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Research Paper

The coupling Navier-stokes/Darcy systems by vorticity-velocity-pressure formulation

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Abstract

In this paper We consider the Coupling Navier-Stokes/Darcy equations in a two or three dimensional domain provided with non standard boundary conditions which involve the normal component of the velocity and the tangential components of the vorticity. We establish a coupled variational formulation of this problem with three independent unknowns: the vorticity, the velocity and the pressure. We discuss coupling conditions and we analyze the global coupled model in order to prove its well-posedness and to characterize effective algorithms to compute the solution of its numerical approximation.

Keywords: *coupled problem, Darcy's equation, Navier-Stokes equation, Finite element.*

1 Introduction

There is an increasing interest in coupling incompressible flow and porous media flow. Applications of such complex phenomena can be found in geosciences (modeling of the interaction of rivers with ground water) and in health sciences (modeling of blood flow and organs). In this work, we consider a domain which is governed by the stationary Navier Stokes system on one part of the domain and by a second order elliptic equation derived form Darcy's law in the rest of the domain, and where the solutions in the two domain are coupled by proper interface conditions. Then we study the vorticity-velocity-pressure Formulation. We discuss some new finite element methods.

A weak solution for the coupled problem is analyzed and is approximated by totally discontinuous elements in [12]. In [13] the coupling of the Navier-Stokes equation with nonhomogeneous boundary conditions is analyzed by using an implicit function theorem. For the Vorticity-velocity-pressure formulation of the Stokes problem and the Navier Stokes problem, we refer to [7], [2] and [5]. The formulation of the problem as an interface equation, and the analyz of the associated (nonlinear) Steklov-Poincaré operators is well presented in [4].
An outline of the paper is as follows:
In Section 2, we present the differential model introducing the Navier-Stokes equations for the fluid and Darcy equations for the porous media. The coupling conditions on the interface between the two domain are well explained.

In section 3 we write the variational formulation of the problem and prove existence and uniqueness results.

In section 4 we introduce the finite elements and a fully discrete system using the curl conforming finite elements for the velocity and the standard continuous elements for the pressure.

Section 5 is devoted to numerical experiments wich confirm the theoretical results.

Consider a fluid occupying a bounded domain $\Omega \in \mathbf{R}^n, (n = 2, 3)$, that is subdivided into two disjoint sub-domains Ω_1 and Ω_2. Let Γ_{12} denote the interface between the sub-domains: $\Gamma_{12} = \partial\Omega_1 \cap \partial\Omega_2$. The fluid motion is modelled by the Navier-Stokes in Ω_1 and the Darcy equations in Ω_2.

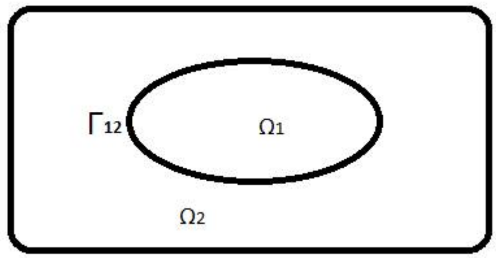

Figure 1: The Presentation of the geometry

We consider the following system of equations:

$$(P)\begin{cases} -\nu\Delta u + u.\nabla u + \nabla p & = f \quad \text{in } \Omega_1 \\ \mu u + \nabla p & = f \quad \text{in } \Omega_2 \\ \text{div} u & = 0 \quad \text{in } \Omega_1 \cup \Omega_2 \\ u.n & = 0 \quad \text{on } \partial\Omega \\ (u\mid_{\Omega_1} - u\mid_{\Omega_2}).n & = 0 \quad \text{in } \Gamma_{12} \\ p\mid_{\Omega_1} - p\mid_{\Omega_2} & = 0 \quad \text{in } \Gamma_{12} \\ \text{curl} u\mid_{\Omega_1} \times n & = 0 \quad \text{in } \Gamma_{12} \end{cases}$$

where u is the velocity, p the pressure, f the density of body forces and μ and ν positive constants.

2 The velocity-Vorticity-pressure formulation of the problem

In this section, we consider the Navier-Stokes equations in a two- or three-dimensional domain provided with a standard boundary conditions.
Using the vorticity $w = curlu$ as new dependent variable, we obtain the first-order velocity-vorticity-pressure formulation of the Navier-Stokes equations in Ω_1.
As usual, $curlu$ denotes the vorticity, namely the vector function:

$$(curlu) : \begin{cases} (\frac{\partial u_3}{\partial x_2} - \frac{\partial u_2}{\partial x_3}, \frac{\partial u_1}{\partial x_3} - \frac{\partial u_3}{\partial x_1}, \frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2}) & \text{if } n = 3 \\ (\frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2}) & \text{if } n = 2 \end{cases}.$$

We introduce the unit outward normal vector $\mathbf{n}$ to Ω on Γ and we consider the system of Navier Stokes equations. We write a variational formulation of this problem with three independent unknowns: the vorticity, the velocity and the pressure, and prove the existence of a solution for this problem.
The basic idea in [7] consists in introducing a new variable that is the vorticity $w = curlu$ as a new unknown and to obtain a new formulation of our problem.

Then, it can be noted that non linear term of the convection $(u.\nabla)u$ can be written as:

$$(u.\nabla)u = \frac{1}{2}\nabla|u|^2 + curlu \wedge u. \tag{1}$$

Also known as $curl\ curlu = -\Delta u + grad\ divu$. Thus, we define a pseudo-pressure $\mathbf{p}$ (usually called the dynamical pressure) by the formula $\mathbf{p} = p + \frac{1}{2}|u|^2$.

We consider now the issue of finding effective coupling conditions across the interface Γ_{12} which separates the fluid flow and the porous medium. This is a classical problem which has been investigated from both a physical and a rigorous mathematical point of view.

Adding the new unknown the system of equations (P) is fully equivalent to:

$$(Q): \begin{cases} \nu\ curlw + w \wedge u + \nabla\mathbf{p} & = f_1 \quad \text{in } \Omega_1 \\ \mu u + \nabla\mathbf{p} & = f_2 \quad \text{in } \Omega_2 \\ \text{div}u & = 0 \quad \text{in } \Omega_1 \cup \Omega_2 \\ curlu & = w \quad \text{in } \Omega_1 \\ u.n & = 0 \quad \text{in } \partial\Omega \\ (u\mid_{\Omega_1} - u\mid_{\Omega_2}).n & = 0 \quad \text{on } \Gamma_{12} \\ p\mid_{\Omega_1} - p\mid_{\Omega_2} & = 0 \quad \text{on } \Gamma_{12} \\ \gamma_t(w) & = 0 \quad \text{on } \Gamma_{12} \end{cases}$$

In the following, n_1 and n_2 denote the unit outward normal vectors to the surfaces $\partial\Omega_1$ and $\partial\Omega_2$, respectively, and we have $n_1 = -n_2$ on $\partial\Omega$.

We recall that the trace operator: $v \to v.n$ is continuous from $H(\text{div}, \Omega)$ to $H^{-\frac{1}{2}}(\partial\Omega)$
and the jump $(v\mid_{\Omega_1} - v\mid_{\Omega_2}).n$ vanishes on Γ_{12}.
To make precise the sense of the operator, γ_t we recall that it is the trace operator and the tangential trace operator on $\partial\Omega$, defined by $\gamma_t(w) = w \times n$ in dimention $n = 2$ and $n = 3$ respectively.

In all the paper, we suppose that f_1 and f_2 the density of body forces in Ω_1 and Ω_2 respectively are in $L^2(\Omega)^3$.In fact, f_1 has a value in Ω_1 and vanish in the rest of the domain, and f_2 has a value in Ω_2 and vanish in the rest of the domain.

In order to write the variational formulation of the previous problem, we introduce the following spaces:

$W^{m,p}(\Omega) = \{v \in L^p(\Omega)^3, \partial^\alpha v \in L^p(\Omega)^3, \forall \mid \alpha \mid \leq m\}$

$H^m(\Omega) = W^{m,2}(\Omega)$

As usual, we shall omit p when $p = 2$ and denote by $(.,.)$ the scalar product of $L^2(\Omega)$. Also, recall the familiar notation:

$H^1_0(\Omega) = \{v \in H^1(\Omega); v = 0 \text{ on } \partial\Omega\}$

with the Poincaré inequality

$\forall v \in H^1_0(\Omega), \| v \|_{0,\Omega} \leq c \mid v \mid_{0,\Omega}$

We consider the domain $H(\text{div}, \Omega)$ of the divergence operator, namely:

$H(\text{div}, \Omega) = \{v \in L^2(\Omega)^3, \text{div}v \in L^2(\Omega)\}$

and also its subspace

$H_0(\text{div}, \Omega) = \{v \in H(\text{div}, \Omega), v.n = 0 \text{ on } \partial\Omega\}$

Where (.) denotes the duality pairing between $H_0(div,\Omega)$ and it's dual.
We introduce the operator space **curl**

$$H(\mathbf{curl},\Omega)=\left\{\vartheta\ \in (L^2(\Omega))^{\frac{n(n-1)}{2}};\ \ \mathbf{curl}\,\vartheta\in L^2(\Omega)^n\right\}. \tag{2}$$

And the subspace of $H(\mathbf{curl},\Omega)$

$$H_0(\mathbf{curl},\Omega)=\{\vartheta\in H(\mathbf{curl},\Omega);\gamma_t(\vartheta)=0\ \text{ in }\ \partial\Omega\}. \tag{3}$$

Remark

- In dimension $n=2$, the space $H(\mathbf{curl},\Omega)$ coincides with the space $H^1(\Omega)$ and the space $H_0(\mathbf{curl},\Omega)$ coincides with the space $H_0^1(\Omega)$.
- In dimension $n=3$, $H^1(\Omega)\subset H(\mathbf{curl},\Omega)$ and $H_0^1(\Omega)\subset H(\mathbf{curl},\Omega)$.

3 Variational formulation

Considering $X=\{v\in(L^2(\Omega))^3, curlv_{|\Omega_1}\in(L^2(\Omega_1))^3\}$, and then we obtain the following weak variational formulation, denoted by (V):

$$(\mathrm{V})\left\{\begin{array}{l}\text{Find } u\in X \text{ and } \mathbf{p}\in H^1(\Omega)/\mathbf{R} \text{ such that :}\\ \mu(u,v)_{\Omega_2}+\nu(curlu,curlv)_{\Omega_1}+(curlu\wedge u,v)_{\Omega_1}+(\nabla\mathbf{p},v)_\Omega=(f_1+f_2\ ,\ v)_\Omega\ \forall v\in X\\ (\nabla\mathbf{q},u)_\Omega=0\quad\ \mathbf{q}\ \in\ H^1(\Omega)/\mathbf{R}\end{array}\right.$$

For simplicity we denote by $\mathbf{f}=f_1+f_2$ such that f_1 has a value in Ω_1 and vanish in the rest of the domain, and f_2 has a value in Ω_2 and vanish in the rest of the domain, so we have:

$$\mathbf{f}=\left\{\begin{array}{l}f_1\ \ in\ \ \Omega_1\\ f_2\ \ in\ \ \Omega_2\end{array}\right.$$

We introduce the problem, establish a decoupled variational formulation and prove its wellposedness. Then we consider the following system of equations:

$$\left\{\begin{array}{l}\text{Find } (\boldsymbol{w},\boldsymbol{u},p)\in H_0(curl,\Omega)\ \times\ H_0(div,\Omega)\ \times\ L_0^2(\Omega) \text{ such that :}\\ \forall\ \boldsymbol{v}\ \in\ H_0(div,\Omega),\ a(\boldsymbol{w},\boldsymbol{u},\boldsymbol{v})+h(\boldsymbol{w},\boldsymbol{u},\boldsymbol{v})+b(\boldsymbol{v},p)=(\mathbf{f}\ ,\ \boldsymbol{v})_\Omega\\ \forall\ q\ \in\ L_0^2(\Omega),\ b(\boldsymbol{u},q)=0\\ \forall\ \vartheta\ \in\ H_0(\mathbf{curl},\Omega),\ c(\boldsymbol{w},\boldsymbol{u},\vartheta)=0.\end{array}\right.$$

The bilinear forms are $a(.,.,.), b(.,.)$ and $c(.,.,.)$ are defined by :

$$a(\boldsymbol{w},\boldsymbol{u},\boldsymbol{v})=\mu\int_{\Omega_2}\mathbf{u}(x).\boldsymbol{v}(x)dx+\nu\int_{\Omega_1}(\mathbf{curl}\boldsymbol{w})(x)\boldsymbol{v}(x)dx \tag{4}$$

$$b(\boldsymbol{v},q)=\int_\Omega(div\boldsymbol{v})(x)q(x)dx \tag{5}$$

$$c(\boldsymbol{w},\boldsymbol{u},\varphi)=\int_\Omega(\boldsymbol{w})(x)\varphi(x)dx-\int_\Omega\boldsymbol{u}(x).(\mathbf{curl}\varphi)(x)dx \tag{6}$$

And the trilinear form $h(.,.,.)$ is defined by:

$$h(\boldsymbol{w},\boldsymbol{u},\boldsymbol{v})=\int_\Omega(\boldsymbol{w}(x)\wedge\boldsymbol{u}(x))\boldsymbol{v}(x)dx \tag{7}$$

We need the next properties which are more developed an demonstrated in [1]:

1. The first property is based on the characterisation of the domain, in fact if Ω is simply-connected there exist a constant c_0 such that:

$$\|\mathbf{v}\|_{L^2(\Omega)^n} \leq c_0 \|curlv\|_{L^2(\Omega)^{\frac{n(n-1)}{2}}} \tag{8}$$

2. The positivity and inf-sup condition for the form $a(.,.,.)$:
There exists a positive constant α, such that the forme $a(.,.,.)$ satisfies

$$\begin{aligned}
\forall\, \mathbf{v} \in \mathrm{V} \setminus 0 \quad \sup_{(\mathbf{w},\mathbf{u}) \in \mathcal{W}} a(\mathbf{w}, \mathbf{u}, \mathbf{v}) &> 0 \\
\forall (\mathbf{w}, \mathbf{u}) \in \mathcal{W} \quad \sup_{\mathbf{v} \in \mathrm{V}} \frac{a(\mathbf{w}, \mathbf{u}, \mathbf{v})}{\|\mathbf{v}\|_{L^2(\Omega)^n}} &\geq \alpha(\|\mathbf{w}\|_{H(\mathbf{curl},\Omega)} + \|\mathbf{v}\|_{L^2(\Omega)^n}).
\end{aligned}$$

where $\mathcal{W} = H_0(\mathbf{curl}, \Omega) \times H_0(div, \Omega)$.
In fact:

$$\begin{aligned}
a(\boldsymbol{w}, \boldsymbol{u}, \boldsymbol{v}) &= \mu \int_{\Omega_2} \mathbf{u}(x).\boldsymbol{v}(x)dx + \nu \int_{\Omega_1} (\mathbf{curl}\boldsymbol{w})(x)\boldsymbol{v}(x)dx \\
&= \mathbf{I}_{\Omega_2}(\boldsymbol{w}, \boldsymbol{u}, \boldsymbol{v}) + \mathbf{I}_{\Omega_1}(\boldsymbol{w}, \boldsymbol{u}, \boldsymbol{v})
\end{aligned}$$

Where

$$\begin{aligned}
\mathbf{I}_{\Omega_1}(\boldsymbol{w}, \boldsymbol{u}, \boldsymbol{v}) &= \nu \int_{\Omega_1} (\mathbf{curl}\boldsymbol{w})(x)\boldsymbol{v}(x)dx \\
&= \nu \int_{\Omega_1} (\mathbf{curl}\ \boldsymbol{w})(x)((\mathbf{curl}\ \boldsymbol{w})(x) + \boldsymbol{u}(x))dx. \\
&= \nu \int_{\Omega_1} (\mathbf{curl}\ \boldsymbol{w})(x)^2 dx + \nu \int_{\Omega_1} (\mathbf{curl}\ \boldsymbol{w})(x)\boldsymbol{u}(\mathbf{x})dx \\
&= \nu \|\mathbf{curl}\ \boldsymbol{w}\|^2_{L^2(\Omega_1)^n} + \nu \int_{\Omega_1} (\mathbf{curl}\ \boldsymbol{w})(x)\boldsymbol{u}(x)dx.
\end{aligned}$$

Which gives:

$$\begin{aligned}
\mathbf{I}_{\Omega_1}(\boldsymbol{w}, \boldsymbol{u}, \boldsymbol{v}) = \nu \|\mathbf{curl}\ \boldsymbol{w}\|^2_{L^2(\Omega_1)^n} &+ \frac{\nu}{2} \int_{\Omega_1} (\mathbf{curl}\ \boldsymbol{w})(x)\boldsymbol{u}(x)dx \\
&+ \frac{\nu}{2} \int_{\Omega_1} (\mathbf{curl}\ \boldsymbol{w})(x)\boldsymbol{u}(x)dx.
\end{aligned}$$

By using Green's formula in a domain Ω_1:

$$\int_{\Omega_1} (\mathbf{curl}\ \boldsymbol{w})(x)\boldsymbol{u}(x)dx = \int_{\Omega_1} (\mathbf{curl}\boldsymbol{u})(x)\boldsymbol{w}(x)dx - \int_{\partial\Omega_1} \gamma_t(\boldsymbol{w})\tilde{\gamma}(\boldsymbol{u})(\tau)d\tau.$$

with the boundary condition on $\partial\Omega_1$.

$$\int_{\partial\Omega_1} \gamma_t(\boldsymbol{w})\tilde{\gamma}(\boldsymbol{u})(\tau)d\tau = 0 \ \ because \ \ \boldsymbol{w} \ \in H_0(\mathbf{curl}, \Omega_1).$$

And with condition $\boldsymbol{w} = \mathbf{curl}\ \boldsymbol{u}$, we obtain

$$\begin{aligned}
\int_{\Omega_1} (\mathbf{curl}\ \boldsymbol{u})(x)\boldsymbol{w}(\mathbf{x})dx &= \int_{\Omega_1} (\mathbf{curl}\ \boldsymbol{u})(x)(\mathbf{curl}\ \boldsymbol{u})(x)dx \\
&= \|\mathbf{curl}\ \boldsymbol{u}\|^2_{L^2(\Omega_1)^{\frac{n(n-1)}{2}}}.
\end{aligned}$$

The form $\mathbf{I}_{\Omega_1}$ becomes:

$$\mathbf{I}_{\Omega_1}(\boldsymbol{w},\boldsymbol{u},\boldsymbol{v}) = \frac{\nu}{2}\|\mathbf{curl}\ \boldsymbol{u}\|^2_{L^2(\Omega_1)^{\frac{n(n-1)}{2}}} + \frac{\nu}{2}\|\boldsymbol{w}\|^2_{L^2(\Omega_1)^{\frac{n(n-1)}{2}}} + \nu\|\mathbf{curl}\ \boldsymbol{w}\|^2_{L^2(\Omega_1)^n}.$$

If Ω is simply connexe [1], we have using (9):

$\forall \boldsymbol{u} \in \mathrm{V}, \quad \|\boldsymbol{u}\|_{L^2(\Omega_1)^n} \leq c_0\|\mathbf{curl}\ \boldsymbol{u}\|^2_{L^2(\Omega_1)^n}.$

For all $\boldsymbol{u} \in \mathrm{V}$, we have:

$$\mathbf{I}_{\Omega_1}(\boldsymbol{w},\boldsymbol{u},\boldsymbol{v}) \geq \frac{\nu}{2}(\|w\|^2_{L^2(\Omega_1)^{\frac{n(n-1)}{2}}} + \|\mathbf{curl}\ \boldsymbol{w}\|^2_{L^2(\Omega_1)^n}) + \frac{\nu}{2c_0^2}\|\boldsymbol{u}\|^2_{L^2(\Omega_1)^n}.$$

and then

$$\mathbf{I}_{\Omega_1}(\boldsymbol{w},\boldsymbol{u},\boldsymbol{v}) \ \geq \ \frac{\nu}{2}\ \|\boldsymbol{w}\|^2_{L^2(\Omega_1)^{\frac{n(n-1)}{2}}} + \frac{\nu}{2c_0^2}\|\boldsymbol{u}\|^2_{L^2(\Omega_1)^n}. \tag{9}$$

In another part we have:

$$\begin{aligned}
\mathbf{I}_{\Omega_2}(\boldsymbol{w},\boldsymbol{u},\boldsymbol{v}) &= \mu\int_{\Omega_2}\mathbf{u}(x).\boldsymbol{v}(x)dx \\
&= \mu\int_{\Omega_2}\mathbf{u}(x).((\mathbf{curl}\ \boldsymbol{w})(x) + \boldsymbol{u}(x))dx \\
&= \mu\int_{\Omega_2}(\ \boldsymbol{u})^2(x)dx + \mu\int_{\Omega_2}(\mathbf{curl}\ \boldsymbol{w})(x)\boldsymbol{u}(\mathbf{x})dx \\
&= \mu.(\|u\|^2_{L^2(\Omega_2)^n} + \|curl\boldsymbol{u}\|^2_{L^2(\Omega_2)^{\frac{n(n-1)}{2}}})
\end{aligned}$$

And then using (9) we obtain the inequality:

$$\mathbf{I}_{\Omega_2}(\boldsymbol{w},\boldsymbol{u},\boldsymbol{v}) \ \geq (\mu + \frac{\mu}{c_0^2}).\|u\|^2_{L^2(\Omega_2)^n} \tag{10}$$

The inequalities (10) and (11) give:

$$a(\boldsymbol{w},\boldsymbol{u},\boldsymbol{v}) \ \geq \ \frac{\nu}{2}\ \|\boldsymbol{w}\|^2_{L^2(\Omega_1)^{\frac{n(n-1)}{2}}} + \frac{\nu}{2c^2}\|\boldsymbol{u}\|^2_{L^2(\Omega_1)^n} + (\mu + \frac{\mu}{c_0^2}).\|u\|^2_{L^2(\Omega_2)^n}. \tag{11}$$

In the other part we have:

$$\begin{aligned}
\|\boldsymbol{v}\|^2_{L^2(\Omega)^n} &= \|\mathbf{curl}\ \boldsymbol{w} + \boldsymbol{u}\|^2_{L^2(\Omega)^n} \\
&\leq (\|\boldsymbol{u}\|_{L^2(\Omega)^n} + \|\mathbf{curl}\ \boldsymbol{w}\|_{L^2(\Omega)^n})^2 \\
&\leq 2(\|\boldsymbol{u}\|^2_{L^2(\Omega)^n} + \|\mathbf{curl}\ \boldsymbol{w}\|^2_{L^2(\Omega)^n}).
\end{aligned}$$

and

$$\frac{a(\boldsymbol{w},\boldsymbol{u},\boldsymbol{v})}{\|\boldsymbol{v}\|_{L^2(\Omega)^n}} \geq \frac{\nu}{2\sqrt{2}}\frac{\|\boldsymbol{w}\|^2_{H(\mathbf{curl},\Omega)} + \frac{1}{c_0^2}\|\boldsymbol{u}\|^2_{L^2(\Omega)^n}}{\|\mathbf{curl}\ \boldsymbol{w}\|^2_{L^2(\Omega)^n} + \|u\|_{L^2(\Omega)^n}}.$$

If we take the condition $0 < c_0 \leq 1$ we obtain:

$$\begin{aligned}
\frac{a(\boldsymbol{w},\boldsymbol{u},\boldsymbol{v})}{\|\boldsymbol{v}\|_{L^2(\Omega)^n}} &\geq \frac{\nu}{2\sqrt{2}}\frac{\|\boldsymbol{w}\|^2_{H(\mathbf{curl},\Omega)} + \|\boldsymbol{u}\|^2_{L^2(\Omega)^n}}{(\|\boldsymbol{w}\|^2_{H(\mathbf{curl},\Omega)} + \|\boldsymbol{u}\|^2_{L^2(\Omega)^n})^{\frac{1}{2}}} \\
&= \frac{\nu}{2\sqrt{2}}(\|\boldsymbol{w}\|^2_{H(\mathbf{curl},\Omega)} + \|\boldsymbol{u}\|^2_{L^2(\Omega)^n})^{\frac{1}{2}} \\
&= \frac{\nu}{2\sqrt{2}}(\|\boldsymbol{w}\|_{H(\mathbf{curl},\Omega)} + \|\boldsymbol{u}\|_{L^2(\Omega)^n}).
\end{aligned}$$

and

$$\sup_{\boldsymbol{v}\in \mathrm{V}}\frac{a(\boldsymbol{w},\boldsymbol{u},\boldsymbol{v})}{\|\boldsymbol{v}\|_{L^2(\Omega)^n}} \geq \alpha(\|\boldsymbol{w}\|_{H(\mathbf{curl},\Omega)}) + \|\boldsymbol{u}\|_{L^2(\Omega)^n}).$$

3. The inf-sup condition for the form $b(.,.)$:
   There exist a positive constant β such that the forme $b(.,.)$ satisfies:

$$\forall q \in L_0^2(\Omega)^n \quad \sup_{\boldsymbol{v}\in H_0(div,\Omega)} \frac{\mathbf{b}(\mathbf{v},q)}{\|\mathbf{v}\|_{H(div,\Omega)}} \geq \beta\|q\|_{L^2(\Omega)}. \tag{12}$$

4. The continuity of the non linear term $h(w,u;v)$ is more presented in [13], [4].We need some Sobolev imbedding theorem and from the Holder's inequality we have:

$$h(w,u;v) \leq c\|w\|_{{L^p(\Omega)}^{\frac{n(n-1)}{2}}}\|\mathbf{u}\|_{L^q(\Omega)^n}\|\mathbf{v}\|_{L^2(\Omega)^n} \tag{13}$$

   with
   $p=6$ and $q=3$ in dimension $n=2$.
   $p=q=4$ in dimension $n=3$.

Theorem 3.1 *For all $\mathbf{f}\in H_0(\mathrm{div},\Omega)$, the problem (V) has an unique solution $(\boldsymbol{w},\boldsymbol{u},p)\in H_0(\boldsymbol{curl},\Omega)\times H_0(\mathrm{div},\Omega)\times L_0^2(\Omega)$, this solution satisfy the next inequality:*

$$\|\boldsymbol{w}\|_{H(\boldsymbol{curl},\Omega)}+\|\boldsymbol{u}\|_{H(div,\Omega)}+\|p\|_{L^2(\Omega)}\leq c_0\|\boldsymbol{f}\|_{H_0(div,\Omega)}. \tag{14}$$

Theorem 3.2 *Problem (Q) and (P) are equivalent in the sence that any $(\boldsymbol{w},\boldsymbol{u},\boldsymbol{p})\in H(\boldsymbol{curl},\Omega)\times H(\mathrm{div},\Omega)\times L_0^2(\Omega)$ such that $(w\wedge u)$ belongs to $L^2(\Omega)$ is a solution of problem (Q) if and only if it is a solution of (P).*

4 Finite element discretization

In what follows and for simplicity, we make the further assumption that both Ω and Ω_F are polyhedra. We introduce a regular family of triangulation $(\tau_h)_h$ in the sense that:

- for each h, $\bar{\Omega}$ is the union of all elements of τ_h;
- for each h, the intersection of two different elements of τ_h, if not empty, is a corner, a whole edge or a whole face of both of them;
- the ratio of the diameter h_k of an element k in τ_h to the diameter of its inscribed sphere is bounded by a constant independent of k and h (As usual, h denotes the maximum of the diameters of the elements of τ_h);

h denotes the maximum of the diameters of the elements of τ_h. We denote by ${\tau_h}^1$ ($resp\ {\tau_h}^2$) the set of elements k of τ_h which are contained in Ω_1 ($resp\ \Omega_2$).

We introduce the spaces:

- $\mathbf{P}_0(k)$ space of the restrictions to k of constant functions on $\mathbf{R}^3$
- $\mathbf{P}_1(k)$ space of the restrictions to k of affine functions on $\mathbf{R}$ and the space $\mathbf{P}_K(k)$ of the restrictions to k of polynomials v of the form:
$$v(x)=a+b\wedge x,\ \ a\in\mathbf{R}^3, b\in\mathbf{R}^3$$

The space $\mathbf{P}_K(k)$ and the corresponding finite elements are studied in [9]. Their degrees of freedom are the average flux along the edges $\int_l(v.t)dl$ for the six edges l of k, t is the direction vector of l. Hence, we associate the operator r_k where $r_k(u)$ is the unique polynomial of $\mathbf{P}_K$ that has the same flux along the edges as u. We define also the operator I_k where $I_k(q)$ is the unique polynomial of $\mathbf{P}_1$ that has the same values on the vertex of k as q.
We have for all $k\in\tau_h$:

$$r_k(\nabla q)=\nabla I_k(q), \quad \forall q\in W^{2,t}(k), \quad \forall t>2. \tag{15}$$

Next, let us introduce the discrete spaces:

$$X_h=\{u_h\in X; u_{h|k}\in\mathbf{P}_K(k),\forall k\in\tau_h\}, \tag{16}$$

$$Q_h = \{q_h \in C^0(\bar{\Omega}); q_{h|k} \in \mathbf{P}_1(k), \forall k \in \tau_h\}, \tag{17}$$

With these spaces, the finite dimensional analogues of U is:

$$U_h = \{v_h \in X_h; (\nabla q_h, v_h) = 0, \forall q_h \in Q_h\}, \tag{18}$$

We define the interpolation operators:

$$\begin{array}{lll} r_h : H^1(\Omega) & \longrightarrow & X_h \\ \quad u & \longmapsto & r_h(u) \end{array} \tag{19}$$

$$\begin{array}{lll} I_h : H^2(\Omega) & \longrightarrow & Q_h \\ \quad u & \longmapsto & I_h(u) \end{array} \tag{20}$$

We discretize (V) by:
Find $u_h \in X_h$ and $p_h \in Q_h/\mathbf{R}$ such that

$$\mu(u_h, v_h)_{\Omega_2} + \nu(curl\, u_h, curl\, v_h)_{\Omega_1} + (curl\, u_h \wedge u_h, v_h)_{\Omega_1} + (\nabla p_h, v_h)_{\Omega} = (\mathbf{f}, v_h)_{\Omega}, \quad \forall v_h \in X_h.$$

Theorem 4.1 *Assume that τ_h is regular family of triangulations. We have:*

$$\|u - r_h(u)\|_{0,\Omega} + h\|curl(u - r_h(u))\|_{0,\Omega} + \|curl\,(u - r_h(u))\, \wedge\, (u - r_h(u))\|_{0,\Omega} \,\leq\, C\,h\,|u|_{1,t,\Omega},$$

$$\forall u \in W^{1,t}(\Omega)^3, \quad \forall t > 2.$$

More over, we have, when $u \in H^2(\Omega)^3$:

$$\|u - r_h(u)\|_{0,\Omega} \,\leq\, C\,h^2\,|u|_{2,\Omega},$$

$$\|curl(u - r_h(u))\|_{0,\Omega} \,\leq\, C\,h\,|u|_{2,\Omega},$$

$$\|curl\,(u - r_h(u))\, \wedge\, (u - r_h(u))\|_{0,\Omega} \,\leq\, C^2\,h^3\,|u|_{2,\Omega}.$$

Theorem 4.2 *Let Ω be a polyhedron and Ω_1 a convex polyhedron. Let τ_h be a uniformly regular family of triangulation of Ω. We have:*

$$\|u_h\|_{0,\Omega_1} \leq \alpha_0(\|u_h\|^2_{0,\Omega_2} + \|curl\, u_h\|^2_{0,\Omega_1} + \|curl\, u_h\|^4_{0,\Omega_1})^{\frac{1}{2}}, \quad u_h \in U_h.$$

For the proof of this theorem we need the following results, [10]:

Theorem 4.3 $\forall\, v \in L^2(\Omega_1)^3$*, satisfying:*

$$div\, v = 0 \quad , \quad curl\, v \in L^2(\Omega_1)^3 \quad , \quad v \cdot n = 0 \quad on \quad \Gamma,$$

verify

$$\|v\|_{0,\Omega_1} \leq c\|curl\, v\|_{0,\Omega_1}.$$

If Ω_1 is convex, $v \in H^1(\Omega_1)$ and we have

$$\|v\|_{1,\Omega_1} \leq c\|curl\, v\|_{1,\Omega_1}.$$

Proof
Let Ω_1 be convex, for every $u_h \in X_h$ we consider the Dirichlet problem:

$$(\nabla z, \nabla \mu)_{\Omega_1} = (u_h, \nabla \mu)_{\Omega_1}, \quad \forall \mu \in H^1(\Omega_1)/\mathbf{R}.$$

We denote

$$\omega = u_h - \nabla z_h \in U_{\Omega_1} = \{v \in H(curl, \Omega_1); (v, \nabla q)_{\Omega_1} = 0,\, \forall q \in H^1(\Omega_1)/\mathbf{R}\},$$

we have

$curl\,\omega = curl\,u_h.$

From theorem 3, It follows that

$\|\omega\|_{1,\Omega_1} \leq c_1 \|curl\,\omega\|_{0,\Omega_1}$

and we refere to [9] and [10] for more detail about the next inequality:

$\|r_h\omega - \omega\|_{0,\Omega_1} \leq c_2 \|curl\,u_h\|_{0,\Omega_1}.$

Then, we apply the operator r_h to ω, u_h is written:

$u_h = r_h\omega + \nabla z_h, \quad z_h \in Q_h.$

Hence

$\|u_h\|_{0,\Omega_1} \leq \|r_h\omega - \omega\|_{0,\Omega_1} + \|\omega\|_{0,\Omega_1} + 2\|curl\,u_h \wedge u_h\|_{0,\Omega_1} + \|\nabla z_h\|_{0,\Omega_1},$

Firstly, we have

$\|r_h\omega - \omega\|_{0,\Omega_1} \leq c_2 \|curl\,u_h\|_{0,\Omega_1},$

$\|\omega\|_{0,\Omega_1} \leq \|\omega\|_{1,\Omega_1} \leq c_1 \|curl\,u_h\|_{0,\Omega_1},$

$\|curl\,u_h \wedge u_h\|_{0,\Omega_1} \leq \|curl\,u_h\|_{0,\Omega_1} \|u_h\|_{0,\Omega_1} \leq c_3 \|curl\,u_h\|^2_{0,\Omega_1}.$

It remains to be seen $\|\nabla z_h\|_{0,\Omega_1}$. For this we take $\mu_h \in Q_h$:

$$\begin{aligned}
(\nabla z_h, \nabla\mu_h)_{\Omega_1} &= (u_h - r_h\omega, \nabla\mu_h)_{\Omega_1} \\
&= (u_h, \nabla\mu_h)_{\Omega_1} + (\omega - r_h\omega, \nabla\mu_h)_{\Omega_1} \\
&= -(u_h, \nabla\mu_h)_{\Omega_2} + (\omega - r_h\omega, \nabla\mu_h)_{\Omega_1} \\
&\leq \|u_h\|_{0,\Omega_2} \|\nabla\mu_h\|_{0,\Omega_2} + c_2 \|curl\,u_h\|_{0,\Omega_1} \|\nabla\mu_h\|_{0,\Omega_1}.
\end{aligned}$$

We choose $\mu_h \in Q_h$ such that:

$\mu_{h\backslash\bar{\Omega_1}} = z_{h\backslash\bar{\Omega_1}}, \qquad \mu_{h\backslash\partial\Omega} = 0,$

so we obtain:

$\|\nabla\mu_h\|_{0,\Omega_2} \leq c\|z_h\|_{\frac{1}{2},\Gamma} \leq c_4 \|\nabla z_h\|_{0,\Omega_1}.$

We deduce that:

$\|\nabla z_h\|_{0,\Omega_1} \leq c'(\|u_h\|_{0,\Omega_2} + \|curl\,u_h\|_{0,\Omega_1}).$

And finally the result.

5 Numerical results

To validate the theoretical results, we performed several numerical simulations using the FreeFem ++ software (see [14]).
The geometry considered is a square $]0, 1.5[\times] - 1, 0[$.
The numerical force and the pressure are taken as

$f(x, y) = x^2 + y^2,$

$p(x, y) = -x - y.$

We take μ and ν equal to 1 for simplicity. In what follows, we present the results obtained. The geometry mesh is given by figure 2.

In figure 3, we can see the isovalues for the velocity.

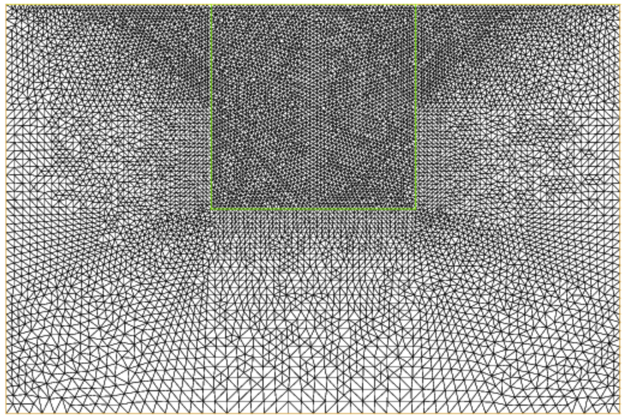

Figure 2: Mesh of the domain

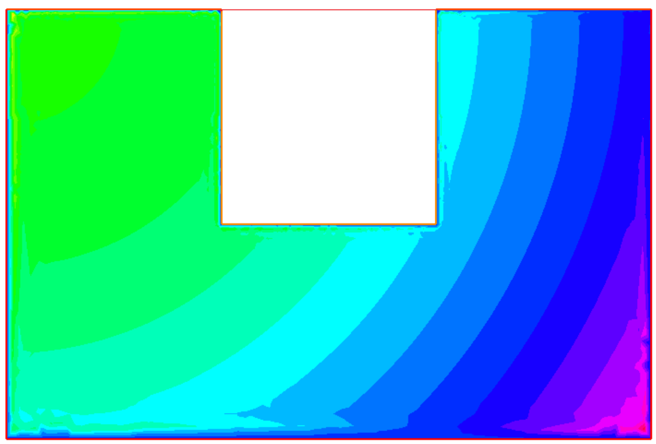

Figure 3: Isovalues

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