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Research Paper

On some BV_σ I-convergent sequence spaces Defined by modulus function

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Abstract

In this article we introduce and study ${}_0BV_\sigma^I(f)$, $BV_\sigma^I(f)$ and ${}_\infty BV_\sigma^I(f)$ sequence spaces with the help of BV_σ [see [23]] and a modulus function f. We study topological, algebraic properties and some inclusion relations on these sequence spaces.

Keywords: *Bounded variation, Invariant mean, σ-Bounded variation, Ideal, Filter, modulus function, I-convergence, I-null, symmetric space, Solid space, Sequence algebra.*

1. Introduction

Let $\mathbb{N}$, $\mathbb{R}$ and $\mathbb{C}$ be the sets of all natural, real and complex numbers respectively.
We denote

$$\omega = \{x = (x_k) : x_k \in \mathbb{R} \text{ or } \mathbb{C}\}$$

the space of all real or complex sequences.

Let ℓ_∞, c and c_0 denote the Banach spaces of bounded, convergent and null sequences respectively with norm

$$\|x\| = \sup_k \mid x_k \mid .$$

Let v denote the space of sequences of bounded variation. That is,

$$v = \left\{ x = (x_k) : \sum_{k=o}^{\infty} \mid x_k - x_{k-1} \mid < \infty = 0 \right\} \qquad (1.1).$$

v is a Banach Space normed by

$$\|x\| = \sum_{k=0}^{\infty} \mid x_k - x_{k-1} \mid \qquad (see[23]).$$

Let σ be a mapping of the set of the positive integers into itself having no finite orbits. A continuous linear functional ϕ on ℓ_∞ is said to be an invariant mean or σ-mean if and only if
(i) $\phi(x) \geq 0$ where the sequence $x = (x_k)$ has $x_k \geq 0$ for all k.
(ii) $\phi(e) = 1$ where $e = \{1, 1, 1, ...\}$,
(iii) $\phi(x_{\sigma(n)}) = \phi(x)$ for all $x \in \ell_\infty$

If $x=(x_k)$, write $Tx=(Tx_k)=(x_{\sigma(k)})$. It can be shown that

$$V_\sigma=\left\{x=(x_k):\lim_{m\to\infty}t_{m,k}(x)=L \text{ uniformly in k}, L=\sigma-\lim x\right\} \tag{1.2}$$

where $m\geq 0,\ k>0$.

$$t_{m,k}(x)=\frac{x_k+x_{\sigma(k)}...+x_{\sigma^m(k)}}{m+1} \text{ and } t_{-1,\ k}=0 \tag{1.3}$$

where $\sigma_m(k)$ denote the $m-th$ iterate of $\sigma(k)$ at k. In case σ is the translation mapping, that is, σ(k)=k+1 σ-mean is called a Banach limit(see,[02]) and V_σ, the set of bounded sequences of all whose invariant means are equal, is the set of almost convergent sequences. The special case of (1.2) in which σ(n)=n+1 was given by Lorentz[18, Theorem 1], and that the general result can be proved in a similar way. It is familiar that a Banach limit extends the limit functional on c (see,[18]) in the sense that

$$\phi(x)=\lim x,\ \text{for all } x\in c \tag{1.4}.$$

Remark 1.1. In view of above discussion we have $c\subset V_\sigma$

Theorem 1.2. [23,Theorem 1.1] A σ-mean extends the limit functional on c in the sense that $\phi(x)=\lim x$ for all $x\in c$ if and only if σ has no finite orbits. That is, if and only if for all $k\geq 0, j\geq 1,\ \sigma^j(k)\neq k$
Put

$$\phi_{m,k}(x)=t_{m,k}(x)-t_{m-1,k}(x) \tag{1.5}$$

assuming that $t_{-1,\ k}=0$

A straight forward calculation shows that (see[22])

$$\phi_{m,k}(x)=\begin{cases}\frac{1}{m(m+1)}\sum_{j=1}^{m} j(x_\sigma^j(k)-x_\sigma^{j-1}(k)), & \text{if}(m\geq 1),\\ x_k & \text{if}(m=0)\end{cases} \tag{1.6}.$$

For any sequence x,y and scalar λ, we have

$$\phi_{m,k}(x+y)=\phi_{m,k}(x)+\phi_{m,k}(y)$$

and

$$\phi_{m,k}(\lambda x)=\lambda\phi_{m,k}(x)$$

Definition 1.3. A sequence $x\in\ell_\infty$ is of σ-bounded variation if and only if
(i) $\sum_{m=0}^{\infty}\mid\phi_{m,k}(x)\mid$ converges uniformly in k.
(ii) $\lim_{m\to\infty}t_{m,k}(x)$, which must exist, should take the same value for all k.

Subsequently invariant means have been studied by Ahmad and Mursaleen [01,22,23], Raimi [25], Khan and Ebadullah [11,12] ,King [13] and many others.

Mursaleen [23] defined the sequence space BV_σ, the space of all sequence of σ -bounded variation as

$$BV_\sigma=\{x\in\ell_\infty:\sum_m\mid\phi_{m,k}(x)\mid<\infty, \text{uniformly in k}\}$$

Theorem 1.4. BV_σ is a Banach space normed by

$$\parallel x\parallel=\sup_k\sum\mid\phi_{m,k}(x)\mid \qquad (c.f.[17],[22],[23],[25],[31]).$$

Definition 1.5. A function $f:[0,\infty)\longrightarrow[0,\infty)$ is called a modulus if
(1) $f(t)=0$ if and only if $t=0$,
(2) f(t+u)$\leq f$(t)+ f(u) for all t, u$\geq$0,
(3) f is increasing, and

(4) f is continuous from the right at zero.

A modulus function f is said to satisfy $\Delta_2 -$ Condition for all values of u if there exists a constant $K > 0$ such that $f(Lu) \leq \mathrm{KL}f(u)$ for all values of $L > 1$.

The idea of modulus was introduced by Nakano in 1953.(See , Nakano, 1953).

Ruckle [26,27,28] used the idea of a modulus function f to construct the sequence space

$$X(f) = \{x = (x_k) : \sum_{k=1}^{\infty} f(|x_k|) < \infty\}.$$

This space is an FK space and Ruckle[26,27,28] proved that the intersection of all such $X(f)$ spaces is ϕ, the space of all finite sequences.
The space $X(f)$ is closely related to the space ℓ_1 which is an $X(f)$ space with $f(x) = x$ for all real $x \geq 0$. Thus Ruckle[26,27,28] proved that, for any modulus f.

$$X(f) \subset \ell_1 \text{ and } X(f)^{\alpha} = \ell_{\infty}$$

Where

$$X(f)^{\alpha} = \{y = (y_k) \in \omega : \sum_{k=1}^{\infty} f(|y_k x_k|) < \infty\}$$

The space $X(f)$ is a Banach space with respect to the norm

$$||x|| = \sum_{k=1}^{\infty} f(|x_k|) < \infty.\text{(See[28])}.$$

Spaces of the type $X(f)$ are a special case of the spaces structured by B.Gramsch [6]. From the point of view of local convexity, spaces of the type $X(f)$ are quite pathological. Symmetric sequence spaces, which are locally convex have been frequently studied by D.J.H Garling[5], G.Köthe[16] , I.J.Maddox [19,20,21] and W.H.Ruckle[26,27,28].

Initially, as a generalization of statistical convergence[4,5], the notation of ideal convergence (I-convergence) was introduced and studied by Kostyrko, Mačaj, Salăt and Wilczyńki ([14,15]). Later on, it was studied by Šalát, Tripathy and Ziman [29,30], Tripathy and Hazarika [3,32,33], Hazarika, *et, al*[7], Khan and Ebadullah [8,9,10,11,12] and many others.

Definition 1.6. A set A is said to have asymptotic density $\delta(A) = \lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \mathcal{X}_{\mathcal{A}}(k)$, if it exists , where $\mathcal{X}_{\mathcal{A}}$ is the characteristic function of A.

Definition 1.7. A sequence x=$(x_k) \in \omega$ is said to be statistically convergent to a limit $L \in \mathbb{C}$ if for every $\epsilon > 0$, we have

$$\lim_{k} \frac{1}{k} |\{n \in \mathbb{N} : |x_k - L| \geq \epsilon, n \leq k\}| = 0.$$

where vertical lines denote the cardinality of the enclosed set.

That is, if $\delta(A(\epsilon)) = 0$, where

$$A(\epsilon) = \left\{k \in \mathbb{N} :| x_k - L |\geq \epsilon\right\}$$

Here we give some definitions and preliminaries about the notion of I-convergence.

Definition 1.8. Let N be a non empty set. Then a family of sets $I \subseteq 2^N$ (power set of N) is said to be an ideal if
1) I is additive i.e $\forall A, B \in I \Rightarrow A \cup B \in I$
2) I is hereditary i.e $\forall A \in I$and $B \subseteq A \Rightarrow B \in I$.

Definition 1.9. A non-empty family of sets $\pounds(I) \subseteq 2^N$ is said to be filter on N if and only if
1) $\Phi \notin \pounds$(I),
2) $\forall\ A,\ B \in \pounds$(I) we have $A \cap B \in \pounds(I)$,
3) $\forall\ A \in \pounds$(I)and $A \subseteq B \Rightarrow B \in \pounds(I)$.

Definition 1.10. An Ideal $I \subseteq 2^N$ is called non-trivial if $I \neq 2^N$.

Definition 1.11. A non-trivial ideal $I \subseteq 2^N$ is called admissible if

$$\{\{x\} : x \in N\} \subseteq I.$$

Definition 1.12. A non-trivial ideal I is maximal if there cannot exist any non-trivial ideal $J \neq I$ containing I as a subset.

Remark 1.13. For each ideal I, there is a filter $\pounds(I)$ corresponding to I.
i.e $\pounds(I) = \{K \subseteq N : K^c \in I\}$, where $K^c = N \setminus K$.

Definition 1.14. A sequence $x = (x_k) \in \omega$ is said to be I-convergent to a number L if for every $\epsilon > 0$, the set $\{k \in N : |x_k - L| \geq \epsilon\} \in I$.
In this case, we write $I - \lim x_k = L$.

Definition 1.15. A sequence $x = (x_k) \in \omega$ is said to be I-null if $L = 0$. In this case, we write $I - \lim x_k = 0$.

Definition 1.16. A sequence $x = (x_k) \in \omega$ is said to be I-cauchy if for every $\epsilon > 0$ there exists a number $m = m(\epsilon)$ such that $\{k \in N : |x_k - x_m| \geq \epsilon\} \in I$.

Definition 1.17. A sequence $x = (x_k) \in \omega$ is said to be I-bounded if there exists some $M > 0$ such that $\{k \in N : |x_k| \geq M\} \in I$.

Definition 1.18. A sequence space E said to be solid(normal) if $(\alpha_k x_k) \in E$ whenever $(x_k) \in E$ and for any sequence(α_k) of scalars with $\mid \alpha_k \mid \leq 1$, for all $k \in \mathbb{N}$.

Definition 1.19. A sequence space E said to be symmetric if $(x_{\pi(k)}) \in E$ whenever $x_k \in E$. where π is a permutation on $\mathbb{N}$

Definition 1.20. A sequence space E said to be sequence algebra if $(x_k) * (y_k) = (x_k.y_k) \in E$ whenever $(x_k), (y_k) \in E$.

Definition 1.21. A sequence space E said to be convergence free if $(y_k) \in E$ whenever $(x_k) \in E$ and $x_k = 0$ implies $y_k = 0$, for all k.

Definition 1.22. Let $K = \{k_1 < k_2 < k_3 < k_4 < k_5 ...\} \subset \mathbb{N}$ and E be a Sequence space. A K-step space of E is a sequence space $\lambda_K^E = \{(x_{k_n}) \in \omega : (x_k) \in E\}$.

Definition 1.23. A canonical pre-image of a sequence $(x_{k_n}) \in \lambda_K^E$ is a sequence $(y_k) \in \omega$ defined by

$$y_k = \begin{cases} x_k, & \text{if } k \in K, \\ 0, & \text{otherwise.} \end{cases}$$

A canonical preimage of a step space λ_K^E is a set of preimages all elements in λ_K^E.i.e. y is in the canonical preimage of λ_K^E iff y is the canonical preimage of some $x \in \lambda_K^E$.

Definition 1.24. A sequence space E is said to be monotone if it contains the canonical preimages of its step space.

Definition 1.25. If $I = I_f$, the class of all finite subsets of N. Then, I is an admissible ideal in N and I_f convergence coincides with the usual convergence.

Definition 1.26. If $I = I_\delta = \{A \subseteq N : \delta(A) = 0\}$. Then, I is an admissible ideal in N and we call the I_δ-convergence as the logarithmic statistical convergence.

Definition 1.27. If $I = I_d = \{A \subseteq N : d(A) = 0\}$. Then, I is an admissible ideal in N and we call the I_d-convergence as the asymptotic statistical convergence.

Remark 1.28. If $I_\delta - \lim x_n = l$, then $I_d - \lim x_n = l$

We used the following lemmas for establishing some results of this article.

Lemma(I). Every solid space is monotone

Lemma(II). Let $K \in \pounds(I)$ and $M \subseteq N$. If $M \notin I$, then $M \cap K \notin I$.

Lemma(III). If $I \subseteq 2^N$ and $M \subseteq N$. If $M \notin I$, then $M \cap N \notin I$.

Khan and K.Ebadullah[18] introduced and studied the following sequence space.

For $m \geq 0$

$$BV_\sigma^I = \left\{ x = x_k \in \omega : \left\{ k \in \mathbb{N} :\mid \phi_{m,k}(x) - L \mid \geq \epsilon \right\} \in I, \text{for some L} \in \mathbb{C} \right\}. \tag{2.1}$$

2. Main results

In this article we introduced and studied the following classes of sequence spaces:

$$BV_\sigma^I(f) = \left\{ x = (x_k) \in \omega : \left\{ k \in \mathbb{N} : \sum_{m=0}^{\infty} f\Bigg(\mid \phi_{m,k}(x) - L \mid \Bigg) \geq \epsilon \right\} \in I, \text{for some L} \in \mathbb{C} \right\}; \tag{2.2}$$

$$_0BV_\sigma^I(f) = \left\{ x = (x_k) \in \omega : \left\{ k \in \mathbb{N} : \sum_{m=0}^{\infty} f\Bigg(\mid \phi_{m,k}(x) \mid \Bigg) \geq \epsilon \right\} \in I, \right\}; \tag{2.3}$$

$$_\infty BV_\sigma^I(f) = \left\{ x = (x_k) \in \omega : \left\{ k \in \mathbb{N} : \exists\ K > 0, \sum_{m=0}^{\infty} f\Bigg(\mid \phi_{m,k}(x) \mid \Bigg) \geq K \right\} \in I \right\}; \tag{2.4}$$

$$_\infty BV_\sigma(f) = \left\{ x = (x_k) \in \omega : \sup_k \sum_{m=0}^{\infty} f\Bigg(\mid \phi_{m,k}(x) \mid \Bigg) < \infty \right\}. \tag{2.5}$$

Also we write

$$\mathcal{M}_{BV_\sigma}(f) = BV_\sigma^I(f) \cap {_\infty BV_\sigma(f)}$$

and

$$_0\mathcal{M}_{BV_\sigma}^I(f) = {_0BV_\sigma^I(f)} \cap {_\infty BV_\sigma(f)}.$$

Theorem 2.1. For any modulus function f, the classes of sequence $_0BV_\sigma^I(f)$, $BV_\sigma^I(f)$, $_0\mathcal{M}_{BV_\sigma}^I(f)$ and $\mathcal{M}_{BV_\sigma}^I(f)$ are the linear spaces.

Proof. We shall prove the result for the space $BV_\sigma^I(f)$. Rests will follow similarly.
For, let $x = (x_k),\ y = (y_k) \in BV_\sigma^I(f)$ be any two arbitrary elements and let α, β be scalars.
Now, since $(x_k), (y_k) \in BV_\sigma^I(f)$, then, there exists $L_1, L_2 \in \mathbb{C}$ such that the sets

$$A_1 = \left\{ k \in \mathbb{N} : \sum_{m=0}^{\infty} f\Bigg(\mid \phi_{m,k}(x) - L_1 \mid \Bigg) \geq \frac{\epsilon}{2} \right\} \in I \tag{2.6}$$

and

$$A_2 = \left\{ k \in \mathbb{N} : \sum_{m=0}^{\infty} f\Bigg(\mid \phi_{m,k}(x) - L_2 \mid \Bigg) \geq \frac{\epsilon}{2} \right\} \in I \qquad (2.7).$$

Since, f is modulus function, we have,

$$\sum_{m=0}^{\infty} f\bigg(\mid (\alpha\phi_{m,k}(x)+\beta\phi_{m,k}(y))-(\alpha L_1+\beta L_2) \mid \bigg)$$
$$\leq \sum_{m=0}^{\infty} f\bigg(\mid \alpha \mid\mid \phi_{m,k}(x)-L_1 \mid) + \sum_{m=0}^{\infty} f\bigg(\mid \beta \mid\mid \phi_{m,k}(y)-L_2 \mid \bigg).$$
$$\leq \sum_{m=0}^{\infty} f\bigg(\mid \phi_{m,k}(x)-L_1 \mid) + \sum_{m=0}^{\infty} f\bigg(\mid \phi_{m,k}(y)-L_2 \mid \bigg) \qquad (2.8).$$

Therefore, by (2.6), (2.7) and (2.8), we have,

$$\bigg\{k\in\mathbb{N}:\sum_{m=0}^{\infty} f\bigg(\mid (\alpha\phi_{m,k}(x)+\beta\phi_{m,k}(y))-(\alpha L_1+\beta L_2) \mid \bigg)\geq\epsilon\bigg\}\subseteq\bigg[A_1\cup A_2\bigg]\in I.$$

implies that

$$\bigg\{k\in\mathbb{N}:\sum_{m=0}^{\infty} f\bigg(\mid (\alpha\phi_{m,k}(x)+\beta\phi_{m,k}(y))-(\alpha L_1+\beta L_2) \mid \bigg)\geq\epsilon\bigg\}\in I.$$

But $(x_k),(y_k)\in BV_\sigma^I(M)$ are the arbitrary elements
Therefore, $\alpha x_k+\beta y_k\in BV_\sigma^I(f),$ for all $(x_k),(y_k)\in BV_\sigma^I(f)$ and for all scalars $\alpha,\ \beta$
Hence, $BV_\sigma^I(f)$ is linear

Theorem 2.2. A sequence $x=(x_k)\in\mathcal{M}_{BV_\sigma}(f)$ I-converges if and only if for every $\epsilon>0$, there exists $N_\epsilon\in\mathbb{N}$ such that

$$\bigg\{k\in\mathbb{N}:\sum_{m=0}^{\infty} f(|\phi_{m,k}(x_k)-\phi_{m,k}(x_{N_\epsilon})|<\epsilon\bigg\}\in\pounds(I) \qquad (2.9)$$

Proof. Let $x=(x_k)\in\mathcal{M}_{BV_\sigma}(f)$. Suppose that $L=I-\lim x$. Then, the set

$$B_\epsilon=\bigg\{k\in\mathbb{N}:\sum_{m=0}^{\infty} f(|\phi_{m,k}(x_k)-L|)<\frac{\epsilon}{2}\bigg\}\in\pounds(I),\quad\text{for all }\epsilon>0.$$

Fix an $N_\epsilon\in B_\epsilon$. Then we have

$$\sum_{m=0}^{\infty} f\bigg(|\phi_{m,k}(x_k)-\phi_{m,k}(x_{N_\epsilon})|\bigg)$$
$$\leq\sum_{m=0}^{\infty} f\bigg(|\phi_{m,k}(x_{N_\epsilon})-L|\bigg)$$
$$+\sum_{m=0}^{\infty} f\bigg(|L-\phi_{m,k}(x_k)|\bigg)<\frac{\epsilon}{2}+\frac{\epsilon}{2}=\epsilon$$

which holds for all $k\in B_\epsilon$.
Hence

$$\{k\in\mathbb{N}:\sum_{m=0}^{\infty} f\bigg(|\phi_{m,k}(x)-\phi_{m,k}(x_{N_\epsilon})|\bigg)<\epsilon\}\in\pounds(I).$$

Conversely, suppose that

$$\bigg\{k\in\mathbb{N}:\sum_{m=0}^{\infty} f\bigg(|\phi_{m,k}(x_k)-\phi_{m,k}(x_{N_\epsilon})|\bigg)<\epsilon\bigg\}\in\pounds(I).$$

Then, being f a modulus function and by using basic triangular inequality, we have

$$\bigg\{k\in\mathbb{N}:\bigg|\sum_{m=0}^{\infty} f\bigg(\mid\phi_{m,k}(x_k)\mid\bigg)-\sum_{m=0}^{\infty} f\bigg(\mid\phi_{m,k}(x_{N_\epsilon})\mid\bigg)\bigg|<\epsilon\bigg\}\in\pounds(I),\text{for all }\epsilon>0.$$

Then, the set

$$C_\epsilon = \{k \in \mathbb{N} : \sum_{m=0}^{\infty} f(|\ \phi_{m,k}(x_k)\ |) \in [\sum_{m=0}^{\infty} f(|\ \phi_{m,k}(x_{N_\epsilon})\ |) - \epsilon, \sum_{m=0}^{\infty} f(|\ \phi_{m,k}(x_{N_\epsilon})\ |) + \epsilon]\} \in \pounds(I).$$

Let $J_\epsilon = \Big[\sum_{m=0}^{\infty} f(|\ \phi_{m,k}(x_{N_\epsilon})\ |) - \epsilon, \sum_{m=0}^{\infty} f(|\ \phi_{m,k}(x_{N_\epsilon})\ |) + \epsilon].$
If we fix an $\epsilon > 0$ then, we have $C_\epsilon \in \pounds(I)$ as well as $C_{\frac{\epsilon}{2}} \in \pounds(I)$.
Hence $C_\epsilon \cap C_{\frac{\epsilon}{2}} \in \pounds(I)$. This implies that

$$J = J_\epsilon \cap J_{\frac{\epsilon}{2}} \neq \phi.$$

That is

$$\{k \in \mathbb{N} : \sum_{m=0}^{\infty} f\Big(\ |\ \phi_{m,k}(x_k)\ |\ \Big) \in J\} \in \pounds(I).$$

That is

$$diam J \leq diam J_\epsilon$$

where the diam of J denotes the length of interval J.
In this way, by induction we get the sequence of closed intervals

$$J_\epsilon = I_0 \supseteq I_1 \supseteq ..... \supseteq I_k \supseteq ....$$

with the property that $diam I_k \leq \frac{1}{2} diam I_{k-1}$ for (k=2,3,4,.....) and
$\{k \in \mathbb{N} : \sum_{m=0}^{\infty} f\Big(\ |\ \phi_{m,k}(x_k)\ |\ \Big) \in I_k\} \in \pounds(I)$ for (k=1,2,3,4,......).
Then there exists a $\xi \in \cap I_k$ where $k \in \mathbb{N}$ such that

$$L = I - \lim_k \sum_{m=0}^{\infty} f\Big(\ |\ \phi_{m,k}(x_k)\ |\ \Big).$$

showing that $x = (x_k) \in \mathcal{M}_{BV_\sigma}(f)$ is I-convergent.
Hence the result.

Theorem 2.3. Let f_1 and f_2 be two modulus functions and satisfying $\Delta_2 - Condition$, then,
(a) $\mathcal{X}(f_2) \subseteq \mathcal{X}(f_1 f_2)$
(b) $\mathcal{X}(f_1) \cap (f_2) \subseteq \mathcal{X}(f_1 + f_2)$ for $\mathcal{X}= {}_0BV_\sigma^I$, BV_σ^I, ${}_0\mathcal{M}_{BV_\sigma}^I$ and $\mathcal{M}_{BV_\sigma}^I$

Proof. (a) Let $x = (x_k) \in {}_0BV_\sigma^I(f_2)$ be any arbitrary element. Then, the set

$$\Big\{k \in \mathbb{N} : \sum_{m=0}^{\infty} f_2\Big(\ |\ \phi_{m,k}(x)\ |\ \Big) \geq \epsilon\Big\} \in I \qquad (2.10).$$

Let $\epsilon > 0$ and choose δ with $0 < \delta < 1$ such that $f_1(t) < \epsilon,\ 0 \leq t \leq \delta$.

Write $y_k = f_2\Big(\ |\ \phi_{m,k}(x)\ |\ \Big)$ and consider

$$\lim_k f_1(y_k) = \lim_{y_k \leq \delta, k \in \mathbb{N}} f_1(y_k) + \lim_{y_k > \delta, k \in \mathbb{N}} f_1(y_k).$$

Now, since f_1 is a modulus function,
Therefore, we have

$$\lim_{y_k \leq \delta, k \in \mathbb{N}} f_1(y_k) \leq f_1(2) \lim_{y_k \leq \delta, k \in \mathbb{N}} (y_k) \qquad (2.11)$$

For $y_k > \delta$, we have $y_k < \frac{y_k}{\delta} < 1 + \frac{y_k}{\delta}$
Now, since f_1 is non-decreasing, it follows that

$$f_1(y_k) < f_1(1 + \frac{y_k}{\delta}) < \frac{1}{2} f_1(2) + \frac{1}{2} f_1(\frac{2y_k}{\delta})$$

Again, since f_1 satisfies Δ_2- Condition, we have

$$f_1(y_k) < \frac{1}{2}K\frac{(y_k)}{\delta}f_1(2) + \frac{1}{2}K\frac{(y_k)}{\delta}f_1(2)$$

Thus,

$$f_1(y_k) < K\frac{(y_k)}{\delta}f_1(2).$$

Hence,

$$\lim_{y_k>\delta, k\in\mathbb{N}} f_1(y_k) \leq \max\{1, K\delta^{-1}f_1(2) \lim_{y_k>\delta, k\in\mathbb{N}}(y_k) \qquad (2.12).$$

Therefore, from (2.10),(2.11) and (2.12), we have,

$$\left\{k\in\mathbb{N}: \sum_{m=0}^{\infty} f_1(y_k) \geq \epsilon\right\} \in I$$

i.e.

$$\left\{k\in\mathbb{N}: \sum_{m=0}^{\infty} f_1 f_2\bigg(\mid \phi_{m,k}(x)\mid\bigg) \geq \epsilon\right\} \in I$$

implies that

$$x=(x_k)\in_0 BV_\sigma^I(f_1 f_2).$$

Thus,

$${}_0BV_\sigma^I(f_2) \subseteq {}_0BV_\sigma^I(f_1 f_2)$$

Hence, $\mathcal{X}(f_2)\subseteq\mathcal{X}(f_1 f_2)$ for $\mathcal{X}= {}_0BV_\sigma^I$
For $\mathcal{X}=BV_\sigma^I$, $\mathcal{X}= {}_0\mathcal{M}_{BV_\sigma}^I$ and $\mathcal{X}=\mathcal{M}_{BV_\sigma^I}$ the inclusions can be established similarly.

(b) Let $x=(x_k)\in {}_0BV_\sigma^I(f_1)\cap {}_0BV_\sigma^I(f_2)$. Let $\epsilon>0$ be given. Then, the sets

$$\left\{k\in\mathbb{N}: \sum_{m=0}^{\infty} f_1\bigg(\mid \phi_{m,k}(x)\bigg) \geq \frac{\epsilon}{2}\right\} \in I,$$

and

$$\left\{k\in\mathbb{N}: \sum_{m=0}^{\infty} f_2\bigg(\mid \phi_{m,k}(x)\mid\bigg) \geq \frac{\epsilon}{2}\right\} \in I,$$

Therefore, the inclusion

$$\left\{k\in\mathbb{N}: \sum_{m=0}^{\infty} (f_1+f_2)\bigg(\mid \phi_{m,k}(x)\mid\bigg) \geq \epsilon\right\}$$
$$\subseteq \left[\left\{k\in\mathbb{N}: \sum_{m=0}^{\infty} f_1\bigg(\mid \phi_{m,k}(x)\mid\bigg) \geq \epsilon\right\}\right.$$
$$\left.\cup \left\{k\in\mathbb{N}: \sum_{m=0}^{\infty} f_2\bigg(\mid \phi_{m,k}(x)\mid\bigg) \geq \epsilon\right\}\right] \in I,$$

implies that

$$\left\{k\in\mathbb{N}: \sum_{m=0}^{\infty} (f_1+f_2)\bigg(\mid \phi_{m,k}(x)\mid\bigg) \geq \epsilon\right\} \in I$$

Thus, $x=(x_k)\in {}_0BV_\sigma^I(f_1+f_2)$
Hence, ${}_0BV_\sigma^I(f_1)\cap {}_0BV_\sigma^I(f_2)\subseteq {}_0BV_\sigma^I(f_1+f_2)$
For $\mathcal{X}=BV_\sigma^I$, $\mathcal{X}= {}_0\mathcal{M}_{BV_\sigma}^I$ and $\mathcal{X}=\mathcal{M}_{BV_\sigma^I}$ the inclusions are similar.

For $f_2(x)=x$ and $f_1(x)=M(x)$, for all $x\in[0,\infty)$, we have the following corollary.

Corollary 2.4. $\mathcal{X} \subseteq \mathcal{X}(f)$ for $\mathcal{X}= {}_0BV_\sigma^I$, BV_σ^I, ${}_0\mathcal{M}_{BV_\sigma}^I$ and $\mathcal{M}_{BV_\sigma}^I$.

Theorem 2.5. The spaces ${}_0BV_\sigma^I(f)$ and ${}_0\mathcal{M}_{BV_\sigma}^I(f)$ are solid and monotone.

Proof. We shall prove the result for ${}_0BV_\sigma^I(f)$. For ${}_0\mathcal{M}_{BV_\sigma}^I(f)$, the result can be proved similarly.
For, let $x = (x_k) \in {}_0BV_\sigma^I(f)$, then the set

$$\left\{k \in \mathbb{N} : \sum_{m=0}^{\infty} f\Bigg(\mid \phi_{m,k}(x) \mid \Bigg) \geq \epsilon \right\} \in I \tag{2.13}$$

Let (α_k) be a sequence of scalars with $\mid \alpha_k \mid\leq 1$ for all $k \in \mathbb{N}$. Then, the result follows from (2.12) and the inequality

$$f\Bigg(\mid \alpha_k \phi_{m,k}(x) \mid \Bigg) \leq\mid \alpha_k \mid f\Bigg(\mid \phi_{m,k}(x) \mid \Bigg) \leq f\Bigg(\mid \phi_{m,k}(x) \mid \Bigg), \text{ for all k} \in \mathbb{N}.$$

The space is monotone follows from lemma(I). Hence the result.

Theorem 2.6. The spaces $BV_\sigma^I(f)$ and $\mathcal{M}_{BV_\sigma^I}(f)$ are not neither solid nor monotone.

Proof. Here we give a counter example for the proof of this result.

Counter example. Let $I = I_f$ and $f(x) = x$ for all $x \in [0, \infty)$. Consider the K-step $\mathcal{Z}_K$ of $\mathcal{Z}$ defined as follows.

Let $(x_k) \in \mathcal{Z}$ and let $(y_k) \in \mathcal{Z}_K$ be such that

$$y_k = \begin{cases} x_k, & \text{if k is even,} \\ 0, & \text{otherwise.} \end{cases}$$

Consider the sequence (x_k) defined as by $x_k = 1$ for all $k \in \mathbb{N}$. Then $(x_k) \in BV_\sigma^I(f)$ and $\mathcal{M}_{BV_\sigma}(f)$ but its K-step preimage does not belong to $BV_\sigma^I(f)$ and $\mathcal{M}_{BV_\sigma}(f)$.
Thus, $BV_\sigma^I(f)$ and $\mathcal{M}_{BV_\sigma}(f)$ are not monotone. Hence $BV_\sigma^I(f)$ and $\mathcal{M}_{BV_\sigma}(f)$ are not solid by lemma(I).

Theorem 2.7. The spaces $BV_\sigma^I(f)$ and ${}_0BV_\sigma^I(f)$ are sequence algebra.

Proof. Let $(x = x_k)$ and $(y = y_k)$ be two elements of ${}_0BV_\sigma^I(f)$.

Then, the sets

$$\left\{k \in \mathbb{N} : \sum_{m=0}^{\infty} f\Bigg(\mid \phi_{m,k}(x) \mid \Bigg) \geq \epsilon \right\} \in I$$

and

$$\left\{k \in \mathbb{N} : \sum_{m=0}^{\infty} f\Bigg(\mid \phi_{m,k}(y) \mid \Bigg) \geq \epsilon \right\} \in I$$

Therefore,

$$\left\{k \in \mathbb{N} : \sum_{m=0}^{\infty} f\Bigg(\mid \phi_{m,k}(x).\phi_{m,k}(y) \mid \Bigg) \geq \epsilon \right\} \in I.$$

Thus, $(x_k).(y_k) \in {}_0BV_\sigma^I(f)$. Hence ${}_0BV_\sigma^I(f)$ is sequence algebra. For $BV_\sigma^I(f)$, the result can be proved similarly.

Theorem 2.8. If I is not maximal and $I \neq I_f$, then the spaces $BV_\sigma^I(f)$ and ${}_0BV_\sigma^I(f)$ are not symmetric.

Proof. Let $A \in I$ be an infinite set and $f(x) = x$ for all $x \in [0, \infty)$. If

$$x_k = \begin{cases} 1, & \text{if k} \in A, \\ 0, & \text{otherwise.} \end{cases}$$

Then, it is clear that $(x_k) \in {}_0BV_\sigma^I(f \subsetneq BV_\sigma^I(f)$
Let $K \subseteq \mathbb{N}$ be such that $K \notin I$ and $\mathbb{N} \setminus K \notin I$
Let $\phi : K \to A$ and $\psi : K^c \to A^c$ be a bijective maps.Then,the mapping $\pi :\to \mathbb{N} \to \mathbb{N}$ defined by

$$\pi(k) = \begin{cases} \phi(k), & \text{if } k \in K, \\ \psi k, & otherwise. \end{cases}$$

is a permutation on $\mathbb{N}$
But $(x_\pi(k)) \notin BV_\sigma^I(f)$ and hence $(x_\pi(k)) \notin {}_0BV_\sigma^I(f)$ showing that $BV_\sigma^I(f)$ and ${}_0BV_\sigma^I(f)$ are not symmetric sequence spaces.

Theorem 2.9. Let f be a modulus function.
Then, ${}_0BV_\sigma^I(f) \subset BV_\sigma^I(f) \subset {}_\infty BV_\sigma^I(f)$ and the inclusions are proper.

Proof. The inclusion ${}_0BV_\sigma^I(f) \subset BV_\sigma^I(f)$ is obvious.
Next, let $(x_k) \in BV_\sigma^I(f)$. Then there exists $L \in \mathbb{C}$ such that

$$\left\{ k \in \mathbb{N} : \sum_{m=0}^{\infty} f\left(\mid \phi_{m,k}(x) - L \mid \right) \geq \epsilon \right\} \in I.$$

We have

$$f\left(\mid \phi_{m,k}(x) \mid \right) \leq \frac{1}{2} f\left(\mid \phi_{m,k}(x) - L \mid \right) + f\left(\frac{1}{2} \mid L \mid \right)$$

Taking supremum over k on both sides, we get $(x_k) \in_\infty BV_\sigma^I(f)$
Hence, ${}_0BV_\sigma^I(f) \subset BV_\sigma^I(f) \subset {}_\infty BV_\sigma^I(f)$
Next,we show that the inclusions are proper.
For, Let us consider $I = I_d$, $f(x) = x^2$ for all $x \in [0, \infty)$. Consider the sequence (x_k) defined by $x_k = 1$. Then $(x_k) \in BV_\sigma^I(f)$ but $(x_k) \notin {}_0BV_\sigma^I(f)$

Again, consider the sequence (y_k) defined by

$$y_k = \begin{cases} 2, & \text{if k is even,} \\ 0, & \text{otherwise.} \end{cases}$$

Then $(y_k) \in {}_\infty BV_\sigma^I(f)$ but $(y_k) \notin BV_\sigma^I(f)$

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