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Research Paper

A best proximity point theorem for generalized Mizoguchi- Takahashi contractions

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Abstract

The purpose of this paper is to provide sufficient conditions for the existence of a unique best proximity point for generalized Mizoguchi- Takahashi contractions.Our paper provides an extension of a result due to Gordji and Ramezani[3].

Keywords: *Fixed point, best proximity point, P-property, Mizoguchi- Takahashi contractions.*

1. Introduction

Let (X, d) be a metric space.Denote by P(X) the set of all nonempty subsets of X and $CB(X)$ the family of all nonempty closed and bounded subsets of X.A point x in X is a fixed point of a multivalued map $T : X \to P(X)$, if $x \in Tx$.Nadler [5] extended the Banach contraction principle to multivalued mappings.

Theorem 1.1 (5) *Let (X, d) be a complete metric spaces and let $T : X \to CB(X)$ be a multivalued map. Assume that there exists $r \in [0, 1)$ such that*

$$H(Tx, Ty) \leq rd(x, y)$$

for all $x, y \in X$,where H is the Hausedorff metric with respect to d.Then T has a fixed point.

The fixed point theory for multivalued mappings developed rapidly after the publication of Nadler's paper [5] in which he established a multivalued version of Banach's contraction principle. A huge number of generalizations of this principle appear in the literature. Particularly, the following generalization of Nadler's fixed point theorem due to Mizoguchi- Takahashi [4].

Theorem 1.2 (4) *Let (X, d) be a complete metric space and $T : X \to CB(X)$ be a multivalued map. Assume that*

$$H(Tx, Ty) \leq \phi(d(x, y))d(x, y) \tag{1}$$

for all $x, y \in X$, where ϕ is a function from $[0, \infty)$ into $[0, 1)$ satisfying $limsup_{s \to t^+}\phi(s) < 1$ for all $t \geq 0$. Then T has a fixed point.

Recently, Amini-Harandi and O'Regan [1] obtained a nice generalization of Mizoguchi and Takahashi's fixed point theorem.Throughout the article, let Ψ be the family of all functions $\psi : [0,\infty) \to [0,\infty)$ satisfying the following conditions:
(a) $\psi(s)= 0 \iff$ s=0,
(b) ψ is nondecreasing,

We denote by Φ the set of all functions $\phi : [0,\infty) \to [0,1)$ satisfying $limsup_{r\to t^+}\phi(r) < 1$ for all $t \geq 0$.
Amini-Harandi and O'Regan generalized the Mizoguchi-Takahashi contraction condition (1) as follows:

Theorem 1.3 (1) *Let (X,d) be a complete metric space and $T : X \to CB(X)$ be a multivalued map. Assume that*

$$\phi(H(Tx,Ty)) \leq \phi(\psi(d(x,y)))\psi(d(x,y))$$

for all $x,y \in X$, where $\psi \in \Psi$ is lower semicontinuous with $limsup_{s\to 0^+} \frac{s}{\phi(s)} < \infty$ and $\phi \in \Phi$. Then T has a fixed point.

Very recently, Gordji and Ramezani [3] established a new fixed point theorem for a self map $T : X \to X$ satisfying a generalized Mizoguchi-Takahashi's condition in the setting of ordered metric spaces. The main result in [3] is the following.

Theorem 1.4 (3) *Let $(X,d,\preceq)$ be a complete ordered metric space and $T : X \to X$ an increasing mapping such that there exists an element $x_0 \in X$ with $x_0 \preceq Tx_0$. Suppose that there exists a lower semicontinuous function $\psi \in \Psi$ and $\phi \in \Phi$ such that*

$$\psi(d(Tx,Ty)) \leq \phi(\psi(d(x,y)))\psi(d(x,y)).$$

for all $x,y \in X$ such that x and y are comparable. Assume that either T is continuous or X is such that the following holds: any $\preceq$-nondecreasing sequence $\{x_n\}$ with $x_n \to x$ implies $x_n \preceq x$ for all n. Then T has a fixed point.

The aim of this paper is to give a generalization of the Theorem 1.4 by considering a non-self map T.

2. Preliminary notes

First, we present a brief discussion about a best proximity point.

Let A be a nonempty subset of ametric space (X,d) and $T : A \to X$ be a mapping.The solutions of the equation $Tx = x$ are fixed point of T.Consequently, T(A)∩A≠ ∅ is a necessary condition for the existence of a fixed point for the operator T.If this necessary condition does not hold, then $d(x,Tx) > 0$ for any $x \in A$ and the mapping $T : A \to X$ does not have any fixed point.In this setting, our aim is to find an element $x \in A$ such that $d(x,Tx)$ is minimum in some sense. A point x in A for which $d(x,Tx) = d(A,B)$ is called a best proximity point of T.

In our context, we consider two nonempty subsets A and B of a complete metric space and a mapping $T : A \to B$ satisfying a generalized Mizoguchi-Takahashi's condition and find a best proximity point of T. We give an example to support our result.

Let A and B be two nonempty subsets of a metric space (X,d). We denote by A_0 and B_0 the following sets:

$$A_0 = \{x \in A : d(x,y) = d(A,B) \ for \ some \ y \in B\},$$

$$B_0 = \{y \in B : d(x,y) = d(A,B) \ for \ some \ x \in A\},$$

where d(A,B) = inf$\{d(x,y) : x \in A \ and \ y \in B\}$.

In[6] authers present sufficient conditions which determine when the sets A_0 and B_0 are nonempty.

Definition 2.1 *Let A, B be two nonempty subsets of a metric space (X, d). A mapping $T : A \to B$ is said to be a generalized Mizoguchi- Takahashi contractions if there exist $\phi \in \Phi$ and $\psi \in \Psi$ such that*

$$\psi(d(Tx,Ty)) \leq \phi(\psi(d(x,y)))\psi(d(x,y))$$

for any $x,y \in A$.

Definition 2.2 (6) *Let (A,B) be a pair of nonempty subsets of a metric space (X, d) with $A_0 \neq \emptyset$.Then the pair (A,B) is said to have the P-property if and only if for any $x_1, x_2 \in A_0$ and $y_1, y_2 \in B_0$,*

$$\left\{ \begin{array}{l} d(x_1, y_1) = d(A, B) \\ d(x_2, y_2) = d(A, B) \end{array} \right\} \implies d(x_1, x_2) = d(y_1, y_2).$$

3. Main results

Theorem 3.1 *Let (A,B) be a pair of nonempty closed subsets of a complete metric space (X,d) such that A_0 is nonempty. Let $T : A \to B$ be a continuous generalized Mizoguchi- Takahashi contraction mapping satisfying $T(A_0) \subset B_0$. Suppose that the pair (A,B) has the P-property.Then there exists a unique x^* in A such that $d(x^*, Tx^*)$ $= d(A, B)$.*

Proof. Since A_0 is nonempty, we take $x_0 \in A$.As $Tx_0 \in T(A_0) \subset B_0$, we can find $x_1 \in A_0$ such that $d(x_1, Tx_0)$ $= d(A, B)$.Similarly, since $Tx_1 \in T(A_0) \subset B_0$, there exists $x_2 \in A_0$ such that $d(x_2, Tx_1) = d(A, B)$. Repeating this process, we can get a sequence $\{x_n\}$ in A_0 satisfying

$$d(x_{n+1}, Tx_n) = d(A, B) \quad for\ any \quad n \in N$$

Since (A,B) has the P-property, we have that

$$d(x_n, x_{n+1}) = d(Tx_{n-1}, Tx_n) \ for\ any\ n \in N.$$

Taking into account that T is a generalized Mizoguchi- Takahashi contraction, for any $n \in N$, we have that

$$\begin{aligned} \psi(d(x_n, x_{n+1})) &\leq \phi(\psi(d(x_{n-1}, x_n)))\psi(d(x_{n-1}, x_n)) \\ &\leq \psi(d(x_{n-1}, x_n)) \end{aligned}$$

Since ψ is nondecreasing, we obtain

$$d(x_n, x_{n+1}) \leq d(x_{n-1}, x_n)$$

This means $d(x_n, x_{n+1})$ is a non-increasing sequence of positive real numbers.Hence there exists $\mu \geq 0$ such that

$$lim_{n\to\infty} d(x_n, x_{n+1}) = \mu$$

Since $\phi \in \Phi$, we have $limsup_{r\to\mu}\phi(r) < 1$.Then, there exist $\alpha \in [0, 1)$ and $\epsilon > 0$ such that $\phi(r) \leq \alpha$ for all $r \in [\mu, \mu + \epsilon)$. We can take $n_0 \in N$ such that $\mu \leq d(x_n, x_{n+1}) \leq \mu + \epsilon$ for all $n \geq n_0$. Then for all $n \geq n_0$, we have

$$\psi(d(x_n, x_{n+1})) \leq \phi(\psi(d(x_{n-1}, x_n)))\psi(d(x_{n-1}, x_n))$$

Letting $r \to \infty$ in the above inequality,we obtain that

$$\psi(\mu) \leq \alpha\psi(\mu) \tag{2}$$

Since $\alpha \in [0, 1)$, this implies that $\mu = 0$.Thus, we have

$$lim_{n\to\infty} d(x_n, x_{n+1}) = 0 \tag{3}$$

Now we claim that the sequence $\{x_n\}$ is a Cauchy sequence.Since $lim_{n\to\infty} d(x_n, x_{n+1}) = 0$, it is sufficient to prove that $\{x_{2n}\}$ is Cauchy sequence.

Suppose on the contrary that $\{x_{2n}\}$ is not a Cauchy sequence.Then there exist $\epsilon > 0$ and subsequences $\{x_{2n_k}\}$ and $\{x_{2m_k}\}$ of $\{x_{2n}\}$ such that $n_k > m_k > k$ and

$$d(x_{2m_k}, x_{2n_k}) \geq \epsilon \tag{4}$$

and

$$d(x_{2m_k}, x_{2n_{k-2}}) < \epsilon \tag{5}$$

Now, from (4) and the triangle inequality, we get

$$\epsilon \leq d(x_{2m_k}, x_{2n_k})$$
$$\leq d(x_{2m_k}, x_{2n_{k-2}}) + d(x_{2n_{k-2}}, x_{2n_{k-1}}) + d(x_{2n_{k-1}}, x_{2n})$$

Letting $k \to \infty$ and using (3),we get

$$lim_{k\to\infty} d(x_{2m_k}, x_{2n_k}) = \epsilon \tag{6}$$

By the fact

$$|d(x_{2m_k}, x_{2n_{k+1}}) - d(x_{2m_k}, x_{2n_k})| \leq d(x_{2n_k}, x_{2n_{k+1}}) \tag{7}$$

$$|d(x_{2m_{k-1}}, x_{2n_k}) - d(x_{2m_k}, x_{2n_k})| \leq d(x_{2m_{k-1}}, x_{2m_k}) \tag{8}$$

using (3) and (6), we obtain

$$lim_{k\to\infty} d(x_{2m_{k-1}}, x_{2n_k}) = lim_{k\to\infty} d(x_{2m_k}, x_{2n_{k+1}}) = \epsilon \tag{9}$$

Moreover, from

$$|d(x_{2m_{k-1}}, x_{2n_{k+1}}) - d(x_{2m_{k-1}}, x_{2n_k})| \leq d(x_{2n_k}, x_{2n_{k+1}}) \tag{10}$$

and combining with (3) and (9), we conclude that

$$lim_{k\to\infty} d(x_{2m_{k-1}}, x_{2n_{k+1}}) = \epsilon \tag{11}$$

from (9), we coclude that

$$lim_{k\to\infty} d(Tx_{2m_{k-1}}, Tx_{2n_k}) = \epsilon \tag{12}$$

$$\psi(d(x_{2m_k}, x_{2n_{k+1}})) = \psi(d(Tx_{2m_{k-1}}, Tx_{2n_k}))$$
$$\leq \phi(\psi(d(x_{2m_{k-1}}, x_{2n_k})))\psi(d(x_{2m_{k-1}}, x_{2n_k}))$$

Letting $k \to \infty$ and using (2) and (12), we have

$$\psi(\epsilon) \leq \alpha\psi(\epsilon)$$

a contradiction.Therefore, $\{x_n\}$ is a Cauchy sequence.Since $\{x_n\} \subset A$ and A is closed subset of a complete metric space (X,d), we can find $x^* \in A$ such that $x_n \to x^*$.
Since T is continuous ,we have $Tx_n \to Tx^*$. Taking into account that the sequence $(d(x_{n+1}, Tx_n))$ is a constant sequence with value d(A,B),we deduce

$$d(x^*, Tx^*) = d(A, B).$$

This means that x^* is a best proximity point of T. For uniqueness, suppose that x_1 and x_2 are two best proximiy points of T with $x_1 \neq x_2$. This means that

$$d(x_1, Tx_1) = d(A, B)$$
$$d(x_2, Tx_2) = d(A, B)$$

Using the P-property, we have

$$d(x_1, x_2) = d(Tx_1, Tx_2)$$

Again, T is generalized Mizoguchi- Takahashi contraction, we have

$$\psi(d(x_1, x_2)) = \psi(Tx_1, Tx_2) \leq \phi(\psi(d(x_1, x_2)))\psi(d(x_1, x_2)) \leq \alpha\psi(d(x_1, x_2))$$

a contradiction.Therefore, $x_1 = x_2$.

Corollary 3.2 *Let (A,B) be a pair of nonempty closed subsets of a complete metric space (X,d) such that* A_0 *is nonempty. Let* $T: A \to B$ *be a continuous mapping satisfying* $T(A_0) \subset B_0$, *and Mizoguchi- Takahashi contraction condition* $d(Tx, Ty) \leq \phi(d(x,y))d(x,y)$ *for any* $x, y \in A$. *Suppose that the pair (A,B) has the P-property.Then there exists a unique* x^* *in A such that* $d(x^*, Tx^*) = d(A,B)$.

Example 3.3 *Consider* $X = \Re^2$ *with the usual metric.*
Let A and B be the subsets of X defined by

$$A = \{0\} \times [0, \infty) \text{ and } B = \{1\} \times [0, 1).$$

Obviously d(A,B) = 1 and B is not closed subset of X.
Note that $A_0 = 0 \times [0, 1)$ *and* $B_0 = B$.
We consider the mapping $T: A \to B$ *defined as*

$$d(0, x) = (1, \frac{x}{1+x}) \text{ for any } (0, x) \in A.$$

In the sequel, we check that T is generalized Mizoguchi- Takahashi contraction.
In fact, for $(0, x), (0, y) \in A$ *with* $x \neq y$, *we have*

$$\begin{aligned} d(T(0,x), T(0,y)) &= d((1, \frac{x}{1+x}), (1, \frac{y}{1+y})) \\ &= |\frac{x}{1+x} - \frac{y}{1+y}| \\ &= |\frac{x-y}{(1+x)(1+y)}| \\ &\leq \frac{|x-y|}{1+|x-y|} \\ &= \phi(\psi(d((0,x),(0,y)))\psi(d((0,x),(0,y))). \end{aligned}$$

Where $\psi(t) = t$ *for* $t > 0$ *and* $\phi(t) = \frac{1}{1+t}$ *with* $limsup_{r \to t^+} \phi(r) < 1$ *for* $t \geq 0$.
Notice that the pair (A,B) satisfies the P-property.
Indeed, if

$$d((0, x_1), (1, y_1)) = \sqrt{1 + (x_1 - y_1)^2} = d(A, B) = 1,$$

$$d((0, x_2), (1, y_2)) = \sqrt{1 + (x_2 - y_2)^2} = d(A, B) = 1,$$

then $x_1 = y_1$ *and* $x_2 = y_2$ *and consequently,*

$$d((0, x_1), (0, x_2)) = |x_1 - x_2| = |y_1 - y_2| = d((1, y_1), (1, y_2)).$$

By Theorem 3.1, T has a unique best proximity point.
Obviously, the point $(0, 0) \in A$ *is a unique best proximity point for T, since*

$$d((0,0), T(0,0)) = d((0,0), (1,0)) = 1 = d(A, B)$$

If $(0, x) \in A$ *is a best proximity point for T, then*

$$1 = d(A, B) = d((0, x), T(0, x)) = d((0, x), (1, \frac{x}{1+x})) = \sqrt{1 + (x - \frac{x}{1+x})^2},$$

and this gives us

$$(x - \frac{x}{1+x}) = 0 \tag{13}$$

the solution of (12) is $x = 0$ *and is unique.Hence* $(0, 0) \in A$ *is unique best proximity point for T.*

References

[1] A. Amini-Harandi, and D. O'Regan , "Fixed point theorem for set-valued contraction type maps in metric spaces", *Fixed Point Theory and Appl*, Vol.7, 2010, (2010), Article ID 390183.

[2] A. A. Eldred, P. Veeraman, " Existence and convergence of best proximity points", *J.Math.Anal. Appl*, Vol.323, (2006), pp.1001-1006.

[3] M. E.Gordji, M. Ramezani , "A generalization of Mizoguchi and Takahashi's theorem for single-valued mappings in partially ordered metric spacesr", *Nonlinear Anal*, Vol.74, (2011), pp.4544-4549.

[4] N. Mizoguchi and W. Takahashi, "Fixed point theorems for multivalued mappings on a complete metric space", *J. Math. Anal. Appl.*, Vol.141, (1989), pp.177-188.

[5] S.B. Nadler, "Multivalued contraction mappings", *Pacific j.Math*, Vol.30, (1969), pp.475-488.

[6] V. Sankar Raj, A best proximity point theorem for weakly contractive non-self-mappings, *Nonlinear Analysis*, vol.74 (2011), 4804-4808.