(IntCl(Y \setminus B)) = F^+(Int(Y \setminus Int(B))) = F^+(Y \setminus ClInt(B)) = X \setminus F^-(ClInt(B))$ is $g^\star b$-open set in X and hence $F^-(ClInt(B))$ is $g^\star b$-closed set in X.
(3) $\Rightarrow$ (4). Let B be any regular closed set of Y. Then B is preclosed set of Y. By (3). $F^-(ClInt(B))$ is $g^\star b$-closed set in X since B is regular closed set, then $F^-(ClInt(B)) = F^-(B)$. Therefore $F^-(B)$ is $g^\star b$-closed set in X.

(4) $\Rightarrow$ (5). Let A be any regular open set of Y. Then $Y \setminus A$ is regular closed set of Y,and by (4) we have $F^{-}(Y \setminus A) = X \setminus F^{+}(A)$ is $g^{\star}b$-closed set in X and hence $F^{+}(A)$ is $g^{\star}b$-open set in X.
(5) $\Rightarrow$ (1). Let $x \in X$ and let A be any regular open set of Y containing $F(x)$. Then $x \in F^{+}(A)$. By (5) we have $F^{+}(A)$ is $g^{\star}b$-open set in X. Therefore we obtain $F(F^{+}(A)) \subseteq A$. Hence by Theorem 3.13, F is upper almost $g^{\star}bp$-continuous.

Theorem 3.15 *If a multifunction $F : X \to Y$ is upper $g^{\star}bp$-continuous, then it is upper almost $g^{\star}bp$-continuous but not conversely.*

proof. Let A be any regular open set in Y, so is preopen in Y. Since F is upper $g^{\star}bp$-continuous then $F^{+}(A)$ is $g^{\star}b$-open in X. Hence by Theorem 3.14, F is upper almost $g^{\star}bp$-continuous.

Remark 3.16 *The converse of the theorem need not be true in general.*

Example 3.17 *Consider $X = Y = \{a, b, c\}$ with the topology τ=$\{\phi, \{c\}, \{a, c\}, \{b, c\}, X\}$,$\sigma = \{\phi, \{a\}, Y\}$ and with the identity multifunction $F : (X, \tau) \to (Y, \sigma)$, F is upper almost $g^{\star}bp$-continuous but not upper $g^{\star}bp$-continuous since for preclosed set $B = \{b, c\}$ in Y $F^{+}(B) = \{b, c\}$ is not $g^{\star}b$-closed in X.*

Theorem 3.18 *If a multifunction $F : X \to Y$ is upper almost α-continuous then F is upper almost $g^{\star}bp$-continuous.*

proof. Let A be any regular open set in Y. Since F is upper almost α-continuous then $F^{+}(A)$ is semi open set in X, hence by Theorem(3.10)[24], is $g^{\star}b$-open in X. Therefore, F is upper almost $g^{\star}bp$-continuous.

Theorem 3.19 *If a multifunction $F : X \to Y$ is upper δ-continuous, then F is upper almost $g^{\star}bp$-continuous.*

proof. Let $x \in X$ and let A be any preopen set in Y, then $A \subseteq IntCl(A)$. Since F is upper δ-continuous, there exists an regular open set U of X containing x such that $F(U) \subseteq IntCl(IntCl(A))$, then $F(U) \subseteq IntCl(A)$. Since U is regular open set, then it is preopen and by Theorem(3.12) [24], U is $g^{\star}b$-open set of X. Therefore, F is upper almost $g^{\star}bp$-continuous.

Theorem 3.20 *If $F : X \to Y$ is upper almost $g^{\star}bp$-continuous function, then we have $F^{-1}(A) \subseteq g^{\star}bInt(F^{+}(IntCl(A)))$ for every preopen set A in Y.*

proof. Let A be any preopen set in Y, then $A \subseteq IntCl(A)$. Since $IntCl(A)$ is regular open set in Y, and Since F is upper almost $g^{\star}bp$-continuous multifunction, so by Theorem 3.14, $F^{+}(intCl(A))$ is $g^{\star}b$-open set in X. So $F^{+}(A) \subseteq F^{+}(intCl(A)) = g^{\star}bInt(F^{+}(IntCl(A)))$.

Corollary 3.21 *If $F : X \to Y$ is lower almost $g^{\star}bp$-continuous function, then we have $g^{\star}bCl(F^{-}(ClInt(E))) \subseteq F^{-}(E)$, for every preclosed set E in Y.*

proof. Let E be any preclosed set in Y, so $Y \setminus E$ is preopen. By Theorem 3.20, $F^{+}(Y \setminus E) \subseteq g^{\star}bInt(F^{+}(IntCl(Y \setminus E)))$ this implies that $X \setminus F^{-}(E) \subseteq g^{\star}bInt(F^{+}(Y \setminus ClInt(E)))$, then $X \setminus F^{-}(E) \subseteq g^{\star}bInt(X \setminus F^{-}(ClInt(E)))$, it follow that $X \setminus F^{-}(E) \subseteq X \setminus g^{\star}bCl(F^{-}(ClInt(E)))$. Hence $g^{\star}bCl(F^{-}(ClInt(E))) \subseteq F^{-}(E)$.

Theorem 3.22 *Let $F : X \to Y$ be an upper almost $g^{\star}bp$-continuous. If Y is preopen set in Z, then $F : X \to Z$ is upper almost $g^{\star}bp$-continuous.*

proof. Let A be any regular open set of Z. Since Y is preopen, then $A \cap Y$ is regular open set in Y [see [10]]. Since F is upper almost $g^{\star}bp$-continuous then $F^{+}(A \cap Y)$ is $g^{\star}b$-open set in X. But $F(x) \in Y$ for each $x \in X$. Thus $F^{+}(A) = F^{+}(A \cap Y)$ is a $g^{\star}b$-open set in X. Therefore F is upper almost $g^{\star}bp$-continuous.

Theorem 3.23 *If $F : X \to Y$ is an upper almost $g^{\star}bp$-continuous multifunction and A is $g^{\star}b$-closed set of X, then the restriction function $F|A : A \to Y$ is almost $g^{\star}bp$-continuous multifunction.*

proof. Let B be any regular closed set of Y. Since F is upper almost $g^{\star}bp$-continuous multifunction, then by Theorem 3.14, $F^{+}(B)$ is $g^{\star}b$-closed set in X, and $(F|A)^{+}(B) = A \cap F^{+}(B)$. Since A is $g^{\star}b$-closed, so $A \cap F^{+}(B)$ is $g^{\star}b$-closed set in A (see Theorem 2.8). Hence $F|A$ is upper almost $g^{\star}bp$-continuous multifunction.

Theorem 3.24 *If $F : X \to Y$ is an upper almost $g^\star bp$-continuous injection and Y is r-T_1, then X is $g^\star b - T_1$.*

proof. Assume that Y is r-T_1. For any distinct points x and y in X, there exists regular open set A and W such that $F(x) \in A$, $F(y) \notin A$, $F(x) \notin W$ and $F(y) \in W$. Since F is upper almost $g^\star bp$-continuous there exists a $g^\star b$-open sets G and H such that $x \in G$, $y \in H$, $F(G) \subseteq A$ and $F(H) \subseteq W$. Thus we obtain $y \notin G$, $x \notin H$. this show that X is $g^\star b - T_1$.

Theorem 3.25 *If $F : X \to Y$ is upper almost $g^\star bp$-continuous and Y is pre-T_2 then X is $g^\star b - T_2$.*

proof. For any pair of distinct points x and y in X, there exists disjoint preopen sets U and V in Y such that $F(x) \in U$ and $F(y) \in V$. Since F is upper almost $g^\star bp$-continuous, there exists $g^\star b$-open sets G and H in X containing x and y, respectively, such that $F(G) \subseteq IntCl(U)$ and $F(H) \subseteq IntCl(V)$. Since U and V are disjoint, we have $IntCl(U) \cap IntCl(V) = \phi$, hence $G \cap H = \phi$. This shows that X is $g^\star b - T_2$.

4. Weakly $g^\star bp$-continuous multifunction

Definition 4.1 *A multifunction $F : X \to Y$ is said to be:*

1. *Upper weakly $g^\star bp$-continuous at a point $x \in X$ if for each preopen set A of Y such that $F(x) \in A$, there exists a $g^\star b$-open set U containing x such that $F(U) \subseteq Cl(A)$.*
2. *Lower weakly $g^\star bp$-continuous at a point $x \in X$ if for each preopen set A of Y such that $F(x) \in A$, there exists a $g^\star b$-open set U of X containing X such that $F(U) \cap Cl(A) \neq \phi$.*
3. *Upper (Lower) almost $g^\star bp$-continuous if it has this property at each point of X.*

Theorem 4.2 *Let $F : X \to Y$ be a multifunction. If $F^+(ClA)$ is $g^\star b$-open set in X for each preopen set A in Y, then F is upper weakly $g^\star bp$-continuous.*

proof. Let $x \in X$ and let A be any preopen set of Y containing $F(x)$. Then $x \in F^+(A) \subseteq F^+(ClA)$. By hypothesis, we have $F^+(ClA)$ is $g^\star b$-open set in X containing x. Therefore, we obtain $F(F^+(ClA)) \subseteq ClA$. Hence F is upper weakly $g^\star bp$-continuous.

It is obvious that upper almost $g^\star bp$-continuous implies upper weakly $g^\star bp$-continuous. However, the converse is not true in general as it shown in the following example.

Example 4.3 *Consider $X = Y = \{a, b, c, d\}$ with the topology $\tau = \sigma = \{\phi, \{b\}, \{c\}, \{a, b\}, \{b, c\}, \{a, b, c\}, X\}$, with identity multifunction $F : (X, \tau) \to (Y, \sigma)$ F is upper weakly $g^\star bp$-continuous but not upper almost $g^\star b$-continuous since for a preopen set $B = \{a, b\}$ in Y $F^+(IntClB) = \{a, b\}$ which is not $g^\star b$-open in X.*

Theorem 4.4 *If $F : X \to Y$ is upper weakly $g^\star bp$-continuous multifunction and Y is almost p-regular, then F is upper almost $g^\star bp$-continuous.*

proof. Let $x \in X$ and let A be preopen set of Y. By the almost p-regularity of Y there exists a regular open set G of Y such that $F(x) \in G \subseteq Cl(G) \subseteq IntCl(A)$. Since F is upper weakly $g^\star bp$-continuous, there exists a $g^\star b$-open set U in X such that $F(U) \subseteq Cl(G) \subseteq IntCl(A)$. Therefore F is almost $g^\star bp$-continuous.

Theorem 4.5 *Let $F : X \to Y$ be a multifunction. If for each $x \in X$ and each regular closed set R of Y containing $F(x)$, there exists a $g^\star b$-open set U in X containing x such that $F(U) \subseteq R$, then F is upper weakly $g^\star bp$-continuous.*

proof. Let $x \in X$ and let A be any preopen set of Y containing $F(x)$. Then put $R = Cl(A)$ which is a regular closed set of Y containing $F(x)$. By hypothesis, there exists a $g^\star b$-open set U in X containing x such that $F(U) \subseteq R$. Hence F is upper weakly $g^\star bp$-continuous.

Theorem 4.6 *Let $F : X \to Y$ be a multifunction. If the inverse image of each regular closed set of Y is a $g^\star b$-open set in X, then F is upper weakly $g^\star bp$-continuous.*

proof. Let A be any preopen set of Y. Then $Cl(A)$ is a regular closed set in Y. By hypothesis, we have $F^+(Cl(A))$ is a $g^\star b$-open set in X. Therefore, by Theorem 4.2, F is upper weakly $g^\star bp$-continuous.

Corollary 4.7 *Let $F : X \to Y$ be a multifunction. If the inverse image of each regular open set of Y is a $g^\star b$-closed set in X, then F is upper weakly $g^\star bp$-continuous.*

Corollary 4.8 *Let $F : X \to Y$ be a multifunction. If $F^+(IntF)$ is $g^\star b$-closed set in X for each preclosed set F in Y, then F is upper weakly $g^\star bp$-continuous.*

Theorem 4.9 *Let $F : X \to Y$ be upper weakly $g^\star bp$-continuous multifunction, if A is $g^\star b$-closed subset of X, then the restriction $F|A : A \to Y$ is upper weakly $g^\star bp$-continuous in the subspace A.*

proof. Let $x \in A$ and let B be a preclosed set of Y containing $F(x)$. Since F is upper weakly $g^\star bp$-continuous,by Corollary 4.8, $F^+(IntB)$ is $g^\star b$-closed set in X, and $(F|A)^+(IntB) = A \cap F^+(IntB)$ is $g^\star b$-closed in X, by Theorem (3.30)[24], it is $g^\star b$-closed in A. Hence $F|A$ is upper weakly $g^\star bp$-continuous.

Theorem 4.10 *Let $F : X \to Y$ be upper weakly $g^\star bp$-continuous multifunction and for each $x \in X$. If Y is any subset of Z containing $F(x)$, then $F : X \to Z$ is upper weakly $g^\star bp$-continuous.*

proof. Let $x \in X$ and A be any preopen set of Z containing $F(x)$. Then $A \cap Y$ is preopen in Y containing $F(x)$. Since $f : X \to Y$ is upper weakly $g^\star bp$-continuous, there exists a $g^\star b$-open set U of X containing x such that $F(U) \subseteq Cl(A \cap Y)$ and hence $F(U) \subseteq ClA$. Therefore, $F : X \to Z$ is upper weakly $g^\star bp$-continuous.

Theorem 4.11 *For a function $f : (X, \tau) \to (Y, \sigma)$, the following statements are equivalent:*

1. *F is upper weakly $g^\star bp$-continuous,*
2. *$g^\star bClF^+(IntpClB) \subseteq F^+(pClB)$, for each $B \subseteq Y$,*
3. *$F^-(pIntB) \subseteq g^\star bIntF^-(ClpIntB)$, for each $B \subseteq Y$,*
4. *$F^-(pIntpClA) \subseteq g^\star bIntF^-(ClA)$, for each preopen set A of Y,*
5. *$F^-(A) \subseteq g^\star bIntF^-(ClA)$, for each regular preopen set A of Y,*
6. *$g^\star bClF^+(IntF) \subseteq F^+(F)$, for each regular preclosed set F of Y,*
7. *$g^\star bClF^+(IntF) \subseteq F^+(ClIntF)$, for each preclosed set F of Y,*
8. *$g^\star bClF^+(A) \subseteq F^+(ClA)$, for each preopen set A of Y,*
9. *$F^-(IntF) \subseteq g^\star bIntF^-(F)$, for each preclosed set F of Y.*

proof. (1) $\Rightarrow$ (2). Let B be any subset of Y. Assume that $x \notin F^+(pClB)$. Then $F(x) \notin pClB$ and there exists a preopen set A containing $F(x)$ such that $A \cap B = \phi$, hence $A \cap IntpClB = \phi$, then $A \subseteq Y \setminus (IntpClB)$ and $ClA \cap IntpClB = \phi$. Hence, by(1), there exists a $g^\star b$-open set U of X containing x such that $F(U) \subseteq ClA$. Therefore, we have $f(U) \cap IntpClB = \phi$ which implies $U \cap F^+(IntpClB) = \phi$ and hence $x \notin g^\star bClF^+(IntpClB)$. Therefore, we obtain $g^\star bClF^+(IntpClB) \subseteq F^+(pClB)$.
(2) $\Rightarrow$ (3). Let B be any subset of Y. Then apply(2) to $Y \setminus B$ we obtain $g^\star bClF^+(IntpCl(Y \setminus B)) \subseteq F^+(pCl(Y \setminus B)) \Rightarrow g^\star bClF^+(Int(Y \setminus pIntB)) \subseteq F^+(Y \setminus pIntB) \Rightarrow g^\star bClF^+(Y \setminus ClpIntB) \subseteq F^+(Y \setminus pIntB) \Rightarrow g^\star bCl(X \setminus F^-(ClpIntB) \subseteq X \setminus F^-(pIntB) \Rightarrow X \setminus g^\star bInt(F^-(ClpIntB)) \subseteq X \setminus F^-(pIntB) \Rightarrow F^-(pIntB) \subseteq g^\star bInt(F^-(ClpIntB))$. Therefore, we obtain $F^-(pIntB) \subseteq g^\star bInt(F^-(ClpIntB))$.
(3) $\Rightarrow$ (4). Let A be any preopen set of Y. Then apply(3) to $pClA$ we obtain $F^-(pIntpClA) \subseteq g^\star bInt(F^-(ClpIntpClA)) \subseteq g^\star bInt(F^-(ClIntClA)) = g^\star bIntF^-(ClA)$. Therefore we obtain $F^-(pIntpClA) \subseteq g^\star bIntF^-(ClA)$.
(4) $\Rightarrow$ (5). Let A be any regular preopen set of Y. Then A is preopen set of Y. By(4) we have $F^-(A) = F^-(pIntpClA) \subseteq g^\star bIntF^-(ClA)$. Therefore we obtain $F^-(A) \subseteq g^\star bIntF^-(ClA)$.
(5) $\Rightarrow$ (6). Let F be any regular preclosed set of Y. Then $Y \setminus F$ is a regular preopen set of Y. By(5), we have $F^-(Y \setminus F) \subseteq g^\star bIntF^-(Cl(Y \setminus F)) \Rightarrow X \setminus F^+(F) \subseteq g^\star bIntF^-(Y \setminus IntF) \Rightarrow X \setminus F^+(F) \subseteq g^\star bInt(X \setminus F^+(IntF)) \Rightarrow X \setminus F^+(F) \subseteq X \setminus g^\star bClF^+(IntF) \Rightarrow g^\star bClF^+(IntF) \subseteq F^+(F)$. Hence $g^\star bClF^+(IntF) \subseteq F^+(F)$.
(6) $\Rightarrow$ (7). Let F be any preclosed set of Y. Then $pClpIntF$ is regular preclosed set of Y. By(6) we have $g^\star bClF^+(IntpClpIntF) = g^\star bClF^+(IntF) \subseteq F^+(pClpIntF)$. Therefore we obtain $g^\star bClF^+(IntF) \subseteq F^+(pClpIntF)$.
(7) $\Rightarrow$ (8). Let A be any preopen set of Y. Then by(7), we have $g^\star bClF^+(A) \subseteq g^\star bClF^+(IntClA) \subseteq F^+(pClpIntClA) \subseteq F^+(ClIntClA) = F^+(ClA)$. Therefore,$g^\star bClF^+(A) \subseteq F^+(ClA)$.
(8) $\Rightarrow$ (9). Let F be any preclosed set of Y. Then $Y \setminus F$ is preopen set of Y. By(8), we have $g^\star bClF^+(Y \setminus F) \subseteq F^+(Cl(Y \setminus F)) \Rightarrow g^\star bCl(X \setminus F^-(F) \subseteq F^-(Y \setminus IntF) \Rightarrow X \setminus g^\star bIntF^-(F) \subseteq X \setminus F^-(IntF) \Rightarrow F^-(IntF) \subseteq g^\star bIntF^-(F)$. Therefore, $F^-(IntF) \subseteq g^\star bIntF^-(F)$.
(9) $\Rightarrow$ (1). Let $x \in X$ and let A be any preopen set in Y containing $F(x)$. Then $x \in F^-(A)$ and ClA is a closed set, hence preclosed, in Y. By (9), we have $x \in F^-(A) \subseteq F^-(IntClA) \subseteq g^\star bIntF^-(ClA)$. If we put $U = g^\star bIntF^-(ClA)$, then we obtain that $x \in U$ and $F(U) \subseteq ClA$. Therefore,, F is weakly $g^\star bp$-continuous.

Theorem 4.12 *The followings are equivalent for a function $f : X \to Y$.*

1. *F is upper weakly $g^\star bp$-continuous,*
2. *$F(g^\star bCl(A)) \subseteq Cl_\theta(F(A))$ for each subset A of X,*
3. *$g^\star bCl(F^+(B)) \subseteq F^+(Cl_\theta(B))$ for each subset B of Y,*
4. *$g^\star bCl(F^+(Int(Cl_\theta(B)))) \subseteq F^+(Cl_\theta(B))$ for every subset B of Y.*

proof. (1) $\Rightarrow$ (2). Let A be any subset of X. Suppose that $F(g^\star bCl(A)) \nsubseteq Cl_\theta(F(A))$. Then there exists $y \in F(g^\star bCl(A))$ such that $y \notin Cl_\theta(F(A))$, so there exists an open set G in Y containing y such that $ClG \cap F(A) = \phi$. If $F^+(y) = \phi$, then there is nothing to prove. Suppose that x be any arbitrary point of $F^+(y)$, so $F(x) \in G$. Since G is open then it is preopen in Y and by(1), there exists a $g^\star b$-open set U of X containing x such that $F(U) \subseteq Cl(G)$. Therefore, we have $F(U) \cap F(A) = \phi$, so $x \notin g^\star bCl(A)$. Hence $y \notin F(g^\star bCl(A))$ which is a contradiction. Therefore, $F(g^\star bCl(A)) \subseteq Cl_\theta(F(A))$.
(2) $\Rightarrow$ (3). Let B be any subset of Y. Set $A = F^+(B)$ in (2), then we have $f(g^\star bCl(F^+(B))) \subseteq Cl_\theta(B)$ and $g^\star bCl(F^+(B)) \subseteq F^+(Cl_\theta(B))$.
(3) $\Rightarrow$ (4). Let B be any subset of Y. Since $Cl_\theta(B)$ is closed in Y hence is preclosed in Y. We have $g^\star bCl(F^+(Int(Cl_\theta(B)))) \subseteq F^+(Cl_\theta(Int(Cl_\theta(B)))) \subseteq F^+(Cl(Int(Cl_\theta(B)))) \subseteq F^+(Cl_\theta(B))$.
(4) $\Rightarrow$ (1). Let G be any preopen set of Y, then $G \subseteq IntCl(G)$. Apply(4) to $IntCl(G)$, we get $g^\star bClF^+(IntCl_\theta(IntCl(G))) \subseteq F^+(Cl_\theta(IntCl(G)))$. By Theorem 2.7, we have $g^\star bClF^+(IntCl(G)) \subseteq F^+(Cl(IntCl(G)))$. So, we get, $g^\star bCl(F^+(G)) \subseteq g^\star bClF^+(IntCl(G)) \subseteq F^+(Cl(IntCl(G))) \subseteq F^+(ClG)$. Hence, by Theorem 4.11, F is upper weakly $g^\star bp$-continuous.

Corollary 4.13 *If a multifunction $F : X \to Y$ is upper weakly $g^\star bp$-continuous, then $F^+(A)$ is $g^\star b$-closed in X for every θ-closed set A in Y.*

proof. If A is θ-closed, so by Theorem 4.12, we obtain that $g^\star bCl(F^+(A)) \subseteq F^+(Cl_\theta A) = F^+(A)$. Therefore, $F^+(A)$ is $g^\star b$-closed.

Corollary 4.14 *Let $F : X \to Y$ be any multifunction. If $F^+(Cl_\theta(B))$ is $g^\star b$-closed in X for every subset B of Y, then $F : X \to Y$ is upper weakly $g^\star bp$-continuous .*

proof. Since $F^+(Cl_\theta(B))$ is $g^\star b$-closed in X, we have $g^\star bCl(F^+(B)) \subseteq g^\star bClF^+(Cl_\theta(B)) = F^+(Cl_\theta(B))$. Therefore, by Theorem 4.12, f is upper weakly $g^\star bp$-continuous.

Theorem 4.15 *A multifunction $F : X \to Y$ is upper weakly $g^\star bp$-continuous if and only if $F^+(A) \subseteq g^\star bIntF^+(Cl(A))$ for each preopen set A in Y.*

proof. Necessity. Let F be upper weakly $g^\star bp$-continuous and let A be any preopen set of Y, then $A \subseteq IntCl(A)$. Therefore, by Theorem 4.11, we get $F^+(A) \subseteq F^+(IntCl(A)) \subseteq g^\star bIntF^+(Cl(A))$. Hence, $F^+(A) \subseteq g^\star bIntF^+(Cl(A))$.
Sufficiency. Let A be any regular preopen set of Y, then A is preopen set in Y. By hypothesis, we have $F^+(A) \subseteq g^\star bIntF^+(Cl(A))$. Therefore, by Theorem 4.11, f is upper weakly $g^\star bp$-continuous.

Corollary 4.16 *A multifunction $F : X \to Y$ is upper weakly $g^\star bp$-continuous if and only if $g^\star bClF^+(Int(F)) \subseteq F^+(F)$ for each preopen set F in Y.*

Theorem 4.17 *If $F : X \to Y$ is a upper weakly $g^\star bp$-continuous function and Y is extremally disconnected space, then F is upper almost $g^\star bp$-continuous.*

proof. Let $x \in X$ and let A be any preopen set of Y containing $F(x)$. Since F is upper weakly $g^\star bp$-continuous, there exists a $g^\star b$-open set U of X containing x such that $F(U) \subseteq Cl(A)$. Since Y is extremally disconnected, then $F(U) \subseteq IntCl(A)$. Therefore, F is upper almost $g^\star bp$-continuous.

Theorem 4.18 *If $F : X \to Y$ is upper weakly $g^\star bp$-continuous injection and Y is pre-T_1 then X is $g^\star b - T_1$.*

proof. Assume that Y is pre-T_1. For any distinct points x and y in X, there exist preopen sets A and W such that $F(x) \in A$, $F(y) \notin A$, $F(x) \notin W$ and $F(y) \in W$. Since F is upper weakly $g^\star bp$-continuous, there exists a $g^\star b$-open sets G and H in X containing x and y respectively, such that $F(G) \subseteq Cl(U)$, $F(H) \subseteq Cl(A)$, $F(H) \subseteq Cl(W)$ since A and W are disjoint then $Cl(A)$ and $Cl(W)$ are disjoint. Thus we obtain $y \notin G$, $x \notin H$. This show that X is $g^\star b - T_1$.

Theorem 4.19 *If $F : X \to Y$ is upper weakly $g^\star bp$-continuous and Y is pre-T_2 then X is $g^\star b - T_2$.*

proof. For any pair of distinct points x and y in X, there exist disjoint preopen sets U and V in Y such that $F(x) \in U$ and $F(y) \in V$. Since F is upper weakly $g^\star bp$-continuous, there exist $g^\star b$-open sets G and H in X containing x and y, respectively, such that $F(G) \subseteq Cl(U)$ and $F(H) \subseteq Cl(V)$. Since U and V are disjoint, we have $Cl(U) \cap Cl(V) = \phi$, hence $G \cap H = \phi$. This shows that X is $g^\star b - T_2$.

5. Contra $g^\star bp$-continuous function

Definition 5.1 *A multifunction $F : X \to Y$ is called:*

1. *Upper contra $g^\star bp$-continuous at $x \in X$ if for each preclosed set A such that $x \in F^+(A)$, there exists a $g^\star b$-open set U containing x such that $U \subseteq F^+(A)$.*

2. *Lower contra $g^\star bp$-continuous at $x \in X$ if for each preclosed set A such that $x \in F^-(A)$, there exists a $g^\star b$-open set U containing x such that $U \subseteq F^-(A)$.*

3. *Lower (upper) contra $g^\star bp$-continuous if F has this property at each point of X.*

Theorem 5.2 *The following are equivalent for a multifunction $F : X \to Y$.*

1. *F is upper contra $g^\star bp$-continuous.*
2. *$F^+(A)$ is $g^\star b$-open set for any preclosed set $A \subseteq Y$.*
3. *$F^-(U)$ is $g^\star b$-closed set for any preopen set $U \subseteq Y$.*
4. *For each $x \in X$ and each preclosed set A containing $F(x)$, there exists a $g^\star b$-open set U containing x such that if $y \in U$, then $F(y) \subseteq A$.*

proof. (1) $\Rightarrow$ (2). Let A be a preclosed set in Y and $x \in F^+(A)$. Since F is upper contra $g^\star bp$-continuous, there exists a $g^\star b$-open set U containing x such that $U \subseteq F^+(A)$. Thus, $F^+(A)$ is $g^\star b$-open. The converse of the proof is similar. (2) $\Rightarrow$ (3). This follows from the fact that $F^+(Y \setminus A) = X \setminus F^-(A)$ for every subset A of Y. (1) $\Leftrightarrow$ (4). Obvious.

Theorem 5.3 *The following are equivalent for a multifunction $F : X \to Y$.*

1. *F is upper contra $g^\star bp$-continuous.*
2. *$F^-(A)$ is $g^\star b$-open set for any preclosed set $A \subseteq Y$.*
3. *$F^+(U)$ is $g^\star b$-closed set for any preopen set $U \subseteq Y$.*
4. *For each $x \in X$ and each preclosed set A such that $F(x) \cap A \neq \phi$, if $y \in U$, then $F(y) \subseteq A$, there exists a $g^\star b$-open set U containing x such that if $y \in U$, then $F(y) \cap A \neq \phi$*

proof. The proof is similar to the proof of Theorem 5.2.

Theorem 5.4 *If a multifunction $F : X \to Y$ is upper contra $g^\star bp$-continuous and Y is preregular, then F is upper $g^\star bp$-continuous.*

proof. Let $x \in X$ and A is preopen set of Y containing $F(x)$. Since Y is preregular, then there exists a preopen set G in Y containing $F(x)$ such that $pCl(G) \subseteq A$. Since F is upper contra $g^\star bp$-continuous, then by Theorem 5.2, there exists a $g^\star b$-open set U in X containing x such that $F(U) \subseteq pCl(G)$. Then $F(U) \subseteq pCl(G) \subseteq A$. Hence F is upper $g^\star bp$-continuous.

Theorem 5.5 *If a multifunction $F : X \to Y$ is upper contra $g^\star bp$-continuous, then F is upper weakly $g^\star bp$-continuous.*

proof. Let A be any preopen set in Y. Since F is upper contra $g^{\star}bp$-continuous, then $F^{+}(A)$ is $g^{\star}b$-closed set of X. Hence, by Theorem 4.2, we obtain that F is upper weakly $g^{\star}bp$-continuous.

The converse of Theorem 5.5 is not true in general as it is shown in the following example.

Example 5.6 *Consider $X = Y = \{a, b, c\}$ with the topology $\tau = \{\phi, \{c\}, \{a, c\}, \{b, c\}, X\}$, $\sigma = \{\phi, \{b\}, \{a, b\}, Y\}$ and a multifunction $F : (X, \tau) \to (Y, \sigma)$ is defined by $F(a) = c$, $F(b) = b$ and $F(c) = a$, F is upper weakly $g^{\star}bp$-continuous but not upper contra $g^{\star}bp$-continuous since for preopen set $B = \{a, b\}$ in Y and $F^{-1}(B) = \{b, c\}$ is not $g^{\star}b$-closed in X.*

Theorem 5.7 *If a multifunction $F : X \to Y$ is upper contra $g^{\star}bp$-continuous and X is $g^{\star}b$-space, then F is upper contra continuous.*

proof. Let A be an open set in Y, then i is preopen. Since F is upper contra $g^{\star}bp$-continuous, so $F^{+}(A)$ is $g^{\star}b$-closed in X. Since X is $g^{\star}b$-space, hence, $F^{+}(A)$ is closed in X. Thus F is upper contra continuous.

6. Multifunctions with $g^{\star}bp$-closed graphs

Definition 6.1 *Let $F : X \to Y$ be any multifunction, the graph of the function F is denoted by $G(F)$ and is said to be $g^{\star}bp$-closed if for each $(x, y) \notin G(F)$, there exists a $g^{\star}b$-open set U in X containing x, and a preopen set V of Y containing y such that $(U \times V) \cap G(F) = \phi$.*

Lemma 6.2 *The multifunction $F : X \to Y$ has a $g^{\star}bp$-closed graph if and only if for each $x \in X$ and $y \in Y$ such that $y \neq F(x)$, there exists a $g^{\star}b$-open set U and a preopen set V containing x and y respectively, such that $F(U) \cap V = \phi$.*

proof. Follows from Definition 6.1.

Proposition 6.3 *If $F : X \to Y$ is upper weakly $g^{\star}bp$-continuous, and Y is pre-T_2 space, then $G(F)$ is a $g^{\star}bp$-closed graph.*

proof. Suppose that $(x, y) \notin G(F)$, then $F(x) \neq y$. By the fact that Y is pre-T_2, there exist preopen sets W and V such that $F(x) \in W$, $y \in V$ and $W \cap V = \phi$. It follow that $ClW \cap V = \phi$. Since F is upper weakly $g^{\star}bp$-continuous, so by Definition 4.1, there exists a $g^{\star}b$-open set U in X containing x such that $F(U) \subseteq ClW$. Hence, we have $F(U) \cap V = \phi$. This means that $G(F)$ is $g^{\star}bp$-closed graph.

Theorem 6.4 *Let $F : X \to Y$ be a preirresolute multifunction where X is an arbitrary topological space and Y is pre-T_2. Then $G(f)$ is $g^{\star}bp$-closed.*

proof. Let $(x, y) \notin G(F)$. Then $F(x) \neq y$. Since Y is pre-T_2, there exists $U \in PO(Y, F(x))$, $V \in PO(Y, y)$ such that $U \cap V = \phi$. Since F is upper preirresolute, this implies that $F^{+}(U) = W \in PO(X, x)$, so $W \in g^{\star}bO(X, x)$. Hence $F(W) = F(F^{+}(U)) \subseteq U$. It follow from above that $F(W) \cap V = \phi$. Therefore, by the Lemma 6.2, we obtain that $G(F)$ is $g^{\star}bp$-closed.

Definition 6.5 *The graph $G(F)$ of a multifunction $F : X \to Y$ is called contra $g^{\star}bp$-closed if for each $(x, y) \notin G(F)$, there exist $U \in g^{\star}bO(X, x)$, $V_{\in}PC(Y, y)$ such that $(U \times V) \cap G(F) = \phi$.*

Lemma 6.6 *The graph $G(F)$ of a multifunction $F : X \to Y$ is contra $g^{\star}bp$-closed if and only if for each $(x, y) \notin G(F)$, there exist $U \in g^{\star}bO(X, x)$, $V_{\in}PC(Y, y)$ such that $F(U) \cap V = \phi$.*

Theorem 6.7 *If a multifunction $F : X \to Y$ is upper contra $g^{\star}bp$-continuous and Y is pre-Urysohn, then $G(F)$ is contra $g^{\star}bp$-closed.*

proof. Let $(x, y) \notin G(F)$. Then $y \neq F(x)$ and there exists preopen sets H_1, H_2 such that $F(x) \in H_1$, $y \in H_2$ and $pCl(H_1) \cap pCl(H_2) = \phi$. From hypothesis, there exists $V \in g^{\star}bO(X, x)$ such that $F(V) \subseteq pCl(H_1)$. Therefore, we obtain $F(V) \cap pCl(H_2) = \phi$. This shows that $G(F)$ is contra $g^{\star}bp$-closed..

Theorem 6.8 *If a multifunction $F : X \to Y$ is upper $g^{\star}bp$-continuous and Y is pre-T_1, then $G(F)$ is upper contra $g^{\star}bp$-closed.*

proof. Let $(x,y) \notin G(F)$. Then $y \neq F(x)$ and there exists preopen set H of Y such that $F(x) \in H$ and $y \notin H$. Since F is upper $g^{\star}bp$-continuous , there exists $g^{\star}b$-open set U in X containing x such that $F(U) \subseteq H$. Therefore we obtain $F(U) \cap (Y-H) = \phi$ and $(Y-H) \in PC(Y,y)$. This show that $G(F)$ is contra $g^{\star}bp$-closed.

Theorem 6.9 *Let $F : X \to Y$ be a multifunction and $G : X \to X \times Y$ the graph function of F, defined by $G(x) = (x, F(x))$ for every $x \in X$. If G is upper contra $g^{\star}bp$-continuous, then F is upper contra $g^{\star}bp$-continuous.*

proof. Let U be any preopen set in Y, then $X \times U$ is preopen set in $X \times Y$. Since G is upper contra $g^{\star}bp$-continuous. It follows that $F^{+}(U) = G^{+}(X \times U)$ is an $g^{\star}b$-closed in X. Thus F is upper contra $g^{\star}bp$-continuous.

Definition 6.10 *Let X and Y be topological spaces. A multifunction $F : X \to Y$ is said to have strongly $g^{\star}bp$-closed graph if for each $(x,y) \notin G(F)$, there exists $U \in g^{\star}bO(X,x)$, $V \in PO(Y,y)$ such that $(U \times Cl(V)) \cap G(F) = \phi$.*

Lemma 6.11 *A multifunction $F : X \to Y$ has strongly $g^{\star}bp$-closed graph if for each $(x,y) \notin G(F)$, there exists $U \in g^{\star}bO(X,x)$, $V \in PO(Y,y)$ such that $F(U) \cap Cl(V) = \phi$.*

Remark 6.12 *Evidently every multifunction has a strongly $g^{\star}bp$-closed graph it has a $g^{\star}bp$-closed graph but the converse is not true as it is shown by the following example.*

Example 6.13 *Let $X = Y = \{a,b\}$ and $\tau = \{\phi, \{a\}, X\}$, $\sigma = \{\phi, \{b\}, Y\}$, then the identity multifunction $I : (X,\tau) \to (Y,\sigma)$ has a $g^{\star}bp$-closed graph but it has not strongly $g^{\star}bp$-closed graph.*

Theorem 6.14 *If $F : X \to Y$ is upper almost $g^{\star}bp$-continuous and Y is pre-T_2, then $G(F)$ is strongly $g^{\star}bp$-closed graph.*

proof. Let $(x,y) \notin G(F)$. Since Y is pre-T_2, then there exists preopen set V of Y containing y such that $F(x) \notin Cl(V)$. Now $Cl(V)$ is regular closed set in Y. So, $Y - Cl(V)$ is regular open in Y containing $F(x)$. Therefore, by the upper almost $g^{\star}bp$-continuous of F there exists $U \in g^{\star}bO(X,x)$ such that $F(U) \subseteq Y - Cl(V)$. Hence $F(U) \cap Cl(V) = \phi$.

Corollary 6.15 *If $F : X \to Y$ is upper $g^{\star}bp$-continuous and Y is pre-T_2 then $G(F)$ is strongly $g^{\star}bp$-closed.*

proof. Since upper $g^{\star}bp$-continuous implies upper almost $g^{\star}bp$-continuous, the result follows.
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