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Research paper

New algorithm method for solving the variational inequality problem in Hilbert space

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Abstract

The purpose of this paper is to introduce a concept of generalized non_spreading and define a new algorithm for infinite families of generalized non_spreading and finite families of resolvent mappings. Also, We study the existence solution of variational inequality to a common fixed point in Hilbert spaces. The main results in this paper extend and generalized of many known results in the literature.

Keywords: *Resolven; Mapping; Non-Spreading; Mapping; Common; Fixed Point; Strong Convergence.*

1. Introduction

Let K be a nonempty bounded closed convex subset of a Hibert space X. A mapping $\mathcal{P}: K \to K$ is said to be nonexpansive if $\|\mathcal{P}x - \mathcal{P}y\| \leq \|x - y\|$, for all $x, y \in K$. On the other hand anyimultivalued mapping $\mathcal{G}$ is called monotone if: $< z_1 - z_2, w_1 - w_1 > \geq 0\ \forall z_i \in D(\mathcal{G})$, for all $w_i \in \mathcal{G}(z_i)$. And it is called maximal monotone, (M,.M), if for all $(z,h) \in X \times X, \langle z - w, h - k\rangle \geq 0$ andifor all $(w,k) \in gph(\mathcal{G})$ then we get, $h \in \mathcal{G}(z)$. The Monotone mappings play important role in optimization Theory see the books([1]-[5]). It has been shown that if X is uniformly convex then every nonexpansive mapping $\mathcal{P} : K \to K$ has a fixed point (see Browder [6], cf. also Kirk [7]). In 1974, Ishikawa [8] introduced a new iteration procedure for approximating fixed points of pseudo-contractive compact mappings in Hilbert spaces Note that the normal Mann iteration procedure [9], is a special case of the Ishikawa one. For a comparison of the two iterative processes in the one-dimensional case we referithe reader to Rhoades [10]. For more details and literature on the convergence of Ishikawa and Mann iterates we refer to [11–18]. Recently, Sastry and Babu [19] introduced the analogs of Mann and Ishikawa iteratesifor multivalued mappings and proved convergence theorems for nonexpansive mappings whose domain is a compact convex subset of a Hilbert space. The convergence of the iteration processes are studied by many researchers, see([20]-[29]). In this paper, we generalize results of Sastry and Babu to uniformly convex Banach spaces. We also introduce both of the iteration processes in a new sense and prove a convergence theorem of Mann iterates for a mapping defined on a non compact domain. Now, we recall some definitions and lemmas which will used in the proofs. Lemma (1.1): [7]
If $\langle a_n\rangle$ be a sequenceiof non-negativeireal numberisuch that:
$a_{n+1} \leq (1 - \gamma_n)a_n + S_n$, $n \geq 0$, where $\langle\gamma_n\rangle$ is a sequence in (0, 1) and $\langle S_n\rangle$ is a sequence in $\mathbb{R}$, $\sum_{n=0}^{\infty} \gamma_n = \infty$ and $\limsup_{n\to\infty} \frac{S_n}{\gamma_n} \leq 0$ or $\sum_{n=1}^{\infty}|S_n| < \infty$. Then $a_n \to 0$ as $n \to \infty$.

Lemma 1.2: *[15] If $\langle a_n\rangle$ be a sequence nonnegative real numbers such that:*

$a_{n+1} \leq (1 - \gamma_n)a_n + \gamma_n S_n + \beta_n$, $n \geq 0$, where $\langle\gamma_n\rangle, \langle\beta_n\rangle$ and $\langle S_n\rangle$ are satisfies the following:

1) $\gamma_n \subset [0,1]$; $\sum_{n=1}^{\infty} \gamma_n = \infty$
2) $\limsup_{n\to\infty} S_n \leq 0$ or $\sum_{n=1}^{\infty}|\gamma_n S_n| < \infty$
3) $\beta_n \geq 0$ for each $n \geq 0$ such that $\sum_{n=0}^{\infty} \beta_n < \infty$. Then $\lim_{n\to\infty} a_n = 0$.

Lemma 1.3: *[16] Let C be a nonempty convex closed subset of real Hilbert space $X$and $\mathcal{P}$ is non-expansive multivalued operator with $Fix(\mathcal{P}) \neq \emptyset$.If $\langle x_n\rangle$ sequence in C such that $x_n \rightharpoonup x$ and $(I - \mathcal{P})x_n \to y, x, y \in C$ .Then we have $(I - \mathcal{P})x = y$.*

2. Main results

In this section, we define a new iterations for sequence of generalized non_spreading mapping and then study the convergence for these schemes.

Definition 2.1: *A mapping $\mathcal{P}$ is generalizedinon_spreading mapping if for each x, y in $D(\mathcal{P})$, there exists positive real sequence $\langle z_n \rangle$ then the following inequality holds*

$$||\mathcal{P}x - \mathcal{P}y||^2 \le ||x - y||^2 + z_n < x - \mathcal{P}x, y - \mathcal{P}y >$$

Lemma 2.1: *Let $\mathcal{G}_1, \mathcal{G}_2, , , , \mathcal{G}_m$; are M.M mappings, $\langle f_n \rangle$ be a sequence of generalized non_spreading mapping on C. If $\sum \alpha_{n,i} = 1$ and $\mathcal{P}_{\sigma_n}^i = \sigma_n f_n(x) + (1-\sigma_n) \sum_{i=1}^m \alpha_{n,i} J_{r_n,i}^i(x)$, where $\langle \alpha_{n,i} \rangle$ be a sequences in (0, 1] , $\langle \sigma_n \rangle$ $\langle \sigma_n \rangle$ be a sequence of positive real numbers and $\langle J_{r_n,i}^i \rangle$ be a sequence of resolvent mapping. Then $\mathcal{P}_{\sigma_n}^i$ are also generalized non_spreading on C $for\ all\ i = 1,2, \ldots\ldots\ldots, m$.*

Proof

For all $x, y \in C$

$$\left\|\mathcal{P}_{\sigma_n}^i(x) - \mathcal{P}_{\sigma_n}^i(y)\right\|^2 = \left\| \sigma_n f_n(x) + (1-\sigma_n) \sum_{i=1}^m \alpha_{n,i} J_{r_n,i}^i(x) - \sigma_n f_n(y) - (1-\sigma_n) \sum_{i=1}^m \alpha_{n,i} J_{r_n,i}^i(y)\right\|^2$$

$$\left\|\mathcal{P}_{\sigma_n}^i(x) - \mathcal{P}_{\sigma_n}^i(y)\right\|^2 \le \left\|\sigma_n \left(f_n(x) - f_n(y)\right) + (1-\sigma_n) \sum \alpha_{n,i} \left(J_{r_n,i}^i(x) - J_{r_n,i}^i(y)\right\|^2 \le \sigma_n \|f_n(x) - f_n(y)\|^2 + (1-\sigma_n) \sum_{i=1}^m \alpha_{n,i} \left\|J_{r_n,i}^i(x) - J_{r_n,i}^i(y)\right\|^2$$

$$= \sigma_n \|x-y\|^2 + \sigma_n z_n < x - \mathcal{P}x, y - \mathcal{P}y > + (1-\sigma_n)\|x-y\|^2 = \|x-y\|^2 + \sigma_n z_n < x - \mathcal{P}x, y - \mathcal{P}y >$$

Theorem 2.2: *let $\mathcal{G}_1, \mathcal{G}_2, , , , \mathcal{G}_m$ are M.M mappings, $\langle f_n \rangle$ be a sequence of generalized non_spreading mapping on C and $(\cap_{i=1}^\infty F(f_n)) \cap \left(\cap_{i=1}^m \mathcal{G}_i^{-1}(0)\right) \ne \emptyset$ $for\ all$ $i = 1,2, \ldots\ldots\ldots, m$. If the iteration process $\langle x_n \rangle$ defined as the following $x_{n+1} = \sigma_n f_n(x_n) + (1-\sigma_n) \sum_{i=1}^m \alpha_{n,i} J_{r_n,i}^i\left(x_{\sigma_n}\right)$, where $\langle \sigma_n \rangle$ be a sequence of positive real numbers convergent to 0.Then there exist a solution of variational inequality*

$$\langle (I - f_n)\left(x_{n_{k+1}}\right), x_{n_k} - \tilde{x} \rangle \le 0 \le 0$$

Proof

Let $v \in (\cap_{i=1}^\infty F(f_n)) \cap \left(\cap_{i=1}^m \mathcal{G}_i^{-1}(0)\right)$

$$\|x_{n+1} - v\|^2 \le \sigma_n \|f_n(x_t) - v\|^2 + (1-\sigma_n) \sum_{i=1}^m \alpha_{n,i} \left\|J_{r_n,i}^i(x_k) - v\right\|^2$$

$$\le \sigma_n \|x_k - v\|^2 + \sigma_n z_n \langle x_k - f_n x, v - f_n v \rangle + (1-\sigma_n) \sum_{i=1}^m \alpha_{n,i} \|x_k - v\|^2$$

$$\|x_{k+1} - v\|^2 \le \sigma_n \|x_k - v\|^2 + (1-\sigma_n)\|x_k - v\|^2 + \sigma_n z_n \langle x - f_n x, v - f_n v \rangle$$

$$\|x_{k+1} - v\|^2 \le \|x_k - v\|^2 + \sigma_n z_n \langle x - f_n x, v - f_n v \rangle$$

Hence, $\|x_{k+1} - v\|^2 \le \|x_k - v\|^2$

So, $\langle x_k \rangle$ is bounded sequence.

$$\left\|x_{k+1} - \sum_{i=1}^m \alpha_{n,i} J_{r_n,i}^i x_k\right\| = \left\|\sigma_n f_n(x_k) + (1-\sigma_n) \sum_{i=1}^m \alpha_{n,i} J_{r_n,i}^i(x_k) - \sum_{i=1}^m \alpha_{n,i} J_{r_n,i}^i(x_k)\right\|$$

$$\le \sigma_n \left\|f_n(x_k) - \sum_{i=1}^m \alpha_{n,i} J_{r_n,i}^i(x_k)\right\|$$

Since $\langle f_n(x_k) \rangle$ and $\langle J_{r_n,i}^i(x_k) \rangle$ are bounded and $\langle \sigma_n \rangle$ be a sequence of positive real numbers convergent to 0, then as $n \longrightarrow 0$ we get

$\left\|x_k - \sum_{i=1}^m \alpha_{n,i} J_{r_n,i}^i(x_k)\right\| \longrightarrow 0$, then we get

$\left\|x_{k+1} - J_{r_n,i}^i x_k\right\| \longrightarrow 0$, for all $i = 1,2, \ldots\ldots\ldots, m$

Now, since $\langle x_k \rangle$ is bounded sequence then there exists $\langle x_{kn} \rangle$ subsequence of $\langle x_k \rangle$ such that

$x_{kn} \rightharpoonup \tilde{x}$. By lemma (1.3) $\Longrightarrow \tilde{x} \in \mathcal{G}_i^{-1}(0)\ \forall\ i = 1,2, \ldots\ldots\ldots, m$

$$x_{k+1} - \tilde{x} = k(f_n(x_k) - \tilde{x}) + (1-k) \sum_{i=1}^m \alpha_{n,i} \left(J_{r_n,i}^i x_k - \tilde{x}\right)$$

Using $x_k - \tilde{x}$ to make inner product, we get

$$\|x_k - \tilde{x}\|^2 = \sigma_n \langle f_n(x_k) - \tilde{x}, x_k - \tilde{x} \rangle + (1-\sigma_n) \sum_{i=1}^m \alpha_{n,i} \langle J_{r_n,i}^i x_k - \tilde{x}, x_k - \tilde{x} \rangle$$

$$\le \sigma_n \langle f_n(x_k) - \tilde{x}, x_k - \tilde{x} \rangle + (1-\sigma_n) \sum_{i=1}^m \alpha_{n,i} \langle x_k - \tilde{x}, x_k - \tilde{x} \rangle$$

$$= \sigma_n \langle f_n(x_k) - \tilde{x}, x_k - \tilde{x} \rangle + (1-\sigma_n)\|x_k - \tilde{x}\|^2.$$

$\sigma_n \ \|x_{k+1} - \breve{x}\|^2 \leq \sigma_n \langle f_n(x_k) - \breve{x}, x_k - \breve{x} \rangle$

$\|x_{k+1} - \breve{x}\|^2 \leq \langle f_n(x_k) - \breve{x}, x_k - \breve{x} \rangle$

$\leq \langle f_n(x_k) - f_n(\breve{x}), x_k - \breve{x} \rangle + \langle f_n(\breve{x}) - \breve{x}, x_k - \breve{x} \rangle$

$\|x_{k+1} - \breve{x}\| \leq \|x_k - \breve{x}\|^2 + z_n \|x_k - \breve{x}\|. \langle x_k - f_n x_k, \breve{x} - f_n \breve{x} \rangle + \langle f_n(\breve{x}) - \breve{x}, x_k - \breve{x} \rangle$

In particular,

$\left\|x_{(k+1)n} - \breve{x}\right\|^2 \leq \|x_{kn} - \breve{x}\|^2 + z_n \|x_{kn} - \breve{x}\|. \langle x_{kn} - f_n x_{kn}, \breve{x} - f_n \breve{x} \rangle + \langle f_n(\breve{x}) - \breve{x}, x_{kn} - \breve{x} \rangle$

But $x_{kn} \rightharpoonup \breve{x}$, then$||x_{kn} - \breve{x}||^2 \longrightarrow 0$ as $n \longrightarrow \infty$

Now, since

$x_{k+1} - f_n(x_{k+1}) = \sigma_n f_n(x_k) - f_n(x_k) + f_n(x_k) + f_n(x_{k+1}) + (1 - \sigma_n) \sum_{i=1}^{m} \alpha_{n,i} J_{r_n,i}^i x_k$

Hence,

$x_{n+1} - f_n(x_n) = (f_n(x_k) + f_n(x_{k+1})) + (\sigma_n - 1) f_n(x_n) + (1 - \sigma_n) \sum_{i=1}^{m} \alpha_{n,i} J_{r_n,i}^i (x_n)$

$x_{n+1} - f_n(x_n) = -(1 - \sigma_n)(f_n - \sum_{i=1}^{m} \alpha_{n,i} J_{r_n,i}^i) x_k + f_n(x_k) + f_n(x_{k+1})$

$For\ all\ z \in (\cap_{i=1}^{\infty} F(f_n)) \cap (\cap_{i=1}^{m} \mathcal{G}_i^{-1}(0))$

$\langle x_{n+1} - f_n(x_n), x_k - z \rangle = -(1 - \sigma_n) \langle f_n - \sum_{i=1}^{m} \alpha_{n,i} J_{r_n,i}^i) x_k + (f_n(x_k) + f_n(x_{k+1})), x_k - z \rangle \leq 0$

(As $(f_n(x_k) + f_n(x_{k+1}) + (f_n - \sum_{i=1}^{m} \alpha_{n,i} J_{r_n,i}^i)(x_k))$ is monotone mapping. It is true for any subsequence of $\langle x_k \rangle$.
And hence, as $n_k \to \infty$ we get, $\tilde{x}$ is a solution of variational inequality

$\langle (I - f_n)(x_{nk+1}), x_{nk} - \tilde{x} \rangle \leq 0$, as $n \to \infty$.

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