*Global Journal of Mathematical Analysis*, 2 (2) (2014) 70-72
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www.sciencepubco.com/index.php/GJMA
doi: 10.14419/gjma.v2i2.2497
Research Paper

# A note on $\delta$-Jordan homomorphism on Banach algebras

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**Abstract**

In this paper we show that, under special hypotheses, each $\delta$-Jordan homomorphism $\varphi$ between Banach algebras $\mathcal{A}$ and $\mathcal{B}$ is continuous and almost multiplicative.

***Keywords***: *$\delta$-Jordan homomorphism, almost multiplicative, Semisimple.*

## 1. Introduction

Let $\mathcal{A}$ and $\mathcal{B}$ be Banach algebras and $\varphi : \mathcal{A} \longrightarrow \mathcal{B}$ be a linear map. Then $\varphi$ is called Jordan homomorphism if $\varphi(a^2) = \varphi(a)^2$ for all $a \in \mathcal{A}$ [4], and it is called $\delta$-Jordan homomorphism if there exist $\delta > 0$ such that

$$\|\varphi(a^2) - \varphi(a)^2\| \leq \delta \ \|a\|^2, \qquad (a \in \mathcal{A}).$$

Moreover, $\varphi$ is said to be multiplicative, if $\varphi(ab) = \varphi(a)\varphi(b)$ for all $a, b \in \mathcal{A}$, and it is said to be almost multiplicative [3], if there exist $\xi > 0$ such that

$$\|\varphi(ab) - \varphi(a)\varphi(b)\| \leq \xi \ \|a\|\|b\|, \qquad (a, b \in \mathcal{A}).$$

It is obvious that if $\varphi$ is multiplicative (almost multiplicative), then it is Jordan homomorphism ($\delta$-Jordan homomorphism), but the converse is false, in general.

In [5], Zelazko show that if $\mathcal{B}$ is commutative and semisimple, then each Jordan homomorphism $\varphi : \mathcal{A} \longrightarrow \mathcal{B}$ is multiplicative. See also [6] for another characterization of this result.

In this paper we investigate a similar result for $\delta$-Jordan homomorphism and then we give a sufficient conditions that each $\delta$-Jordan homomorphism $\varphi : \mathcal{A} \longrightarrow \mathcal{B}$ to be almost multiplicative (Theorem 2.5 below).

It is well-known that every multiplicative linear functional $\varphi$ on Banach algebra $\mathcal{A}$ is continuous and $\|\varphi\| \leq 1$, see [2] for example.

In [3], Jarosz generalized this result and proved the following Theorem.

**Theorem 1.1** *Let $\varphi : \mathcal{A} \longrightarrow \mathbb{C}$ be an almost multiplicative linear functional, then $\varphi$ is continuous and $\|\varphi\| \leq 1+\xi$.*

The next result, which is a generalization of Jarosz's theorem, obtained in [1].

**Theorem 1.2** *Let $\varphi : \mathcal{A} \longrightarrow \mathcal{B}$ be an almost multiplicative linear map. If $\mathcal{B}$ is semisimple, then $\varphi$ is continuous and $\|\varphi\| \leq 1 + \xi$.*

## 2. $\delta$-Jordan homomorphism

The next result show that Theorem 1.1 is valid for $\delta$-Jordan homomorphism instead of almost multiplicative.

**Theorem 2.1** *Let $\varphi$ be a $\delta$-Jordan homomorphism from Banach algebra $\mathcal{A}$ into $\mathbb{C}$. Then $\varphi$ is continuous and $\|\varphi\| \leq 1 + \delta$.*

Proof. By definition we have $\|\varphi\| = \sup\{|\varphi(a)| : a \in \mathcal{A},\ \|a\| = 1\}$, thus for $0 < \lambda < \sqrt{\delta}$, there exist $a \in \mathcal{A}$ with $\|a\| = 1$ and $\|\varphi\| - \lambda < |\varphi(a)|$. Then

$$|\varphi(a)|^2 - |\varphi(a^2)| = |\varphi(a)^2| - |\varphi(a^2)| \leq |\varphi(a^2) - \varphi(a)^2| \leq \delta,$$

therefore

$$|\varphi(a)|^2 \leq |\varphi(a^2)| + \delta.$$

Since $\|a\| = 1$, we have

$$(\|\varphi\| - \lambda)^2 < |\varphi(a)|^2 \leq |\varphi(a^2)| + \delta \leq \|\varphi\| + \delta.$$

Letting $\lambda \longrightarrow 0$, so $\|\varphi\|^2 - \|\varphi\| \leq \delta$. Then

$$(2\|\varphi\| - 1)^2 \leq 1 + 4\delta.$$

It follows that

$$\|\varphi\| \leq \frac{1 + \sqrt{1 + 4\delta}}{2} \leq 1 + \delta.$$

**Corollary 2.2** *Let $\varphi : \mathcal{A} \longrightarrow \mathbb{C}$ be a $\delta$-Jordan homomorphism. Then for all $\lambda \in \mathbb{C}$, $(1 + \lambda)\varphi$ is $\delta$-Jordan homomorphism.*

**Theorem 2.3** *Let $\mathcal{A}$ and $\mathcal{B}$ be two Banach algebras and $\varphi : \mathcal{A} \longrightarrow \mathcal{B}$ be a $\delta$-Jordan homomorphism. If $\mathcal{B}$ is semisimple, then $\varphi$ is continuous.*

Proof. Let $\psi : \mathcal{B} \longrightarrow \mathbb{C}$ be a $\delta$-Jordan homomorphism. Then $\psi$ is bounded by Theorem 2.1, so we have

$$|\psi \circ \varphi(a^2) - (\psi \circ \varphi(a))^2| \leq \|\psi\|\ \|\varphi(a^2) - \varphi(a)^2\| \leq (1 + \delta)\ \delta\ \|a\|^2.$$

Therefore $\psi \circ \varphi$ is a $\eta$-Jordan homomorphism, where $\eta = \delta(1+\delta)$, thus it is continuous by above Theorem. Suppose that $(a_n)$ be a sequence in $\mathcal{A}$ such that $\lim_n a_n = a$ and $\lim_n \varphi(a_n) = b$. Then

$$\psi(b) = \psi(\lim_n \varphi(a_n)) = \lim_n \psi \circ \varphi(a_n) = \psi \circ \varphi(a),$$

thus, $\psi(b - \varphi(a)) = 0$. Since $\mathcal{B}$ is semisimple, we get $\varphi(a) = b$. Therefore $\varphi$ is continuous by the close graph Theorem.

The norm $\|.\|$ on a Banach algebra $\mathcal{A}$ is called uniform, if $\|a^2\| = \|a\|^2$ for all $a \in \mathcal{A}$. The uniform Banach algebra is a Banach algebra with uniform norm.

The proof of the next result is same as of Theorem 2.1.

**Theorem 2.4** *Let $\varphi$ be a $\delta$-Jordan homomorphism from Banach algebra $\mathcal{A}$ into a uniform Banach algebra $\mathcal{B}$. Then $\|\varphi\| \leq 1 + \delta$.*

The following theorem, which is the main one in the paper, is criterion for a $\delta$-Jordan homomorphism to be almost multiplicative.

**Theorem 2.5** *Let $\mathcal{A}$ and $\mathcal{B}$ be two commutative Banach algebras and $\varphi : \mathcal{A} \longrightarrow \mathcal{B}$ be a $\delta$-Jordan homomorphism. Then $\varphi$ is almost multiplicative.*

Proof. Let $a, b \in \mathcal{A}$ and $\|a\| = \|b\| = 1$. Since $\mathcal{A}$ and $\mathcal{B}$ are commutative, we get

$$\varphi(ab) - \varphi(a)\varphi(b) = \frac{1}{4}[\varphi(u^2) - \varphi(u)^2 - \varphi(v^2) + \varphi(v)^2],$$

where $u = a + b$ and $v = a - b$. Hence

$$\|\varphi(ab) - \varphi(a)\varphi(b)\| \leq \frac{1}{4}\ \|\varphi(u^2) - \varphi(u)^2\| + \frac{1}{4}\ \|\varphi(v^2) - \varphi(v)^2\| \leq \frac{\delta}{4}\ (\|u\|^2 + \|v\|^2) \leq 2\delta.$$

Therefore,

$$\|\varphi(ab) - \varphi(a)\varphi(b)\| \leq 2\delta\ \|a\|\|b\|.$$

Put $\xi = 2\delta$, then for all $a, b \in \mathcal{A}$, with $\|a\| = \|b\| = 1$, we have

$$\|\varphi(ab) - \varphi(a)\varphi(b)\| \leq \xi\ \|a\|\|b\|.$$

Now suppose that $a, b \in \mathcal{A}$ be arbitrary. Take $x = a/\|a\|$ and $y = b/\|b\|$. Then $\|x\| = \|y\| = 1$, so by above argument we get

$$\|\varphi(xy) - \varphi(x)\varphi(y)\| \leq \xi\ \|x\|\|y\|.$$

Since $\|x\| = \|y\| = 1$, we deduce

$$\|\varphi(ab) - \varphi(a)\varphi(b)\| \leq \xi\ \|a\|\|b\|,$$

for all $a, b \in \mathcal{A}$. This complete the proof.

**Theorem 2.6** *Let $\varphi$ be a almost multiplicative linear functional on Banach algebra $\mathcal{A}$, and $\psi \in \mathcal{A}'$. If for all $a \in \mathcal{A}$,*

$$|\varphi(a) - \psi(a)| < \varepsilon,$$

*then $\psi$ is almost multiplicative.*

Proof. Suppose that $a, b \in \mathcal{A}$ and $\|a\| = \|b\| = 1$. Then

$$\begin{aligned} |\varphi(a)\varphi(b) - \psi(a)\psi(b)| &\leq |\varphi(b)||\varphi(a) - \psi(a)| + |\varphi(a) - \psi(a)||\varphi(b) - \psi(b)| + |\varphi(a)||\varphi(b) - \psi(b)| \\ &\leq \varepsilon(|\varphi(a)| + |\varphi(b)|) + \varepsilon^2 \\ &\leq 2\varepsilon\ \|\varphi\| + \varepsilon^2 \\ &\leq 2\varepsilon\ (1 + \xi) + \varepsilon^2. \end{aligned}$$

Take $\eta = 2\varepsilon\ (1 + \xi) + \varepsilon^2$, then

$$|\psi(ab) - \psi(a)\psi(b)| \leq |\psi(ab) - \varphi(ab)| + |\varphi(ab) - \varphi(a)\varphi(b)| + |\varphi(a)\varphi(b) - \psi(a)\psi(b)| \leq \varepsilon + \xi + \eta.$$

Hence, for all $a, b \in \mathcal{A}$ with $\|a\| = \|b\| = 1$, we have

$$|\psi(ab) - \psi(a)\psi(b)| \leq (\varepsilon + \xi + \eta)\ \|a\|\|b\|.$$

Thus $\psi$ is almost multiplicative. The result follows for arbitrary non-zero elements $a, b \in \mathcal{A}$, if we replacing $a$ by $a/\|a\|$ and $b$ by $b/\|b\|$ in above inequity.

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