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Research Paper

On paranorm BV_σ I-convergent sequence spaces defined by an Orlicz function

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Abstract

In this article we introduce and study ${}_0BV_\sigma^I(M,p)$, $BV_\sigma^I(M,p)$ and ${}_\infty BV_\sigma^I(M,p)$ sequence spaces where $p=(p_k)$ is the sequence of strictly positive real numbers with the help of BV_σ space [see [23]] and an Orlicz function M. We study some topological and algebraic properties and decompostion theorem. Further we prove some inclusion relations related to these new spaces.

Keywords: *Bounded variation, Invariant mean, σ-Bounded variation, Ideal, Filter, Orlicz function, I-convergence, I-null, Solid space, Sequence algebra, paranorm.*

1. Introduction

Let $\mathbb{N}$, $\mathbb{R}$ and $\mathbb{C}$ be the sets of all natural, real and complex numbers respectively.
We denote

$$\omega = \{x=(x_k): x_k \in \mathbb{R} \text{ or } \mathbb{C}\}$$

the space of all real or complex sequences.

Let ℓ_∞, c and c_0 denote the Banach spaces of bounded, convergent and null sequences respectively with norm

$$\|x\| = \sup_k \mid x_k \mid$$

Let v denote the space of sequences of bounded variation. That is,

$$v = \left\{ x=(x_k): \sum_{k=o}^{\infty} \mid x_k - x_{k-1} \mid < \infty = 0 \right\} \tag{1.1}$$

v is a Banach Space normed by

$$\|x\| = \sum_{k=0}^{\infty} \mid x_k - x_{k-1} \mid \qquad (see[23])$$

Let σ be a mapping of the set of the positive integers into itself having no finite orbits. A continuous linear functional ϕ on ℓ_∞ is said to be an invariant mean or σ-mean if and only if
(i) $\phi(x) \geq 0$ where the sequence $x=(x_k)$ has $x_k \geq 0$ for all k.
(ii) $\phi(e)=1$ where $e=\{1,1,1,...\}$,

(iii) $\phi(x_{\sigma(n)}) = \phi(x)$ for all $x \in \ell_\infty$

If $x = (x_k)$, write $Tx = (Tx_k) = (x_{\sigma(k)})$. It can be shown that

$$V_\sigma = \left\{ x = (x_k) : \lim_{m\to\infty} t_{m,k}(x) = L \text{ uniformly in k}, \ L = \sigma - \lim x \right\} \tag{1.2}$$

where $m \geq 0, k > 0$.

$$t_{m,k}(x) = \frac{x_k + x_{\sigma(k)} ... + x_{\sigma^m(k)}}{m+1} \text{ and } t_{-1,\ k} = 0 \tag{1.3}$$

where $\sigma_m(k)$ denote the m-th iterate of $\sigma(k)$ at k. In case σ is the translation mapping, that is, σ(k)=k+1, σ-mean is called a Banach limit(see,[2]) and V_σ, the set of bounded sequences of all whose invariant means are equal, is the set of almost convergent sequences. The special case of (1.2) in which σ(n)=n+1 was given by Lorentz[19, Theorem 1], and that the general result can be proved in a similar way. It is familiar that a Banach limit extends the limit functional on c (see,[19]) in the sense that

$$\phi(x) = \lim x, \text{ for all } x \in c \qquad (1.4),$$

Remark 1.1. In view of above discussion we have $c \subset V_\sigma$.

Theorem 1.2. [23,Theorem 1.1] A σ-mean extends the limit functional on c in the sense that $\phi(x) = \lim x$ for all $x \in c$ if and only if σ has no finite orbits. That is, if and only if for all $k \geq 0, j \geq 1,\ \sigma^j(k) \neq k$
Put

$$\phi_{m,k}(x) = t_{m,k}(x) - t_{m-1,k}(x) \tag{1.5}$$

assuming that $t_{-1,\ k} = 0$

A straight forward calculation shows that (see[22])

$$\phi_{m,k}(x) = \begin{cases} \frac{1}{m(m+1)} \sum\limits_{j=1}^{m} j(x_\sigma^j(k) - x_\sigma^{j-1}(k)), & \text{if}(m \geq 1), \\ x_k & \text{if}(m = 0) \end{cases} \tag{1.6}$$

For any sequence x, y and scalar λ, we have

$$\phi_{m,k}(x+y) = \phi_{m,k}(x) + \phi_{m,k}(y)$$

and

$$\phi_{m,k}(\lambda x) = \lambda \phi_{m,k}(x).$$

Definition 1.3. A sequence $x \in \ell_\infty$ is of σ-bounded variation if and only if
(i) $\sum\limits_{m=0}^{\infty} \mid \phi_{m,k}(x) \mid$ converges uniformly in k.
(ii) $\lim\limits_{m\to\infty} t_{m,k}(x)$, which must exist, should take the same value for all k.

Subsequently invariant means have been studied by Ahmad and Mursaleen [23,1,22], J.P.King [14],Raimi [26], Khan and Ebadullah [12,13] and many others. Mursaleen [23] defined the sequence space BV_σ, the space of all sequence of σ -bounded variation as

$$BV_\sigma = \{x \in \ell_\infty : \sum_m \mid \phi_{m,k}(x) \mid < \infty, \text{uniformly in k}\}$$

Theorem 1.4. BV_σ is a Banach space normed by

$$\| x \| = \sup_k \sum | \phi_{m,k}(x) | \qquad (c.f.[23],[26],[29],[22])$$

Definition 1.5. A function $M : [0, \infty) \to [0, \infty)$ is said to be an Orlicz function if it satisfies the following conditions
(i) M is continuous, convex and non-decreasing
(ii) $M(0) = 0, M(x) > 0$ and $M(x) \to \infty$ as $x \to \infty$

Remark 1.6. If the convexity of an Orlicz function is replaced by $M(x + y) \leq M(x) + M(y)$, then this function is called modulus function.

Remark 1.7. If M is an Orlicz function, then $M(\lambda x) \leq \lambda M(x)$ for all λ with $0 < \lambda < 1$.

An Orlicz function M is said to satisfy Δ_2- Condition for all values of u if there exists a constant $K > 0$ such that $M(Lu) \leq$ KL$M(u)$ for all values of $L > 1$.

Lindenstrauss and Tzafriri[18] used the idea of an Orlicz function to construct the sequence space

$$\ell_M = \Big\{x \in \omega : \sum_{k=1}^{\infty} M(\frac{| x_k |}{\rho}) < \infty, \text{for some } \ \rho > 0\Big\}. \tag{1.7}$$

The space ℓ_M becomes a Banach space with the norm

$$\| x \| = \inf\Big\{\rho > 0 : \sum_{k=1}^{\infty} M(\frac{| x_k |}{\rho}) \leq 1\Big\} \tag{1.8}$$

which is called an Orlicz sequence space. The space ℓ_M is closely related to the space ℓ_p which is an Orlicz sequence space with $M(t) = t^P$ for $1 < p < \infty$.

Later on some Orlicz sequence spaces were investigated by Parashar and Choudhury [25], Maddox [20],Khan [10], Kamthan and Gupta [9],Bhardwaj and Singh [3], and many others.

Definition 1.8. Let X be a linear space. A function $g : X \longrightarrow R$ is called paranorm, if for all $x, y \in X$,
(PI) $g(x) = 0 \ if \ x = \theta$,
(P2) $g(-x) = g(x)$,
(P3) $g(x + y) \leq g(x) + g(y)$,
(P4) If (λ_n) is a sequence of scalars with $\lambda_n \to \lambda$ $(n \to \infty)$ and $x_n, a \in X$ with $x_n \to a$ $(n \to \infty)$ in the sense that $g(x_n - a) \to 0$ $(n \to \infty)$, then $g(\lambda_n x_n - \lambda a) \to 0$ $(n \to \infty)$.

The concept of paranorm is closely related to linear metric spaces. It is a generalization of that of absolute value(see,[21]). The notion of paranormed sequence space was studied at the initial stage by Nakano[24]. Later on, it was further investigated by Maddox[20,21], Lascarides[17], Tripathy[30] and many others. A paranorm g for which $g(x) = 0$ implies $x = \theta$ is called a total paranorm on X, and the pair (X, g) is called a totally paranormed space.

Initially, as a generalization of statistical convergence[6,7], the notation of ideal convergence (I-convergence) was introduced and studied by Kostyrko, Mačaj, Salăt and Wilczyńki ([15,16]). Later on, it was studied by Šalát and Tripathy [30], Hazarika [8,32], Khan and Ebadullah [11,12,13],Demirci [4] and many others.

Here we give some important definitions.

Definition 1.9. A sequence x=$(x_k) \in \omega$ is said to be statistically convergent to a limit $L \in \mathbb{C}$ if for every $\epsilon > 0$, we have

$$\lim_k \frac{1}{k} |\{n \in \mathbb{N} : |x_k - L| \geq \epsilon, n \leq k\}| = 0$$

where vertical lines denote the cardinality of the enclosed set.

Definition 1.10. Let N be a non empty set. Then a family of sets $I \subseteq 2^N$ (power set of N) is said to be an ideal if
1) I is additive i.e $\forall A, B \in I \Rightarrow A \cup B \in I$
2) I is hereditary i.e $\forall A \in I$and $B \subseteq A \Rightarrow B \in I$.

Definition 1.11. A non-empty family of sets $\pounds(I) \subseteq 2^N$ is said to be filter on N if and only if
1) $\Phi \notin \pounds$(I),
2) $\forall\ A,\ B \in \pounds$(I) we have $A \cap B \in \pounds(I)$,
3) $\forall\ A \in \pounds$(I)and $A \subseteq B \Rightarrow B \in \pounds(I)$.

Definition 1.12. An Ideal $I \subseteq 2^N$ is called non-trivial if $I \neq 2^N$.

Definition 1.13. A non-trivial ideal $I \subseteq 2^N$ is called admissible if $\{\{x\} : x \in N\} \subseteq I$.

Definition 1.14. A non-trivial ideal I is maximal if there cannot exist any non-trivial ideal $J \neq I$ containing I as a subset.

Remark 1.15. For each ideal I, there is a filter $\pounds(I)$ corresponding to I.
i.e $\pounds(I) = \{K \subseteq N : K^c \in I\}$, where $K^c = N \setminus K$.

Definition 1.16. A sequence $x = (x_k) \in \omega$ is said to be I-convergent to a number L if for every $\epsilon > 0$, the set $\{k \in N : |x_k - L| \geq \epsilon\} \in I$.
In this case, we write $I - \lim x_k = L$.

Definition 1.17. A sequence $x = (x_k) \in \omega$ is said to be I-null if $L = 0$. In this case, we write $I - \lim x_k = 0$.

Definition 1.18. A sequence $x = (x_k) \in \omega$ is said to be I-cauchy if for every $\epsilon > 0$ there exists a number $m = m(\epsilon)$ such that $\{k \in N : |x_k - x_m| \geq \epsilon\} \in I$.

Definition 1.19. A sequence $x = (x_k) \in \omega$ is said to be I-bounded if there exists some $M > 0$ such that $\{k \in N : |x_k| \geq M\} \in I$.

Definition 1.20. A sequence space E said to be solid(normal) if $(\alpha_k x_k) \in E$ whenever $(x_k) \in E$ and for any sequence(α_k) of scalars with $|\ \alpha_k\ | \leq 1$, for all $k \in \mathbb{N}$.

Definition 1.21. A sequence space E said to be symmetric if $(x_{\pi(k)}) \in E$ whenever $x_k \in E$. where π is a permutation on $\mathbb{N}$

Definition 1.22. A sequence space E said to be sequence algebra if $(x_k) * (y_k) = (x_k.y_k) \in E$ whenever $(x_k), (y_k) \in E$.

Definition 1.23. A sequence space E said to be convergence free if $(y_k) \in E$ whenever $(x_k) \in E$ and $x_k = 0$ implies $y_k = 0$, for all k.

Definition 1.24. Let $K = \{k_1 < k_2 < k_3 < k_4 < k_5 ...\} \subset \mathbb{N}$ and E be a Sequence space.A K-step space of E is a sequence space $\lambda_K^E = \{(x_{k_n}) \in \omega : (x_k) \in E\}$.

Definition 1.25. A canonical pre-image of a sequence $(x_{k_n}) \in \lambda_K^E$ is a sequence $(y_k) \in \omega$ defined by

$$y_k = \begin{cases} x_k, & \text{if } k \in K, \\ 0, & \text{otherwise.} \end{cases}$$

A canonical preimage of a step space λ_K^E is a set of preimages all elements in λ_K^E.i.e. y is in the canonical preimage of λ_K^E iff y is the canonical preimage of some $x \in \lambda_K^E$.

Definition 1.26. A sequence space E is said to be monotone if it contains the canonical preimages of its step space.

Definition 1.27. If $I = I_f$, the class of all finite subsets of N. Then, I is an admissible ideal in N and I_f convergence coincides with the usual convergence.

Definition 1.28. If $I = I_\delta = \{A \subseteq N : \delta(A) = 0\}$. Then, I is an admissible ideal in N and we call the I_δ-convergence as the logarithmic statistical convergence.

Definition 1.29. If $I = I_d = \{A \subseteq N : d(A) = 0\}$. Then, I is an admissible ideal in N and we call the I_d-convergence as the asymptotic statistical convergence.

Remark 1.30. If $I_\delta - \lim x_n = l$, then $I_d - \lim x_n = l$

The following lemmas remained an important tool for the establishment of some results of this article.

Lemma(I). Every solid space is monotone

Lemma(II). Let $K \in \pounds(I)$ and $M \subseteq N$. If $M \notin I$, then $M \cap K \notin I$.

Lemma(III). If $I \subseteq 2^N$ and $M \subseteq N$. If $M \notin I$, then $M \cap N \notin I$.
Khan and K.Ebadullah[18] introduced and studied the following sequence space.
For $m \geq 0$

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$$BV_\sigma^I = \left\{ x = (x_k) \in \omega : \{k \in \mathbb{N} :\mid \phi_{m.k}(x) - L \mid\geq \epsilon\} \in I, \quad \text{for some L} \in \mathbb{C} \right\}. \tag{2.1}$$

2. Main results

In this article we introduce the following classes of sequence spaces :

For $m \geq 0$

$$BV_\sigma^I(M,p) = \left\{ x = (x_k) \in \omega : \left\{ k \in \mathbb{N} : M\left(\frac{\mid \phi_{m,k}(x) - L \mid}{\rho} \right)^{p_k} \geq \epsilon \right\} \in I; \quad \text{for some L} \in \mathbb{C},\ \rho > 0 \right\}; \tag{2.2}$$

$$_\circ BV_\sigma^I(M,p) = \left\{ x = (x_k) \in \omega : \left\{ k \in \mathbb{N} : M\left(\frac{\mid \phi_{m,k}(x) \mid}{\rho} \right)^{p_k} \geq \epsilon \right\} \in I, \quad \text{for some } \rho > 0 \right\}; \tag{2.3}$$

$$\ell_\infty(M,p) = \left\{ x = (x_k) \in \omega : \sup_k M\left(\frac{\mid \phi_{m,k}(x) \mid}{\rho} \right)^{p_k} < \infty, \quad \text{for some } \rho > 0 \right\}; \tag{2.4}$$

$$_\infty BV_\sigma^I(M,p) = \left\{ x = (x_k) \in \omega : \left\{ k \in \mathbb{N} : \exists K > 0,\ M\left(\frac{\mid \phi_{m,k}(x) \mid}{\rho} \right)^{p_k} \geq K \right\} \in I,\ \text{for some } \rho > 0 \right\}. \tag{2.5}$$

We also denote

$$\mathcal{M}_{BV_\sigma}^I(M,p) = BV_\sigma^I(M,p) \cap \ell_\infty(M,p)$$

and

$$_0\mathcal{M}_{BV_\sigma}^I(M,p) = {_0BV_\sigma^I}(M,p) \cap \ell_\infty(M,p).$$

Throughout the article, if required, we denote
$\phi_{m,k}(x)$=$x^{'}$**,** $\phi_{m,k}$**(y)**=$y^{'}$ **and** $\phi_{m,k}$**(z)**=$z^{'}$ **where** x,y,z **are** (x_k)**,** (y_k) **and** (z_k) **respectively.**

Theorem 2.1. Let $p=(p_k)\in l_\infty$. For an Orlicz function M, the classes of sequence ${}_0BV_\sigma^I(M,p), BV_\sigma^I(M,p)$, ${}_0\mathcal{M}_{BV_\sigma}^I(M,p)$ and $\mathcal{M}_{BV_\sigma}^I(M,p)$ are the linear spaces.

Proof. We shall prove the result for the space $BV_\sigma^I(M,p)$. Rests will follow similarly.

For, let $x=(x_k), y=(y_k)\in BV_\sigma^I(M,p)$ be any two arbitrary elements and let $\alpha,\ \beta$ are scalars.
Now,since
$x=(x_k), y=(y_k)\in BV_\sigma^I(M,p)$. $\Rightarrow$ For $\epsilon>0,\ \exists$ some +ve numbers ρ_1 and ρ_2 such that
the sets

$$A_1=\left\{k\in\mathbb{N}: M\left(\frac{\mid x_k'-L_1\mid}{\rho_1}\right)^{p_k}\geq\frac{\epsilon}{2}\right\}\in I,\ \text{for some } L_1\in\mathbb{C} \tag{2.6}$$

and

$$A_2=\left\{k\in\mathbb{N}: M\left(\frac{\mid y_k'-L_2\mid}{\rho_1}\right)^{p_k}\geq\frac{\epsilon}{2}\right\}\in I,\ \text{for some } L_2\in\mathbb{C} \qquad (2.7).$$

Let

$$\rho_3=\max\{2\mid\alpha\mid\rho_1, 2\mid\beta\mid\rho_2\} \qquad (2.8).$$

Since, M is non-decreasing and convex, we have,

$$\begin{aligned} M\left(\frac{|(\alpha x_k'+\beta y_k')-(\alpha L_1+\beta L_2)|}{\rho_3}\right)^{p_k} &\leq M\left(\frac{|\alpha||x_k'-L_1|}{\rho_3}\right)^{p_k}+M\left(\frac{|\beta||y_k'-L_2|}{\rho_3}\right)^{p_k}\\ &\leq M\left(\frac{|x_k'-L_1|}{\rho_1}\right)^{p_k}+M\left(\frac{|y_k'-L_2|}{\rho_2}\right)^{p_k} \end{aligned} \tag{2.9}$$

Therefore, from (2.6), (2.7) and (2.9), we have

$$\left\{k\in\mathbb{N}: M\left(\frac{\mid(\alpha x_k'+\beta y_k')-(\alpha L_1+\beta L_2)\mid}{\rho_3}\right)^{p_k}\geq\epsilon\right\}\subseteq A_1\cup A_2\in I.$$

implies that

$$\left\{k\in\mathbb{N}: M\left(\frac{\mid(\alpha x_k'+\beta y_k')-(\alpha L_1+\beta L_2)\mid}{\rho_3}\right)^{p_k}\geq\epsilon\right\}\in I$$

Therefore, $\alpha(x_k)+\beta(y_k)\in BV_\sigma^I(M,p)$
But $x=(x_k), y=(y_k)\in BV_\sigma^I(M,p)$ are the arbitrary elements
Therefore, $\alpha x_k+\beta y_k\in BV_\sigma^I(M),$ for all $x=(x_k), y=(y_k)\in BV_\sigma^I(M,p)$ and for all scalars $\alpha,\ \beta$
Hence, $BV_\sigma^I(M,p)$ is linear

Theorem 2.2. Let $p=(p_k)\in l_\infty$. For an Orlicz function M, the spaces $\mathcal{M}_{BV_\sigma}^I(M,p)$ and ${}_0\mathcal{M}_{BV_\sigma}^I(M,p)$ are paranormed spaces, paranormed by

$$g(x)=\inf_{k\geq1}\left\{\rho^{\frac{p_k}{H}}:\sup_k M\left(\frac{\mid\phi_{m,k}(x)\mid}{\rho}\right)^{p_k}\leq1,\ \text{for some } \rho>0\right\}$$

where $H=\max\{1,\sup_k p_k\}$.

Proof. (PI) Clearly $g(x)=0\ if\ x=\theta$,
(P2) It is obvious that $g(-x)=g(x)$,

(P3) Let $x=(x_k)$ and $y=(y_k)$ be two elements in $\mathcal{M}^I_{BV_\sigma}(M,p)$. Now for $\rho_1,\rho_2>0$, we denote

$$A_1=\left\{\rho_1:\sup_k M\left(\frac{\mid \phi_{m,k}(x)\mid}{\rho}\right)^{p_k}\le 1\right\} \tag{2.10}$$

and

$$A_2=\left\{\rho_2:\sup_k M\left(\frac{\mid \phi_{m,k}(x)\mid}{\rho}\right)^{p_k}\le 1\right\} \tag{2.11}.$$

Let us take $\rho=\rho_1+\rho_2$. Then by using the convexity of M, we have

$$M\left(\frac{\mid \phi_{m,k}(x+y)\mid}{\rho}\right)\le \frac{\rho_1}{\rho_1+\rho_2}M\left(\frac{\mid \phi_{m,k}(x)\mid}{\rho_1}\right)+\frac{\rho_2}{\rho_1+\rho_2}M\left(\frac{\mid \phi_{m,k}(y)\mid}{\rho_2}\right)$$

which in terms give us

$$\sup_k M\left(\frac{\mid \phi_{m,k}(x+y)\mid}{\rho}\right)^{p_k}\le 1$$

and

$$\begin{aligned}g(x+y)&=\inf\left\{(\rho_1+\rho_2)^{\frac{p_k}{H}}:\rho_1\in A_1,\ \rho_2\in A_2\right\}\\&\le\inf\left\{(\rho_1)^{\frac{p_k}{H}}:\rho_1\in A_1\right\}+\inf\left\{(\rho_1)^{\frac{p_k}{H}}:\rho_1\in A_1\right\}\\&=g(x)+g(y).\end{aligned}$$

(P4) Let (λ_k) be a sequence of scalars with $\lambda_k\to L$ where $\lambda_k,L\in\mathbb{C}$ and let $(x_k),\ x\in\mathcal{M}^I_{BV_\sigma}(M,p)$ be such that $g(x_k-x)\to 0$ as $k\to\infty$. To prove that $g(\lambda_k x_k-Lx)\to 0$ as $k\to\infty$.
We put

$$A_3=\left\{\rho_r>0:\sup_k M\left(\frac{\mid \phi_{m,k}(x_k)\mid}{\rho_r}\right)^{p_k}\le 1\right\} \tag{2.12}$$

and

$$A_4=\left\{\rho_s>0:\sup_k M\left(\frac{\mid \phi_{m,k}(x_k-x)\mid}{\rho_s}\right)^{p_k}\le 1\right\} \tag{2.13}$$

By convexity and continuity of M, we observe that

$$\begin{aligned}M\left(\frac{|\phi_{m,k}(\lambda_k x_k-Lx)|}{|\lambda_k-L|_{\rho_r}+|L|_{\rho_s}}\right)&\le M\left(\frac{|\phi_{m,k}(\lambda_k x_k-Lx_k)|}{|\lambda_k-L|_{\rho_r}+|L|_{\rho_s}}\right)+M\left(\frac{|\phi_{m,k}(Lx_k-Lx)|}{|\lambda_k-L|_{\rho_r}+|L|_{\rho_s}}\right)\\&\le\frac{|\lambda_k-L|_{\rho_r}}{|\lambda_k-L|_{\rho_r}+|L|_{\rho_s}}M\left(\frac{|\phi_{m,k}(x_k)}{\rho_r}\right)+\frac{|L|_{\rho_s}}{|\lambda_k-L|_{\rho_r}+|L|_{\rho_s}}M\left(\frac{|\phi_{m,k}(x_k-x)|}{\rho_r}\right)\end{aligned}$$

From the above inequality, it follows that

$$\sup_k M\left(\frac{\mid \phi_{m,k}(\lambda_k x_k-Lx)\mid}{\mid\lambda_k-L\mid_{\rho_r}+\mid L\mid_{\rho_s}}\right)^{p_k}\le 1$$

and consequently, we have

$$\begin{aligned}g(\lambda_k x_k-Lx)&=\inf\left\{\left(\mid\lambda_k-L\mid_{\rho_r}+\mid L\mid_{\rho_s}\right)^{\frac{p_k}{H}}:\rho_r\in A_3,\rho_s\in A_4\right\}\\&\le\mid\lambda_k-L\mid^{\frac{p_k}{H}}\inf\left\{(\rho_r)\frac{p_k}{H}:\rho_r\in A_3\right\}+\mid L\mid^{\frac{p_k}{H}}\inf\left\{(\rho_s)\frac{p_k}{H}:\rho_r\in A_4\right\}\\&\le\max\left\{1,\mid\lambda_k-L\mid^{\frac{p_k}{H}}\right\}g(x_k)+\max\left\{1,\mid L\mid^{\frac{p_k}{H}}\right\}g(x_k-x)\end{aligned} \tag{2.14}$$

Notice that $g(x_k)\le g(x)+g(x_k-x)$ for all $k\in\mathbb{N}$. Hence by our assumption, the right hand side of (2.14) tends

to 0 as $k \to \infty$ and the result follows.
For ${}_0\mathcal{M}^I_{BV_\sigma}(M,p)$, the result is similar and hence omitted.

Theorem 2.3 Let M_1 and M_2 be two Orlicz functions and satisfying Δ_2- Condition, then
(a) $\mathcal{X}(M_2,p) \subseteq \mathcal{X}(M_1M_2,p)$
(b) $\mathcal{X}(M_1,p) \cap (M_2,p) \subseteq \mathcal{X}(M_1+M_2,p)$
where $\mathcal{X}= {}_0BV^I_\sigma$, BV^I_σ, ${}_0\mathcal{M}^I_{BV_\sigma}$, $\mathcal{M}^I_{BV_\sigma}$.

Proof. (a). Let $x=(x_k) \in {}_0BV^I_\sigma(M_2)$ be any arbitrary element. Let $\epsilon > 0$ be given$\Rightarrow \exists\ \rho > 0$ such that

$$\left\{k \in \mathbb{N} : M_2\left(\frac{\mid \phi_{m,k}(x) \mid}{\rho}\right)^{p_k} \geq \epsilon\right\} \in I.$$

i.e.

$$\left\{k \in \mathbb{N} : M_2\left(\frac{\mid x'_k \mid}{\rho}\right)^{p_k} \geq \epsilon\right\} \in I, \qquad (2.15).$$

Let $\epsilon > 0$ and choose δ with $0 < \delta < 1$ such that $M_1(t) < \epsilon,$ for $0 \leq t \leq \delta$.
Let us write
$y_k = M_2\left(\frac{|x'_k|}{\rho}\right)^{p_k}$
and consider

$$\lim_k M_1(y_k) = \lim_{y_k \leq \delta, k \in \mathbb{N}} M_1(y_k) + \lim_{y_k > \delta, k \in \mathbb{N}} M_1(y_k).$$

Now, since M_1 is an Orlicz function, we have
$M_1(\lambda x) \leq \lambda M_1(x)$ for all λ with $0 < \lambda < 1$.
Therefore, $\lim_{y_k \leq \delta, k \in \mathbb{N}} M_1(y_k) \leq M_1(2) \lim_{y_k \leq \delta, k \in \mathbb{N}} (y_k)$

(2.16)

For $y_k > \delta$, we have $y_k < \frac{y_k}{\delta} < 1 + \frac{y_k}{\delta}$
Now, since M_1 is non-decreasing and convex, it follows that

$$M_1(y_k) < M_1(1+\frac{y_k}{\delta}) < \frac{1}{2}M_1(2) + \frac{1}{2}M_1(\frac{2y_k}{\delta})$$

Again, since M_1 satisfies Δ_2- Condition, we have

$$M_1(y_k) < \frac{1}{2}K\frac{(y_k)}{\delta}M_1(2) + \frac{1}{2}K\frac{(y_k)}{\delta}M_1(2).$$

Thus,

$$M_1(y_k) < K\frac{(y_k)}{\delta}M_1(2).$$

Hence,

$$\lim_{y_k > \delta, k \in \mathbb{N}} M_1(y_k) \leq max\{1, K\delta^{-1}M_1(2) \lim_{y_k > \delta, k \in \mathbb{N}} (y_k). \qquad (2.17)$$

Therefore, from (2.15), (2.16) and (2.17), it follows that

$$\left\{k \in \mathbb{N} : M_1M_2\left(\frac{\mid \phi_{m,k}(x) \mid}{\rho}\right)^{p_k} \geq \epsilon\right\} \in I,$$

implies that $x=(x_k) \in {}_0BV^I_\sigma(M_1M_2,p)$
Therefore, ${}_0BV^I_\sigma(M_2,p) \subseteq {}_0BV^I_\sigma(M_1M_2,p)$. Hence, $\mathcal{X}(M_2,p) \subseteq \mathcal{X}(M_1M_2,p)$ for $\mathcal{X}= {}_0BV^I_\sigma$
For $\mathcal{X}=BV^I_\sigma$,$\mathcal{X}= {}_0\mathcal{M}^I_{BV_\sigma}$ and $\mathcal{X}=\mathcal{M}^I_{BV_\sigma}$ the inclusions can be established similarly.

(b). Let $x=(x_k) \in {}_0BV^I_\sigma(M_1,p) \cap {}_0BV^I_\sigma(M_2,p)$. Let $\epsilon > 0$ be given. Then there exists $\rho > 0$ such that the sets

$$\left\{k \in \mathbb{N} : M_1\left(\frac{\mid \phi_{m,k}(x) \mid}{\rho}\right)^{p_k} \geq \epsilon\right\} \in I,$$

and

$$\left\{k \in \mathbb{N} : M_2\left(\frac{|\ \phi_{m,k}(x)\ |}{\rho}\right)^{p_k} \geq \epsilon\right\} \in I,$$

Therefore, the inclusion

$$\left\{k \in \mathbb{N} : (M_1 + M_2)\left(\frac{|\ \phi_{m,k}(x)\ |}{\rho}\right)^{p_k} \geq \epsilon\right\}$$

$$\subseteq \left[\left\{k \in \mathbb{N} : M_1\left(\frac{|\ \phi_{m,k}(x)\ |}{\rho}\right)^{p_k} \geq \epsilon\right\}\right.$$

$$\left.\cup\left\{k \in \mathbb{N} : M_2\left(\frac{|\ \phi_{m,k}(x)\ |}{\rho}\right)^{p_k} \geq \epsilon\right\}\right]$$

implies that

$$\left\{k \in \mathbb{N} : (M_1 + M_2)\left(\frac{|\ \phi_{m,k}(x)\ |}{\rho}\right)^{p_k} \geq \epsilon\right\} \in I.$$

showing that $x = (x_k) \in {}_0BV_\sigma^I(M_1 + M_2, p)$
Hence, ${}_0BV_\sigma^I(M_1, p) \cap {}_0BV_\sigma^I(M_2, p) \subseteq {}_0BV_\sigma^I(M_1 + M_2, p)$
For $\mathcal{X} = BV_\sigma^I$,$\mathcal{X} = {}_0\mathcal{M}_{BV_\sigma}^I$ and $\mathcal{X} = \mathcal{M}_{BV_\sigma}^I$ the inclusions are similar.

For $M_2(x) = x$ and $M_1(x) = M(x)$, for all $x \in [0, \infty)$, we have the following corollary.

Corollary. $\mathcal{X} \subseteq \mathcal{X}(M, p)$ for $\mathcal{X} = {}_0BV_\sigma^I$, BV_σ^I, ${}_0\mathcal{M}_{BV_\sigma}^I$ and $\mathcal{M}_{BV_\sigma}^I$.

Theorem 2.4. For any orlicz function M, the spaces ${}_0BV_\sigma^I(M, p)$ and ${}_0\mathcal{M}_{BV_\sigma}^I(M, p)$ are solid and monotone.

Proof. Here we consider ${}_0BV_\sigma^I(M, p)$. For ${}_0\mathcal{M}_{BV_\sigma}^I(M, p)$, the proof shall be similar.
For,let $x = (x_k) \in {}_0BV_\sigma^I(M, p)$ be any arbitrary element.$\Rightarrow$ For $\epsilon > 0$, $\exists\ \rho > 0$ with

$$\left\{k \in \mathbb{N} : M\left(\frac{|\ \phi_{m,k}(x)\ |}{\rho}\right)^{p_k} \geq \epsilon\right\} \in I$$

Let(α_k) be a sequence of scalars such that

$$|\ \alpha_k\ | \leq 1, \text{for all k} \in \mathbb{N}.$$

Now, since M is an Orlicz function
We have,

$$M\left(\frac{|\ \alpha_k \phi_{m,k}(x)\ |}{\rho}\right)^{p_k} \leq |\ \alpha_k\ |^{p_k}\ M\left(\frac{|\ \phi_{m,k}(x)\ |}{\rho}\right)^{p_k} \leq M\left(\frac{|\ \phi_{m,k}(x)\ |}{\rho}\right) p_k.$$

Therefore,

$$\left\{k \in \mathbb{N} : M\left(\frac{|\ \alpha_k \phi_{m,k}(x)\ |}{\rho}\right)^{p_k} \geq \epsilon\right\} \subseteq \left\{k \in \mathbb{N} : M\left(\frac{|\ \phi_{m.k}(x)\ |}{\rho}\right)^{p_k} \geq \epsilon\right\} \in I$$

implies that

$$\left\{k \in \mathbb{N} : M\left(\frac{|\ \alpha_k \phi_{m,k}(x)\ |}{\rho}\right)^{p_k} \geq \epsilon\right\} \in I$$

Thus, $(\alpha_k x_k) \in {}_0BV_\sigma^I(M, p)$.
Hence ${}_0BV_\sigma^I(M, p)$ is solid
Therefore, by lemma(I) ${}_0BV_\sigma^I(M)$ is monotone. Hence the result.

Theorem 2.5. The spaces $\mathcal{M}^I_{BV_\sigma}(M,p)$ and ${}_0\mathcal{M}^I_{BV_\sigma}(M,p)$ are not seperable.

Proof. By a counter example we prove the result for the space $\mathcal{M}^I_{BV_\sigma}(M,p)$.
For ${}_0\mathcal{M}^I_{BV_\sigma}(M,p)$, the result follows similarly.

Counter Example.
Let A be an infinite subset of increasing natural numbers such that $A \in I$.
Let

$$p_k = \begin{cases} 1, \text{if } \ \text{k} \in A, \\ 2, \text{otherwise}. \end{cases}$$

Let $P_0 = \{(x_k) : x_k = 0 \text{ or } 1, \text{for } k \in M \text{ and } x_k = 0, \text{ otherwise}\}$.
Since A is infinite, so P_0 is uncountable. Consider the class of open balls $B_1 = \{B(z, \frac{1}{2}) : z \in P_0\}$.
Let C_1 be an open cover of $\mathcal{M}^I_{BV_\sigma}(M,p)$ containing B_1.
Since B_1 is uncountable, so C_1 cannot be reduced to a countable subcover for $\mathcal{M}^I_{BV_\sigma}(M,p)$. Thus $\mathcal{M}^I_{BV_\sigma}(M,p)$ is not seperable.

Theorem 2.6. Let $H = \sup\limits_k p_k < \infty$ and I an admissible ideal. Then the following are equivalent.
(a) $x = (x_k) \in BV^I_\sigma(M,p)$;
(b) there exists $y = (y_k) \in BV_\sigma(M,p)$ such that $x_k = y_k$, for a.a.k.r.I;
(c) there exists $y = (y_k) \in BV_\sigma(M,p)$ and $z = (z_k) \in {}_0BV^I_\sigma(M,p)$ such that $x_k = y_k + z_k$ for all $k \in \mathbb{N}$ and $\left\{k \in \mathbb{N} : M\left(\frac{|y'_k - L|}{\rho}\right)^{p_k} \geq \epsilon\right\} \in I$;
(d) there exists a subset $K = \{k_1 < k_2 ....\}$ of $\mathbb{N}$ such that $K \in \pounds(I)$
and $\lim\limits_{n\to\infty} M\left(\frac{|x'_{k_n} - L|}{\rho}\right)^{p_{k_n}} = 0$.

Proof. (a) implies (b). Let $x = (x_k) \in BV^I_\sigma(M,p)$. Then there exists $L \in \mathbb{C}$ such that

$$\left\{k \in \mathbb{N} : M\left(\frac{|x'_k - L|}{\rho}\right)^{p_k} \geq \epsilon\right\} \in I.$$

Let (m_t) be an increasing sequence with $m_t \in \mathbb{N}$ such that

$$\left\{k \leq m_t : M\left(\frac{|x'_k - L|}{\rho}\right)^{p_k} \geq t^{-1}\right\} \in I.$$

Define a sequence(y_k) as

$$y_k = x_k, \text{ for all } \ k \leq m_1.$$

For $m_t < k \leq m_{t+1}, \ t \in \mathbb{N}$.

$$y_k = \begin{cases} x_k, & \text{if } M\left(\frac{|x'_k - L|}{\rho}\right)^{p_k} < t^{-1} \\ \text{L}, & \text{otherwise}. \end{cases}$$

Then $y = (y_k) \in BV_\sigma(M,p)$ and form the following inclusion

$$\left\{k \leq m_t : x_k \neq y_k\right\} \subseteq \left\{k \leq m_t : M\left(\frac{|x'_k - L|}{\rho}\right)^{p_k} \geq \epsilon\right\} \in I.$$

We get $x_k = y_k$, for a.a.k.r.I.

(b) implies (c). For $(x_k) \in BV^I_\sigma(M,p)$. Then there exists $(y_k) \in BV_\sigma(M,p)$ such that $x_k = y_k$, for a.a.k.r.I. Let $K = \{k \in \mathbb{N} : x_k \neq y_k\}$, then $K \in I$.
Define a sequence (z_k) as

$$z_k = \begin{cases} x_k - y_k, & \text{if } k \in K, \\ 0, otherwise. \end{cases}$$

Then $z_k \in {}_0BV_\sigma^I(M,p)$ and $y_k \in BV_\sigma(M,p)$.

(c) implies (d). Suppose (c) holds. Let $\epsilon > 0$ be given. Let $P_1 = \{k \in \mathbb{N} : M\left(\frac{|x'_{k_n} - L|}{\rho}\right)^{p_k} \geq \epsilon\} \in I$ and

$$K = P_1^c = \{k_1 < k_2 < k_3 < ...\} \in \pounds(I).$$

Then, we have $\lim\limits_{n\to\infty} M\left(\frac{|x'_{k_n} - L|}{\rho}\right)^{p_{k_n}} = 0.$

(d) implies (a). Let $K = \{k_1 < k_2 < k_3 < ...\} \in \pounds(I)$ and $\lim\limits_{n\to\infty} M\left(\frac{|x'_{k_n} - L|}{\rho}\right)^{p_{k_n}} = 0.$

Then, for any $\epsilon > 0$, and Lemma (II), we have

$$\left\{k \in \mathbb{N} : M\left(\frac{|x'_k - L|}{\rho}\right)^{p_k} \geq \epsilon\right\} \subseteq K^c \cup \left\{k \in \mathbb{N} : M\left(\frac{|x'_{k_n} - L|}{\rho}\right)^{p_{k_n}} \geq \epsilon\right\} \in I$$

implies that

$$\left\{k \in \mathbb{N} : M\left(\frac{|x'_k - L|}{\rho}\right)^{p_k} \geq \epsilon\right\} \in I$$

Therefore, $(x_k) \in BV_\sigma^I(M,p)$.
Hence the result.

Theorem 2.7. Let $h = \inf\limits_k p_k$ and $H = \sup\limits_k p_k$. Then, the following results are equivalent. (a) $H < \infty$ and $h > 0$. (b) ${}_0BV_\sigma^I(M,p) = BV_{\circ\sigma}^I$.

Proof. Suppose that $H < \infty$ and $h > 0$, then the inequalities $min\{1, s^h\} \leq s^{p_k} \leq max\{1, s^H\}$ hold for any $s > 0$ and for all $k \in \mathbb{N}$.
Therefore the equivalent of (a) and (b) is obvious.

Theorem 2.8. Let $p = (q_k)$ and $q = (q_k)$ be two sequences of positive real numbers. Then ${}_0\mathcal{M}_{BV_\sigma}^I(M,p) \supseteq {}_0\mathcal{M}_{BV_\sigma}^I(M,q)$ if and only if $\lim\limits_{k\in K} \inf \frac{p_k}{q_k} > 0$, where $K^c \subseteq \mathbb{N}$ such that $K \in I$.

Proof. Let $\lim\limits_{k\in K} \inf \frac{p_k}{q_k} > 0$. and $(x_k) \in {}_0\mathcal{M}_{BV_\sigma}^I(M,p)$. Then, there exists $\beta > 0$ such that $p_k > \beta q_k$, for all sufficiently large $k \in K$.
Since $(x_k) \in {}_0\mathcal{M}_{BV_\sigma}^I(M,p)$.
For a given $\epsilon > 0$, $\exists\ \rho > 0$ such that

$$B_0 = \left\{k \in \mathbb{N} : M\left(\frac{|x'_k|}{\rho}\right)^{p_k} \geq \epsilon\right\} \in I.$$

Let $G_0 = K^c \cup B_0$ Then $G_0 \in I$.
Then, for all sufficiently large $k \in G_0$,

$$\left\{k \in \mathbb{N} : M\left(\frac{|x'_k|}{\rho}\right)^{p_k} \geq \epsilon\right\} \subseteq \{k \in \mathbb{N} : \left\{k \in \mathbb{N} : M\left(\frac{|x'_k|}{\rho}\right)^{\beta q_k} \geq \epsilon\right\} \in I.$$

implies that

$$\left\{k \in \mathbb{N} : M\left(\frac{|x'_k|}{\rho}\right)^{p_k} \geq \epsilon\right\} \in I$$

Therefore $(x_k) \in {}_0\mathcal{M}_{BV_\sigma}^I(M,p)$.

Converse part of the result follows obviously.

Theorem 2.9. Let $p = (p_k)$ and $q = (q_k)$ be two sequences of positive real numbers. Then

$$_0\mathcal{M}^I_{BV_\sigma}(M, q) \supseteq {_0\mathcal{M}^I_{BV_\sigma}}(M, p)$$

if and only if $\lim\limits_{k\in K} \inf \frac{q_k}{p_k} > 0$, where $K^c \subseteq \mathbb{N}$ such that $K \in I$.

Proof. The proof follows similarly as the proof of Theorem 2.8.

Theorem 2.10. Let $p = (p_k)$ and $q = (q_k)$ be two sequences of positive real numbers. Then $_0\mathcal{M}^I_{BV_\sigma}(M, p) = {_0\mathcal{M}^I_{BV_\sigma}}(M, q)$ if and only if $\lim\limits_{k\in K} \inf \frac{p_k}{q_k} > 0$, and $\lim\limits_{k\in K} \inf \frac{q_k}{p_k} > 0$, where $K^c \subseteq \mathbb{N}$ such that $K \in I$.

Proof. On combining Theorem 2.9 and 2.10 we get the required result.

Theorem 2.11. The set $\mathcal{M}^I_{BV_\sigma}(M, p)$ is closed subspace of $\ell_\infty(M, p)$.

Proof. Let $(x_k^{(i)})$ be a Cauchy sequence in $\mathcal{M}^I_{BV_\sigma}(M, p)$ such that $x^{(i)} \to x$.
We show that $x \in \mathcal{M}^I_{BV_\sigma}(M, p)$
Since $(x_k^{(i)}) \in \mathcal{M}^I_{BV_\sigma}(M, p)$, then there exists a sequence a_i and $\rho > 0$ such that

$$\{k \in \mathbb{N} : M\bigg(\frac{\mid (x_k^{(i)})' - a_i \mid}{\rho}\bigg)^{p_k} \geq \epsilon\} \in I$$

We need to show that
(1) (a_i) converges to a.
(2) If $U = \{k \in \mathbb{N} : M\Big(\frac{|(x_k^{(i)})' - a|}{\rho}\Big)^{p_k} < \epsilon\}$, then $U^c \in I$.
(1) Since $(x_k^{(i)})$ is Cauchy sequence in $\mathcal{M}^I_{BV_\sigma}(M, p) \Rightarrow$ for a given $\epsilon > 0$, there exists $k_0 \in \mathbb{N}$ such that

$$\sup_k M\bigg(\frac{\mid (x_k^{(i)})' - (x_k^{(j)})' \mid}{\rho}\bigg)^{p_k} < \frac{\epsilon}{3}, \text{for all i, j} \geq k_0.$$

For $\epsilon > 0$, we have

$$B_{ij} = \{k \in \mathbb{N} : M\bigg(\frac{\mid (x_k^{(i)})' - (x_k^{(j)})' \mid}{\rho}\bigg)^{p_k} < \frac{\epsilon}{3}\bigg\}$$

$$B_i = \{k \in \mathbb{N} : M\bigg(\frac{\mid (x_k^{(i)})' - a_i \mid}{\rho}\bigg)^{p_k} < \frac{\epsilon}{3}\bigg\}$$

$$B_j = \{k \in \mathbb{N} : M\bigg(\frac{\mid (x_k^{(j)})' - a_j \mid}{\rho}\bigg)^{p_k} < \frac{\epsilon}{3}\bigg\}$$

Then, B^c_{ij},B^c_i, $B^c_j \in I$
Let $B^c = B^c_{ij} \cup B^c_i \cup B^c_j$, where $B = \bigg\{k \in \mathbb{N} : M\Big(\frac{|a_i - a_j|}{\rho}\Big)^{p_k} < \epsilon\bigg\}$.
Then, $B^c \in I$.
We choose $k_0 \in B^c$.
Then for each $i, j \geq k_0$,
we have

$$\bigg\{k \in \mathbb{N} : M\Big(\frac{|a_i - a_j|}{\rho}\Big)^{p_k} < \epsilon\bigg\} \supseteq \Bigg[\bigg\{k \in \mathbb{N} : M\Big(\frac{|a_i - a_j|}{\rho}\Big)^{p_k} < \frac{\epsilon}{3}\bigg\}$$

$$\cap \bigg\{k \in \mathbb{N} : M\bigg(\frac{\mid (x_k^{(i)})' - a_i \mid}{\rho}\bigg)^{p_k} < \frac{\epsilon}{3}\bigg\}$$

$$\cap\left\{k \in \mathbb{N} : M\left(\frac{| a_j - (x_k^{(j)})' |}{\rho}\right)^{p_k} < \frac{\epsilon}{3}\right\}\Bigg]$$

implies that
(a_i) is a Cauchy sequence of scalars in C, so there exists a scalar a in C such that $a_i \to a$, as $n \to \infty$.
(2) Let $0 < \delta < 1$ be given. Then we show that if
$U = \{k \in \mathbb{N} : M\left(\frac{|(x_k^{(i)})' - a|}{\rho}\right)^{p_k} \leq \epsilon\}$, then $U^c \in I$.
Since $x^{(i)} \to x$, then there exists $q_0 \in \mathbb{N}$ such that

$$P = \left\{k \in \mathbb{N} : M\left(\frac{| (x_k^{(q_0)})' - x_k' |}{\rho}\right)^{p_k} < (\frac{\delta}{3D})^H\right\} \tag{2.21}$$

where $D = \max\{1, 2^{G-1}\}, G = \sup\limits_k p_k \geq 0$ and $H = \max\{1, \sup\limits_k p_k\}$
implies $P^c \in I$.
The number q_0 can be chosen that together with (2.21), we have

$$Q = \left\{k \in \mathbb{N} : M\left(\frac{| a_{q_0} - a |}{\rho}\right)^{p_k} < (\frac{\delta}{3D})^H\right\}$$

such that $Q^c \in I$.
Since

$$\left\{k \in \mathbb{N} : M\left(\frac{| (x_k^{(q_0)})' - a_{q_0} |}{\rho}\right)^{p_k} \geq \delta\right\} \in I.$$

Then, we have a subset S of $\mathbb{N}$ such that $S^c \in I$, where

$$S = \left\{k \in \mathbb{N} : M\left(\frac{| (x_k^{(q_0)})' - a_{q_0} |}{\rho}\right)^{p_k} < (\frac{\delta}{3D})^H\right\}.$$

Let $U^c = P^c \cup Q^c \cup S^c$, where

$$U = \left\{k \in \mathbb{N} : M\left(\frac{| (x_k' - a |}{\rho}\right)^{p_k} < \delta\right\}$$

Therefore, for each $k \in U^c$, we have

$$\left\{k \in \mathbb{N} : M\left(\frac{|(x_k' - a|}{\rho}\right)^{p_k} < \delta\right\} \supseteq \Bigg[\left\{k \in \mathbb{N} : M\left(\frac{|(x_k^{(q_0)})' - x_k'|}{\rho}\right)^{p_k} < (\frac{\delta}{3D})^H\right\}$$

$$\cap\left\{k \in \mathbb{N} : M\left(\frac{| a_{q_0} - a |}{\rho}\right)^{p_k} < (\frac{\delta}{3D})^H\right\}$$

$$\cap\left\{k \in \mathbb{N} : M\left(\frac{| (x_k^{(q_0)})' - a_{q_0} |}{\rho}\right)^{p_k} < (\frac{\delta}{3D})^H\right\}\Bigg].$$

Then the result follows.
Since the inclusions $\mathcal{M}_{BV_\sigma}^I(M,p) \subset \ell_\infty(M,p)$ and ${}_0\mathcal{M}_{BV_\sigma}^I(M,p) \subset \ell_\infty(M,p)$ are strict so in view of Theorem (2.11) we have the following result.

Theorem 2.12. The spaces $\mathcal{M}_{BV_\sigma}^I(M,p)$ and ${}_0\mathcal{M}_{BV_\sigma}^I(M,p)$ are nowhere dense subsets of $\ell_\infty(M,p)$.

Theorem 2.13. For an Orlicz function M, the spaces ${}_0BV_\sigma^I(M,p)$ and $BV_\sigma^I(M,p)$ are sequence algebra.

Proof. Here we consider ${}_0BV_\sigma^I(M,p)$. For the other result the proof is similar.
Let $x = (x_k), y = (y_k) \in {}_0BV_\sigma^I(M,p)$ be any two arbitrary elements.

$\Rightarrow \exists\ \rho_1, \rho_2 > 0$ such that

$$\left\{k \in \mathbb{N} : M\left(\frac{|\ \phi_{m,k}(x)\ |}{\rho_1} \geq \epsilon\right)^{p_k}\right\} \in I. \tag{2.22}$$

and

$$\left\{k \in \mathbb{N} : M\left(\frac{|\ \phi_{m,k}(y)\ |}{\rho_1} \geq \epsilon\right)^{p_k}\right\} \in I. \tag{2.23}$$

Let $\rho = \rho_1\rho_2 > 0$
Then, it is obvious from (2.22) and (2.23) that

$$\left\{k \in \mathbb{N} : M\left(\frac{|\ \phi_{m,k}(x)\phi_{m,k}(y)\ |}{\rho} \geq \epsilon\right)^{p_k}\right\} \in I.$$

which further implies that $(x_k.y_k) = (x_k y_k) \in {}_0BV_\sigma^I(M,p)$
Hence, ${}_0BV_\sigma^I(M,p)$ is a Sequence algebra.

Theorem 2.11. Let M be an Orlicz function. Then, ${}_\circ BVI_\sigma(M,p) \subset BVI_\sigma(M,p) \subset {}_\infty BV_\sigma^I(M,p)$.

Proof. Let M be an Orlicz function. Then, we have to show that
${}_0BV_\sigma^I(M,p) \subseteq BV_\sigma^I(M,p) \subseteq {}_\infty BV_\sigma^I(M,p)$
Firstly, ${}_0BV_\sigma^I(M) \subseteq BV_\sigma^I(M)$ is obvious.
Let $x = (x_k) \in BV_\sigma^I(M,p)$. Then there exists $L \in \mathbb{C}$ and $\rho > 0$ such that

$$\left\{k \in \mathbb{N} : M\left(\frac{|x'_k - L|}{\rho}\right)^{p_k} \geq \epsilon\right\} \in I.$$

That is

$$I - \lim M\left(\frac{|x'_k - L|}{\rho}\right)^{p_k} = 0.$$

Therefore, we have

$$M\left(\frac{|x'_k|}{2\rho}\right)^{p_k} \leq \frac{1}{2}M\left(\frac{|x'_k - L|}{\rho}\right)^{p_k} + \frac{1}{2}M\left(\frac{|L|}{\rho}\right)^{p_k}.$$

Taking supremum over k both sides, we get $x = (x_k) \in {}_\infty BV_\sigma^I(M,p)$.
Hence, ${}_\circ BV_\sigma^I(M,p) \subset BV_\sigma^I(M,p) \subset {}_\infty BV_\sigma^I(M,p)$.

Theorem 2.15. If I is not maximal and $I \neq I_f$. Then, the space ${}_0BV_\sigma^I(M,p)$ and $BV_\sigma^I(M,p)$ are not symmetric.

Proof. Let $A \in I$ be any infinite set and $M(x) = x,$ for all $x \in [0, \infty)$.
Define a sequence (x_k) as

$$x_k = \begin{cases} 1, & \text{if } k \in A, \\ 0, & \text{otherwise.} \end{cases}$$

Then, it is clear that $(x_k) \in {}_0BV_\sigma^I(M,p) \subsetneq BV_\sigma^I(M,p)$
Let $K \subseteq \mathbb{N}$ be such that $K \notin I$ and $\mathbb{N} \setminus K \notin I$.
Let $\phi : K \to A$ and $\psi : K^c \to A^c$ be bijective maps. Then, the mapping $\pi :\to \mathbb{N} \to \mathbb{N}$ defined by

$$\pi(k) = \begin{cases} \phi(k), & \text{if } k \in K, \\ \psi k, & \text{otherwise.} \end{cases}$$

is a permutation on $\mathbb{N}$

But $(x_\pi(k)) \notin BV_\sigma^I(M,p)$ and hence $(x_\pi(k)) \notin {}_0BV_\sigma^I(M,p)$ showing that

$$BV_\sigma^I(M,p) \text{ and } {}_0BV_\sigma^I(M,p)$$

are not symmetric sequence spaces.

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