Global Journal of Mathematical Analysis, 2 (3) (2014) 105-110
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www.sciencepubco.com/index.php/GJMA
doi: 10.14419/gjma.v2i3.2795
Research Paper

Discrete prey-predator model with Beddington-DeAngelis functional response: simple vs. complex dynamics

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Abstract

In this paper the dynamics of a discrete-time prey-predator system is investigated in the closed first quadrant $\mathcal{R}_+^2$. The existence and stability of fixed points are analyzed algebraically .The conditions of existence for flip bifurcation is derived by busing center manifold theorem and bifurcation theory. Numerical simulations not only illustrate our results but also exhibit complex dynamical behaviors of the model, such as the periodic-doubling bifurcation in periods 2,4 and 8 and quasi-periodic orbits and chaotic sets.

Keywords: *Discrete Model, Beddington-Deangelis Functional Response, Stability, Flip Bifurcation, Center Manifold Theorem, Numerical Simulation*

1. Introduction

It is well known the Lotka-Voltera prey-predator model is one of the fundamental population models, a predator -prey interaction has been described firstly by two pioneers Lotka [18] and Voltera [22] in two independent works. After them, more realistic prey-predator model were introduced by Holling suggesting three types of functional responses for different species to model the phenomena of predation [9]. Predator-prey models have already received much attention from many authors. For example, the stability, and the existence of periodic solutions of the predator prey models are studied in [16], [19], [20], [23]. The dynamics of a prey-predator differential equations model is studied by many authors see [6], [7], [15]. Another possible way to understand a prey-predator model is by using discrete models. Actually these models are more reasonable than the continuous time models when populations have non-overlapping generations see [14]. Discrete-time models can give rise to more efficient computational models for numerical simulations and it exhibits more plentiful dynamical behaviors than a continuous-time model of the same type. For continuous-time predator-pry models, many authors have chosen delay as the bifurcation parameter to discuss the Hopf bifurcation in [13, 21]. However, there are few articles discussing the dynamical behaviors of predator-prey models, which include bifurcations and chaos phenomena for the discrete-time models. Liu and Xiao [17], Jing and Yang [11], He and Lai [8], Hu et al. [10] obtained the flip bifurcation by using the center manifold theorem and bifurcation theory. But Agiza et al. [1] and Celik et al. [3] only showed the flip bifurcation and Hopf bifurcation by using numerical simulations. A functional response is called of Beddington-DeAngelis type if it takes the form $P(x,y) = \frac{kx}{(1+by+ax)}$. This type of functional response was introduced by Beddington [2] and DeAngelis et al. [4]. The term by measures the mutual interference between predators. In this paper we consider the following continuous-time prey-predator model with Beddington-DeAngelis type of functional response described by differential equations

$$\frac{dx_1}{dT} = rx_1\left(1 - \frac{x_1}{K}\right) - \frac{mx_1x_2}{Bx_2+hx_1+1}$$
$$\frac{dx_2}{dT} = \frac{c_1mx_1x_2}{Bx_2+hx_1+1} - d_1x_2 \quad (1)$$

The model parameters $r, K, m, B, h, c_1,$ and d_1 are assuming only positive values. The prey x_1 grows with intrinsic growth rate r and carrying capacity K in the absence of predation. The predator x_2 consumes the prey x_1 with functional response Beddington-DeAngelis type. The constant c_1 is conversion rates of prey to predator,d_1 is predator death rate for species x_2. by using the following transformation $t = rT, x_1 = Kx, x_2 = \frac{ry}{m}$ we get the following system

$$\frac{dx}{dt} = x(1-x) - \frac{xy}{1+ax+by}$$
$$\frac{dy}{dt} = \frac{cxy}{1+ax+by} - ey \quad (2)$$

Where $a = hk, b = \frac{Br}{m}, c = \frac{c_1 mK}{r}$, and $e = \frac{d_1}{r}$. Applying the forward Euler scheme to system (2), we obtain the discrete-time system as follows:

$$x \to x + dx\left[1 - x - \frac{y}{1+ax+by}\right]$$
$$y \to y + dy\left[\frac{cx}{1+ax+by} - e\right] \quad (3)$$

Where d is the step size. The main goal of this paper is to investigate this version as a discrete-time dynamical system in the interior of the first quadrant by using bifurcation theory and center manifold theory. The organization of this paper is as follows. In the second section we discuss the existence and local stability of fixed points in model (3). In the third section study the numerical simulations ,which not only illustrate our results with theoretical analysis but also exhibit complex dynamical behaviors such as the cascade periodic-doubling bifurcation inperiods 2,4 and 8 and quasi-periodic orbits and chaotic sets .In section four we give the discussion for our work.

2. The existence and stability of fixed points

In this section, we consider the discrete-time model (3) in the closed first quadrant $\mathcal{R}_+^2$on the xy-plane. We first discuss the existence of the fixed points for (3), and then study the stability of the fixed point by the eigenvalues for the variational matrix of (3) at the fixed point.

To determine the fixed points of (3) we have to solve the non- linear system given by

$$x \to x + dx\left[1 - x - \frac{y}{1+ax+by}\right]$$
$$y \to y + dy\left[\frac{cx}{1+ax+by} - e\right] \quad (4)$$

By composition of the above algebraic system we obtain the following results:

Lemma 2.1 *System (3):*

1) Has two fixed points; for all parameter values,
a) $E_0(0,0)$Is origin?
b) $E_1(1,0)$Is the axial fixed point.
2) Has the interior fixed point $\mathrm{E}_2(\mathrm{x}^*, \mathrm{y}^*)$ where

$y^* = \frac{(c-ae)x^* - e}{be}$ And $0 < \frac{e}{(c-ae)} < x^* < 1$.

Now we study the stability of these fixed points. Note that the local stability of the fixed point(x, y) is determined by the modules of eigenvalues ofthe characteristic equation at the fixed point.

The Jacobian matrix of (3) evaluated at the point (x, y) is given by

$$J(x,y) = \begin{bmatrix} j_{11} & j_{12} \\ j_{21} & j_{22} \end{bmatrix}$$

Where

$$j_{11} = 1 + dx\left(-1 + \frac{ay}{(1+ax+by)^2} + d\left(1 - x - \frac{y}{1+ax+by}\right)\right);$$
$$j_{12} = \frac{-dx(1+ax)}{(1+ax+by)^2};$$
$$j_{21} = \frac{dcy(1+by)}{(1+ax+by)^2};$$
$$j_{22} = 1 - \left(\frac{bcdxy}{(1+ax+by)^2}\right)\left(\frac{cx}{(1+ax+by)} - e\right)d$$

And the characteristic equation of the Jacobian matrix $J(x,y)$ can be writtenas

$\lambda^2 + P(x,y)\lambda + Q(x,y) = 0(5)$

Where

$$P(x,y) = -(j_{11} + j_{22})$$
$$Q(x,y) = j_{11}j_{22} - j_{12}j_{21}.$$

In order to discuss the stability of the fixed points of (3), we also need the following Lemma.

Lemma 2.2 *Let*$F(\lambda) = \lambda^2 + P\lambda + Q$. *Suppose that* $F(1) > 0, \lambda_1, \lambda_2$*are two roots of* $F(\lambda) = 0$. *Then:*

1) $|\lambda_1| < 1$ and $|\lambda_2| < 1$ if and only if $F(-1) > 0$ and $Q < 1$;
2) $|\lambda_1| < 1$ and $|\lambda_2| > 1$ (or $|\lambda_1| > 1$ and $|\lambda_2| < 1$) if and only if $F(-1) < 0$;

3) $|\lambda_1| > 1$ and $|\lambda_2| > 1$ if and only if $F(-1) > 0$ and $Q > 1$;
4) $|\lambda_1| = -1$ and $|\lambda_2| \neq 1$ if and only if $F(-1) = 0$ and $P \neq 0, 2$;
5) If λ_1 and λ_2 are complex and $|\lambda_1| = 1$ and $|\lambda_2| = 1$ if and only if $P^2 - 4Q < 0$ and $Q = 1$.

Let λ_1 and $\lambda_2$be two roots of (5). We recall some definitions of topological types for a fixed point(x, y).

Definition 2.3: A fixed point $(x, y)$is called a sink if $|\lambda_1| < 1$and$|\lambda_2| < 1$, (x, y) is called a source if $|\lambda_1| > 1$ and $|\lambda_2| > 1$, (x, y) is called a saddleif $|\lambda_1| > 1$ and $|\lambda_2| < 1$ (or $|\lambda_1| < 1$ and $|\lambda_2| > 1$), and (x, y) is called a Non-hyperbolic if either $|\lambda_1| = 1$ or$|\lambda_2| = 1$.

Proposition 2.4: The fixed point $E_0$is a source if $d > \frac{2}{e}$,E_0 is a saddle if$0 < d < \frac{2}{e}$, and E_0 is a non-hyperbolic if $d = \frac{2}{e}$.
At the fixed point $E_0$the roots of equation (5) are $\lambda_1 = 1 + d$with $d > 0$and$\lambda_2 = 1 - de$, soλ_1 is not one with model and the second eigenvalue $\lambda_2 = -1$when $d = \frac{2}{e}$. This periodic doubling bifurcation may occur where parameters vary in the neighborhood$d = \frac{2}{e}$.
The next proposition shows the local dynamics of fixed point $E_1$from Lemma (2.2).

Proposition 2.5: There are at least four different topological types of $E_1$for all permissible values of parameters
1) $E_1$is a sink if $\frac{c}{1+a} < e$and $0 < d < min\left\{2, \frac{2(1+a)}{e(1+a)-c}\right\}$;
2) $E_1$is a source if $\frac{c}{1+a} > e$ and $2 < d$ or $\left(\frac{c}{1+a} < e \text{ and } d > max\left\{2, \frac{2(1+a)}{e(1+a)-c}\right\}\right)$;
3) $E_1$is a saddle if $d < 2$ *and* $\frac{c}{1+a} > e$ and $0 < \frac{2(1+a)}{e(1+a)-c} < d < 2$ *or* $\left(2 < d < \frac{2(1+a)}{e(1+a)-c}\right)$;
4) E_1isnon-hyperbolic if either $d = 0$ or$\frac{c}{1+a} = e$or $d = \frac{2(1+a)}{e(1+a)-c}$;

From the above proposition we obtain, for a fixed point E_1 if $(a, b, c, d, e) \in \mathcal{R}$
Where
$R = \left\{(a, b, c, d, e)\middle| d = 2, e \neq \frac{c}{1+a}, d \neq \frac{2(1+a)}{e(1+a)-c}, a, b, c, d, e > 0\right\}$,
then one of the eigen values of $J(E_1)$ is -1 and the other $\lambda = 1 + \left(\frac{c}{1+a} - e\right) d$isneither 1 nor -1.Therefore, there may be flip bifurcation of a fixed point E_1, if d varies in the small neighbourhood of d = 2 and $(a, b, c, d, e) \in \mathcal{R}$.

Proposition 2.6: Let $E_2(x^*, y^*)$be the positive fixed point of (3). Then
1) E_2 Is a sink if one of the following conditions holds?
a) $P^* \geq 0$ And $0 < d < \frac{N^* - \sqrt{P^*}}{M^*}$;
b) $P^* < 0$ And $0 < d < \frac{N^*}{M^*}$;
2) *2.* E_2 *Is a source if one of the following conditions holds?*
a) $P^* \geq 0$ And $d > \frac{N^* + \sqrt{P^*}}{M^*}$;
b) $P^* < 0$ And $d > \frac{N^*}{M^*}$;
3) *3.* E_2 *Is a saddle if one of the following conditions holds?*
a) $P^* \geq 0$ And $0 < d < \frac{N^* + \sqrt{P^*}}{M^*}$;
b) $P^* \geq 0$ And $0 < \frac{N^* - \sqrt{P^*}}{M^*} < d$;
4) *4.* E_2 *is non hyperbolic if one of the following conditions holds*
a) $P^* \geq 0$ And $d = \frac{N^* \mp \sqrt{P^*}}{M^*}$;
b) $P^* < 0$ And $d = \frac{N^*}{M^*}$;

Where
$M^* = e(1 - x^*)\left(1 - \frac{ae}{c} - b + 2bx^*\right)$
$N^* = -e(1 - x^*)\left(\frac{a}{c} - b\right) + x^*$
$P^* = N^{*2} - 4M^*$.

From Lemma (2.2), we can see that one of the eigen values of E_2 is -1 and the other is neither 1 nor -1if the term (4a) in proposition (2.6) holds. Therefore there may be flip bifurcation of E_2 if d varies in the small neighborhood of$\mathcal{B}_1^*$ or $\mathcal{B}_2^*$ where
$\mathcal{B}_1^* = \left\{(a, b, c, d, e)\middle| d = \frac{N^* - \sqrt{P^*}}{M^*}, P^* > 0, a > b, b, c, d, e > 0\right\}$
$\mathcal{B}_1^* = \left\{(a, b, c, d, e)\middle| d = \frac{N^* + \sqrt{P^*}}{M^*}, P^* > 0, a, b, c, d, e > 0\right\}$

Also when the term (4b) in proposition (2.6) holds, we can obtain that the eigenvalues of E_2 are pair of conjugate complex numbers with $|\lambda_1| = 1, |\lambda_2| = 1$and hence the conditions in term (4b) of proposition (2.6) can be written as
$\mathcal{C} = \left\{(a,b,c,d,e)\middle| d = \frac{N^*}{M^*}, P^* < 0, a,b,c,d,e > 0\right\}$
If the parameter d varies in the small neighborhood of $\mathcal{C}$, then the Hopf bifurcation will appear.

3. Numerical simulations

In this section, we give the bifurcation diagrams, phase portraits of model (3) to confirm the above theoretical analysis and show the dynamical behaviors by using numerical simulations. The bifurcation parameters are considered in the following three cases:
Case 1: Choosing a = 0.3, b = 0.5, c = 0.1, and e = 0.2, initial value$(x_0, y_0) = (0.5, 0.6)$and $d \in [0, 2.96]$. We see that model (3) has a fixed point $E_1(1,0)$ and$(a,b,c,d) \in R$. Figure1 show the correctness of proposition (2.5). From Figure 1 we see that the fixed point $E_1(1,0)$ is stable for *d*< 2, and loses its stability when *d* = 2. Further, when *d* > 2there is the period- doubling bifurcation. Moreover a chaotic set is emerged with increasing of *d*.

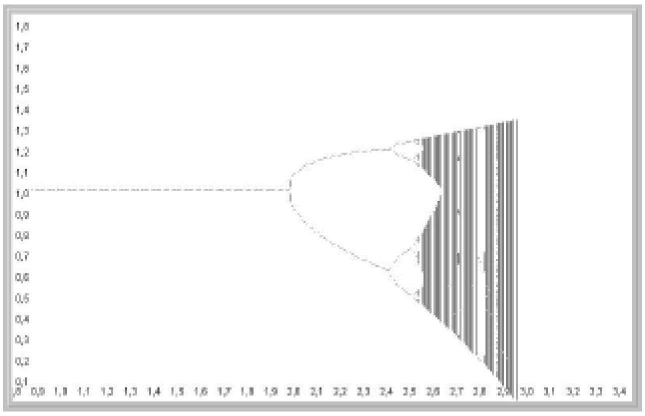

Fig. 1: The Flip Bifurcation of *X* (*N*) With A=0.3, B=0.5, C=0.1, And E=0.2,$d \in [0, 2.96]$ And Initial$(x_0, y_0) = (0.5, 0.6)$.

Case 2: Choosing a = 1.3, b = 3, c = 2, and e = 0.4, initial value $(x_0, y_0) = (0.4, 0.6)$and $d \in [1.5, 3.5]$ and we only vary parameter *d* to see the variation of dynamics behaviors about model (3), on the basis of proposition (2.6), we know that the system has only one positive fixed point. After calculation for the positive fixed point of map (3), the flip bifurcation emerges from the fixed point (5/6, 25/36) at *d* = 2.6855 and (a, b, c, d, e) = (1.3, 3, 2, 2.6855, 0.4) ∈ $\mathcal{B}_1^*$. It shows the correctness of proposition (2.6). From Figure 2 we see that the fixed point is stable for *d*< 2.6855, and loses its stability when *d* = 2.6855. We also observe that there is a cascade of period-doubling bifurcation.

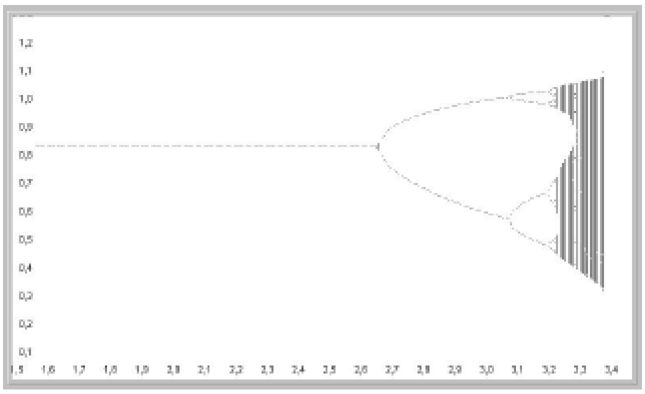

Fig. 2: The Bifurcation Diagram of Model (3) In The (*D*, *X*)-Plane For A = 1.3, B=3, C=2, And E = 0.4,$d \in [1.5, 3.5]$And Initial $(x_0, y_0) = (0.4, 0.6)$.

Case 3: Choosing a=1, b = 0.5, c = 4, and e = 9/10, initial value $(x_0, y_0) = (0.4, 0.9)$ and $d \in [1, 2.2]$according proposition (2.6), we know that the system has only one positive fixed point. After calculation of the positive fixed point of the system, the flip bifurcation emerges from the fixed point (9/20, 11/10) at d =1.598 and $(a,b,c,d,e) = (1, 0.5, 4, 1.598, 0.9) \in \boldsymbol{\mathcal{C}}$. It shows the correctness of proposition (2.6).

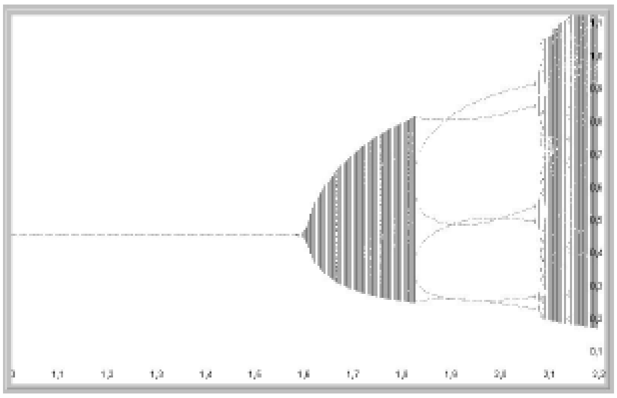

Fig. 3: The Bifurcation Diagram of the System with $d \in [1, 2.2]$And $(x_0, y_0) = (0.4, 0.9)$.

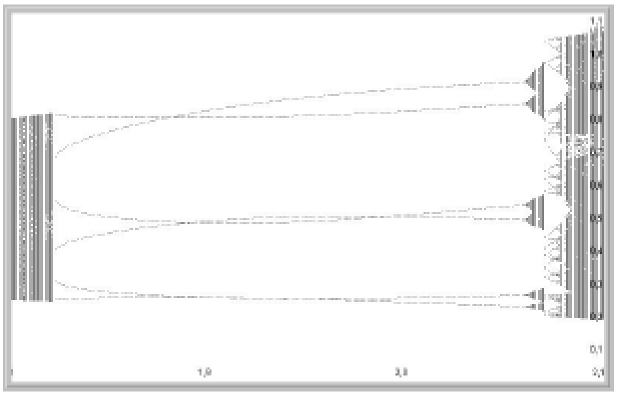

Fig. 4: Local Amplification Corresponding to Fig.4

From Figure 3 we observe that the fixed point (9/20, 11/10) is stable for d<1.598, that it loses its stability when d=1.598 and that an invariant circle appears when the parameter d exceeds 1.598. Figure 4 is local amplification or $d \in [1.82, 2.1]$.

4. Conclusions

In this paper we conclude that the discrete-time prey-predator model (3) has complex dynamics and we show that the unique positive fixed point of model (3) can undergo flip bifurcation and Hopf bifurcation. Moreover system (3) displays much interesting dynamical behaviors, including cascade of periodic-doubling, quasi-periodic orbits and the chaotic sets. These results reveal far richer dynamics of the discrete model compared to the continuous model.

References

[1] H.N. Agiza, E.M. Elabbasy, H. El-Metwally, A.A. Elsadany, Chaotic dynamics of a discrete prey-predator model with holling type II,NonliearAnal. 10 (2009) 116-129.

[2] J.R. Beddington, Mutual interference between parasites or predators andits effect on searching e_ciency,J. Animal Ecol. 44 (1975) 331-340.

[3] C. Celik, O. Duman, Allee effect in a discrete-time predator-prey system,Chaos Solitons Fractals 40 (2009) 1956-1962.

[4] D.L. DeAngelis, R.A. Goldstein, R.V. O'Neill, A model for trophic interaction,Ecology, 56 (1975) 881-892.

[5] J. Guckenheimer, P. Holmes, Nonlinear Oscillations, dynamical systemsand bifurcation of vector fields, Springer- Verlag, New York, (1983).

[6] Hainzl J. 1988. Stability and Hopf bifurcation a predator-prey system with several parameters. SIAM Journal on Applied Mathematics, 48: 170-180 .

[7] He X. 1996.Stability and delays in a predator-prey system. Journal of Mathematical Analysis and Applications, 198: 355-370.

[8] Z.He, X.Lai , Bifurcation and chaos behavior of a discrete-time predator-prey system,Nonliear Anal. 12 (2011) 403-417.

[9] C.S. Holling, The functional response of predator to prey density and itsrole in mimicry and population regulation,Mem. Ent. Soc. Canada, 45(1965) 1-60.

[10] Z. Hu, Z.Teng, L. Zhang, Stability and bifurcation analysis of a discretepredator-prey model with nonmonotonic functional response,NonliearAnal. 12 (2011) 2356-2377.

[11] Z. Jing, J. Yang, Bifurcation and chaos in discrete-time predator-prey sys-tem, Chaos Solitons Fractals 27 (2006) 259-277.

[12] T.K. Kar, A. Batabyal, Stability and bifurcation of a prey-predator modelwith time delay, C. R. Biol. 332 (2009) 642-651.

[13] F. Lian, Y. Xu, Hopf bifurcation analysis of a predator-prey system withHolling IV functional response and time delay,Appl. Math. Comput. 215(2009) 1484-1495.

[14] R. M. May, 1975. Biological populations obeying difference equations: stable points, stable cycles and chaos. Journal of Theoretical Biology, 51(2): 511-524 .

[15] Murray JD. 1998. Mathematical Biology. Springer-Verlag, Berlin, Germany.

[16] X. Liao, S. Zhou, Y. Chen, On permanence and global stability in a general Gilpin-Ayala competition predator-prey discrete system,Appl. Math.Comput. 190 (2007) 500-509.
[17] X. Liu, D. Xiao, Complex dynamic behaviors of a discrete-time predator-prey system,Chaos Solitons Fractals 32 (2007) 80-94.
[18] A.J. Lotka, Elements of Mathematical Biology, Dover, New York, 1962.
[19] R.Naji , A. Balasim, Dynamical behavior of a three species food chainmodel with Beddington-DeAngelis functional response,Chaos SolitonsFractals, 32 (2007) 1853-1866.
[20] R. Redheffer, Nonautonomous Lotka-Volterra systems, I,J. DifferentiaEquations 127 (1996) 519-541.
[21] Y. Song, S. Yuan, J. Zhang, Bifurcation analysis in the delayed Leslie-Gower predator-prey system,Appl. Math. Model. 33 (2009) 4049-4061.
[22] V. Voltera, Operematematiche: mmemorie e note, vol. V, Acc. Naz. DeiLincei, Roma, Cremon, 1962.
[23] S. Zhang , D Tan , L. Chen, Dynamic complexities of a food chain modelwith impulsive perturbations and Beddington-DeAngelis functional response, Chaos, Solitons and Fractals, 27 (2006) 768-777