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$\frac{G'}{G}-$Expansion method for related equations to the Zhiber–Shabat equation

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Abstract

In this work $\frac{G'}{G}-$expansion method has been employed for related equations to Zhiber–Shabat equation: Liouville equation, sinh-Gordon equation, Dodd–Bullough–Mikhailov (DBM) equation. It is shown that $\frac{G'}{G}-$expansion method provides a very effective and a powerful mathematical tool, for solving these equations.

Keywords: $\frac{G'}{G}-$*expansion method, Dodd–Bullough–Mikhailov equation, Liouville equation, sinh-Gordon equation, Zhiber–Shabat equation..*

1 Introduction

In this paper, we consider Zhiber–Shabat equation as the following [16]

$$u_{xt}+pe^{u}+qe^{-u}+re^{-2u}=0, \tag{1}$$

where p, q, and r are arbitrary constant. When $q=r=0$, Eq. (1) reduces to the well-known Liouville equation. For $q=0, r\neq 0$, Eq. (1) reduces to the well-known Dodd–Bullough–Mikhailov equation (DBM), while $q\neq 0, r=0$, Eq. (1) reduces to the well-known Sinh-Gordon equation. The above equations play a significant role in many scientific applications such as solid state physics, nonlinear optics, plasma physics, fluid dynamics, mathematical biology, nonlinear optics, dislocations in crystals, and chemical kinetics, and quantum field theory [16–18].

2 The $\frac{G'}{G}-$expansion method

The Consider a nonlinear partial differential equation, in two independent variables say x and t , in the form

$$p(u,u_t,u_x,u_{xx},u_{tt},\ldots)=0. \tag{2}$$

Where $u=u(x,t)$ is an unknown function, and p is non-linear equation in $u=u(x,t)$ and its various partial derivatives. To apply the method following steps showed be fallowed.

Step 1. Using the transformation

$$\xi=x-wt, \tag{3}$$

where w is constant ,which converts the PDE to an ODE:

$$Q(u,u',u'',u''',\ldots)=0. \tag{4}$$

Where the superscripts stands for the derivatives with respect to ξ .

Step 2. Suppose that the solution of ODE (4) can be expressed by a polynomial in $\frac{G'}{G}$ as the following:

$$u(\xi)=\sum_{i=0}^{m}\alpha_i\left(\frac{G'}{G}\right)^i, \tag{5}$$

where $G=G(\xi)$ satisfies a second order LODE in the form

$$G''+\lambda G'+\mu G=0, \tag{6}$$

$\alpha_i,\lambda,$ and μ are constants to be determined later with $\alpha_m\neq 0$. The positive integer m can be determined by considering the homogeneous balance the highest order derivatives and the highest order nonlinear terms, appearing in ODE (4).

Step 3. Substituting Eq. (5) into Eq. (4) and using the second order LODE, Eq. (6), yields an algebraic equation involving powers of $\frac{G'}{G}$. Equating the coefficient of each power of $\frac{G'}{G}$ to zero leads to a system of algebraic equations for determining $\alpha_i, w, \lambda,$ *and* μ.

Step 4. Having the values of $\alpha_i, w, \lambda,$ *and* μ, from Step 3 and the solutions of LODE (6), which can be obtained easily, we are closed to the solutions of the nonlinear evolution Eq. (2).

3 Application $\frac{G'}{G}$ – expansion method

Detailed We first use $u(x,t)=u(\xi)$ that will carry out the Zhiber–Shabat Eq. (1) into the following form

$$-wu''+pe^{u}+qe^{-u}+re^{-2u}=0. \tag{7}$$

Applying the transformation $u=\ln(v)$, Eq. (7) terns to

$$-w\left(vv''-(v')^2\right)+pv^3+qv+r=0. \tag{8}$$

Suppose that the solution of Eq. (8) can be expressed by a polynomial in terms of $\frac{G'}{G}$ as the following:

$$v(\xi)=\sum_{i=0}^{m}\alpha_i\left(\frac{G'}{G}\right)^i, \tag{9}$$

where $G=G(\xi)$ satisfies the second order LODE (6). If $p\neq 0$, then by considering the homogeneous balance between vv'' and v^3 in Eq. (8), reads to $m=2$, and Eq. (9) turns to the following simple form

$$v(\xi)=\alpha_2\left(\frac{G'}{G}\right)^2+\alpha_1\left(\frac{G'}{G}\right)+\alpha_0,\quad \alpha_2\neq 0. \tag{10}$$

3.1 The Liouville equation

Let us consider $q=r=0$ and $p=1$, by using Eq. (8), we derive

$$-w\left(vv''-(v')^2\right)+v^3=0. \tag{11}$$

By substituting (10) into Eq. (11) and collecting all terms with the same power of $\frac{G'}{G}$ together, the left-hand side of Eq. (11) is converted into another polynomial in $\frac{G'}{G}$. Equating each coefficient of this polynomial to zero yields a set of simultaneous algebraic equations for determining $\alpha_2,\alpha_1,\alpha_0,w,\lambda,$ and μ as follows:

$$\left(\frac{G'}{G}\right)^0 : -2w\,\alpha_0\alpha_2\mu^2+\alpha_0^3-w\,\alpha_0\alpha_1\lambda\mu+w\,\alpha_1^2\mu^2=0\,,$$

$$\left(\frac{G'}{G}\right)^1 : 2w\,\alpha_1\alpha_2\mu^2+w\,\alpha_1{}^2\lambda\mu-2w\,\alpha_0\alpha_1\mu-w\,\alpha_0\alpha_1\lambda^2-6w\,\alpha_0\alpha_2\lambda\mu+3\alpha_0{}^2\alpha_1=0,$$

$$\left(\frac{G'}{G}\right)^2 : 2w\,\alpha_2{}^2\mu^2-8w\,\alpha_0\alpha_2\mu-3w\,\alpha_0\alpha_1\lambda-4w\,\alpha_0\alpha_2\lambda^2+w\,\alpha_1\alpha_2\lambda\mu+3\alpha_0{}^2\alpha_2+3\alpha_0\alpha_1{}^2=0,$$

$$\left(\frac{G'}{G}\right)^3 : -w\,\alpha_1{}^2\lambda-2w\,\alpha_0\alpha_1+\alpha_1^3-2w\,\alpha_1\alpha_2\mu-w\,\alpha_1\alpha_2\lambda^2+2w\,\alpha_2{}^2\lambda\mu-10w\,\alpha_0\alpha_2\lambda+6\alpha_0\alpha_1\alpha_2=0,$$

$$\left(\frac{G'}{G}\right)^4 : -6w\,\alpha_0\alpha_2-w\,\alpha_1{}^2-5w\,\alpha_1\alpha_2\lambda+3\alpha_1{}^2\alpha_2+3\alpha_0\alpha_2{}^2=0,$$

$$\left(\frac{G'}{G}\right)^5 : -2w\,\alpha_2{}^2\lambda-4w\,\alpha_1\alpha_2+3\alpha_1\alpha_2{}^2=0,$$

$$\left(\frac{G'}{G}\right)^6 : -2w\,\alpha_2{}^2+\alpha_2{}^3=0.$$

Solving this algebraic equations, yields to

$$\alpha_0=2w\,\mu,\ \alpha_1=2w\,\lambda,\ \alpha_2=2w\,, \tag{12}$$

where $\lambda,\mu,$ and w are arbitrary constants.

By substituting (12) into (10), we drive

$$v\,(\xi)=2w\left(\frac{G'}{G}\right)^2+2w\,\lambda\left(\frac{G'}{G}\right)+2w\,\mu. \tag{13}$$

Substituting the general solutions of Eq. (6) into (13), three types of traveling wave solutions of Liouville equation will be obtained:

When $\lambda^2-4\mu>0$,

$$v_1(\xi)=\frac{1}{2}w\left(\lambda^2-4\mu\right)\left(\frac{A\sinh\frac{1}{2}\sqrt{\lambda^2-4\mu}\,\xi+B\cosh\frac{1}{2}\sqrt{\lambda^2-4\mu}\,\xi}{A\cosh\frac{1}{2}\sqrt{\lambda^2-4\mu}\,\xi+B\sinh\frac{1}{2}\sqrt{\lambda^2-4\mu}\,\xi}\right)^2-\frac{1}{2}w\,\lambda^2+2w\,\mu. \tag{14}$$

In particular case of $A=0,\ B\neq 0,\lambda=0,$ and $\sqrt{-\mu}=\mu'$, Eq. (14) turns to the following

$$\begin{aligned} v_1(x,t) &= 2w\,\mu'^2\tanh^2\mu'\,(x-wt)-2w\,\mu'^2 \\ &= -2w\,\mu'^2\,\mathrm{sec}h^2\left[\mu'(x-wt)\right], \qquad \mu'>0, w<0. \end{aligned}$$

This is the same as the result obtained by wazwaz [16].

While $\lambda^2-4\mu<0$,

$$v_2(\xi)=\frac{1}{2}w\left(\lambda^2-4\mu\right)\left(\frac{-A\sin\frac{1}{2}\sqrt{4\mu-\lambda^2}\,\xi+B\cos\frac{1}{2}\sqrt{4\mu-\lambda^2}\,\xi}{A\cos\frac{1}{2}\sqrt{4\mu-\lambda^2}\,\xi+B\sin\frac{1}{2}\sqrt{4\mu-\lambda^2}\,\xi}\right)^2-\frac{1}{2}w\,\lambda^2+2w\,\mu. \tag{15}$$

And for $\lambda^2-4\mu=0$,

$$v_3(\xi)=2w\left(\frac{2B}{A+B\xi}\right)^2-\frac{1}{2}w\,\lambda^2+2w\,\mu. \tag{16}$$

Where A and B are arbitrary constants.

3.2 The Dodd–Bullough–Mikhailov equation

This equation can be obtained if we set $p=r=1,$ and $q=0,$ in Eq. (1). Then Eq. (8) turns to the following

$$-w\left(vv''-(v')^2\right)+v^3+1=0. \tag{17}$$

Proceeding a similar way as illustrated in 3.1, by substituting (17) into Eq. (9) and collecting all terms with the same power of $\frac{G'}{G}$ together, the left-hand side of Eq. (9) is converted into a polynomial in $\frac{G'}{G}$. Equating each coefficient of this polynomial to zero yields a set of simultaneous algebraic equations for determining $\alpha_2, \alpha_1, \alpha_0, w, \lambda,$ and μ as follows:

$$\left(\frac{G'}{G}\right)^0 : 1\text{-}2w\,\alpha_0\alpha_2\mu^2+\alpha_0^3-w\,\alpha_0\alpha_1\lambda\mu+w\,\alpha_1^2\mu^2=0\,,$$

$$\left(\frac{G'}{G}\right)^1 : 2w\,\alpha_1\alpha_2\mu^2+w\,\alpha_1^2\lambda\mu-2w\,\alpha_0\alpha_1\mu-w\,\alpha_0\alpha_1\lambda^2-6w\,\alpha_0\alpha_2\lambda\mu+3\alpha_0^2\alpha_1=0,$$

$$\left(\frac{G'}{G}\right)^2 : 2w\,\alpha_2^2\mu^2-8w\,\alpha_0\alpha_2\mu-3w\,\alpha_0\alpha_1\lambda-4w\,\alpha_0\alpha_2\lambda^2+w\,\alpha_1\alpha_2\lambda\mu+3\alpha_0^2\alpha_2+3\alpha_0\alpha_1^2=0,$$

$$\left(\frac{G'}{G}\right)^3 : -w\,\alpha_1^2\lambda-2w\,\alpha_0\alpha_1+\alpha_1^3-2w\,\alpha_1\alpha_2\mu-w\,\alpha_1\alpha_2\lambda^2+2w\,\alpha_2^2\lambda\mu-10w\,\alpha_0\alpha_2\lambda+6\alpha_0\alpha_1\alpha_2=0,$$

$$\left(\frac{G'}{G}\right)^4 : -6w\,\alpha_0\alpha_2-w\,\alpha_1^2-5w\,\alpha_1\alpha_2\lambda+3\alpha_1^2\alpha_2+3\alpha_0\alpha_2^2=0,$$

$$\left(\frac{G'}{G}\right)^5 : -2w\,\alpha_2^2\lambda-4w\,\alpha_1\alpha_2+3\alpha_1\alpha_2^2=0,$$

$$\left(\frac{G'}{G}\right)^6 : -2w\,\alpha_2^2+\alpha_2^3=0.$$

Solving this algebraic equations, reads two sets of the solutions;
The first solution set:

$$\alpha_0=\frac{2\mu+\lambda^2}{4\mu-\lambda^2},\alpha_1=\frac{6\lambda}{4\mu-\lambda^2},\ \alpha_2=\frac{6}{4\mu-\lambda^2},w=\frac{3}{4\mu-\lambda^2}. \tag{18}$$

The second solution set:

$$\alpha_0=\frac{2\mu+\lambda^2}{4\mu-\lambda^2}\left(\frac{-1}{2}\pm\frac{\sqrt{3}}{2}i\right),\quad \alpha_1=\frac{6\lambda}{4\mu-\lambda^2}\left(\frac{-1}{2}\pm\frac{\sqrt{3}}{2}i\right),\quad \alpha_2=\frac{6}{4\mu-\lambda^2}\left(\frac{-1}{2}\pm\frac{\sqrt{3}}{2}i\right),$$
$$w=\frac{3}{4\mu-\lambda^2}\left(\frac{-1}{2}\pm\frac{\sqrt{3}}{2}i\right). \tag{19}$$

Where λ and μ are arbitrary constants.
By substituting (18) into (10), we drive

$$v(\xi)=\frac{6}{4\mu-\lambda^2}\left(\frac{G'}{G}\right)^2+\frac{6\lambda}{4\mu-\lambda^2}\left(\frac{G'}{G}\right)+\frac{2\mu+\lambda^2}{4\mu-\lambda^2}, \tag{20}$$

Where

$$\xi=x-\frac{3}{4\mu-\lambda^2}t. \tag{21}$$

Substituting the general solutions of Eq. (6) into (20) we would have three types of traveling wave solutions of Dodd–Bullough–Mikhailov equation as follows:
For $\lambda^2-4\mu>0$,

$$v_1(\xi)=\frac{-3}{2}\left(\frac{A\sinh\frac{1}{2}\sqrt{\lambda^2-4\mu}\,\xi+B\cosh\frac{1}{2}\sqrt{\lambda^2-4\mu}\,\xi}{A\cosh\frac{1}{2}\sqrt{\lambda^2-4\mu}\,\xi+B\sinh\frac{1}{2}\sqrt{\lambda^2-4\mu}\,\xi}\right)^2-\frac{1}{4}\frac{6\lambda^2}{4\mu-\lambda^2}+\frac{2\mu+\lambda^2}{4\mu-\lambda^2}. \tag{22}$$

In particular, if $A=0,\ B\neq 0, \lambda=0,$ and $\sqrt{-\mu}=\mu'$, v_1 turns to

$$v_1(x,t)=\frac{1}{2}\left(1-3\tan h^2\left(\frac{1}{2}\sqrt{\frac{-3}{w}}(x-wt)\right)\right),\ w<0.$$

This is the same as the result obtained by wazwaz [16].

While $\lambda^2-4\mu<0$,

$$v_2(\xi)=\frac{-3}{2}\left(\frac{-A\sin\frac{1}{2}\sqrt{4\mu-\lambda^2}\,\xi+B\cos\frac{1}{2}\sqrt{4\mu-\lambda^2}\,\xi}{A\cos\frac{1}{2}\sqrt{4\mu-\lambda^2}\,\xi+B\sin\frac{1}{2}\sqrt{4\mu-\lambda^2}\,\xi}\right)^2-\frac{1}{4}\frac{6\lambda^2}{4\mu-\lambda^2}+\frac{2\mu+\lambda^2}{4\mu-\lambda^2}. \tag{23}$$

Where A and B are arbitrary constants.

By substituting (19) into (10), we drive

$$v(\xi)=\frac{6}{4\mu-\lambda^2}\left(\frac{-1}{2}\pm\frac{\sqrt{3}}{2}i\right)\left(\frac{G'}{G}\right)^2+\frac{6\lambda}{4\mu-\lambda^2}\left(\frac{-1}{2}\pm\frac{\sqrt{3}}{2}i\right)\left(\frac{G'}{G}\right)+\frac{2\mu+\lambda^2}{4\mu-\lambda^2}\left(\frac{-1}{2}\pm\frac{\sqrt{3}}{2}i\right), \tag{24}$$

Where

$$\xi=x-\frac{3}{4\mu-\lambda^2}\left(\frac{-1}{2}\pm\frac{\sqrt{3}}{2}i\right)t. \tag{25}$$

Substituting the general solutions of Eq. (6) into (24) we would have three types of traveling wave solutions of Dodd–Bullough–Mikhailov equation as follows:

For $\lambda^2-4\mu>0$,

$$v_{4,5}(\xi)=\left(\frac{-1}{2}\pm\frac{\sqrt{3}}{2}i\right)\left(\frac{-3}{2}\left(\frac{A\sinh\frac{1}{2}\sqrt{\lambda^2-4\mu}\,\xi+B\cosh\frac{1}{2}\sqrt{\lambda^2-4\mu}\,\xi}{A\cosh\frac{1}{2}\sqrt{\lambda^2-4\mu}\,\xi+B\sinh\frac{1}{2}\sqrt{\lambda^2-4\mu}\,\xi}\right)^2-\frac{1}{4}\frac{6\lambda^2}{4\mu-\lambda^2}+\frac{2\mu+\lambda^2}{4\mu-\lambda^2}\right). \tag{26}$$

While $\lambda^2-4\mu<0$,

$$v_{6,7}(\xi)=\left(\frac{-1}{2}\pm\frac{\sqrt{3}}{2}i\right)\left(\frac{-3}{2}\left(\frac{-A\sin\frac{1}{2}\sqrt{4\mu-\lambda^2}\,\xi+B\cos\frac{1}{2}\sqrt{4\mu-\lambda^2}\,\xi}{A\cos\frac{1}{2}\sqrt{4\mu-\lambda^2}\,\xi+B\sin\frac{1}{2}\sqrt{4\mu-\lambda^2}\,\xi}\right)^2-\frac{1}{4}\frac{6\lambda^2}{4\mu-\lambda^2}+\frac{2\mu+\lambda^2}{4\mu-\lambda^2}\right). \tag{27}$$

3.3 The Sinh-Gordon equation

This equation can be obtained if we set $r=0$. let's take $q=-1$ and $p=1$. Then Eq. (8) turns to

$$-w\left(vv''-(v')^2\right)+v^3-v=0. \tag{28}$$

Proceeding a similar procedure as illustrated in 3.1, a set of simultaneous algebraic equations for determining $\alpha_2,\alpha_1,\alpha_0,w,\lambda,$ and μ will be obtained as follows:

$$\left(\frac{G'}{G}\right)^0:\ -2w\,\alpha_0\alpha_2\mu^2-\alpha_0+\alpha_0^3-w\,\alpha_0\alpha_1\lambda\mu+w\,\alpha_1^2\mu^2=0,$$

$$\left(\frac{G'}{G}\right)^1:-\alpha_1+2w\,\alpha_1\alpha_2\mu^2+w\,\alpha_1^2\lambda\mu-2w\,\alpha_0\alpha_1\mu-w\,\alpha_0\alpha_1\lambda^2-6w\,\alpha_0\alpha_2\lambda\mu+3\alpha_0^2\alpha_1=0,$$

$$\left(\frac{G'}{G}\right)^2:\ 2w\,\alpha_2^2\mu^2-\alpha_2-8w\,\alpha_0\alpha_2\mu-3w\,\alpha_0\alpha_1\lambda-4w\,\alpha_0\alpha_2\lambda^2+w\,\alpha_1\alpha_2\lambda\mu+3\alpha_0^2\alpha_2+3\alpha_0\alpha_1^2=0,$$

$$\left(\frac{G'}{G}\right)^3:-w\,\alpha_1^2\lambda-2w\,\alpha_0\alpha_1+\alpha_1^3-2w\,\alpha_1\alpha_2\mu-w\,\alpha_1\alpha_2\lambda^2+2w\,\alpha_2^2\lambda\mu-10w\,\alpha_0\alpha_2\lambda+6\alpha_0\alpha_1\alpha_2=0,$$

$$\left(\frac{G'}{G}\right)^4:\ -6w\,\alpha_0\alpha_2-w\,\alpha_1^2-5w\,\alpha_1\alpha_2\lambda+3\alpha_1^2\alpha_2+3\alpha_0\alpha_2^2=0,$$

$$\left(\frac{G'}{G}\right)^5 : -2w\,\alpha_2{}^2\lambda - 4w\,\alpha_1\alpha_2 + 3\alpha_1\alpha_2{}^2 = 0,$$

$$\left(\frac{G'}{G}\right)^6 : -2w\,\alpha_2{}^2 + \alpha_2{}^3 = 0.$$

By the solution of these algebraic equations, the following results are obtained.

$$\alpha_0 = \pm\frac{\lambda^2}{4\mu-\lambda^2},\ \alpha_1 = \pm\frac{4\lambda}{4\mu-\lambda^2},\ \alpha_2 = \pm\frac{4}{4\mu-\lambda^2},\ w = \pm\frac{2}{4\mu-\lambda^2}. \tag{29}$$

Where λ and μ are arbitrary constants.

By substituting (29) into (10), we drive

$$v(\xi) = \pm\frac{4}{4\mu-\lambda^2}\left(\frac{G'}{G}\right)^2 \pm \frac{4\lambda}{4\mu-\lambda^2}\left(\frac{G'}{G}\right) \pm \frac{\lambda^2}{4\mu-\lambda^2}, \tag{30}$$

where

$$\xi = x \pm \frac{2}{4\mu-\lambda^2}t. \tag{31}$$

Substituting the general solutions of Eq. (6) into (30) we would have three types of traveling wave solutions of the Sinh-Gordon equation as follows:

For $\lambda^2 - 4\mu > 0$,

$$v_{1,2}(\xi) = \pm\left(\frac{A\sinh\frac{1}{2}\sqrt{\lambda^2-4\mu}\,\xi + B\cosh\frac{1}{2}\sqrt{\lambda^2-4\mu}\,\xi}{A\cosh\frac{1}{2}\sqrt{\lambda^2-4\mu}\,\xi + B\sinh\frac{1}{2}\sqrt{\lambda^2-4\mu}\,\xi}\right)^2. \tag{32}$$

While $\lambda^2 - 4\mu < 0$,

$$v_{3,4}(\xi) = \pm\left(\frac{-A\sin\frac{1}{2}\sqrt{4\mu-\lambda^2}\,\xi + B\cos\frac{1}{2}\sqrt{4\mu-\lambda^2}\,\xi}{A\cos\frac{1}{2}\sqrt{4\mu-\lambda^2}\,\xi + B\sin\frac{1}{2}\sqrt{4\mu-\lambda^2}\,\xi}\right)^2. \tag{33}$$

4 Conclusion

In this article, we have been looking for the exact solutions of the Liouville, sinh-Gordon, and Dodd–Bullough–Mikhailov (DBM) equations. We have achieved this goal by applying $\frac{G'}{G}$-expansion method. The free parameters can be determined by applying given initial or boundary conditions. The result shows that $\frac{G'}{G}$-expansion method is a powerful tool for obtaining exact solutions. Applications of $\frac{G'}{G}$-expansion method for other kinds of nonlinear equations and the problem of convergence of this method are under study in our research group. The computations associated in this work were performed by using Maple 13.

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