Global Journal of Mathematical Analysis, 2 (4) (2014) 249-258
©Science Publishing Corporation
www.sciencepubco.com/index.php/GJMA
doi: 10.14419/gjma.v2i4.3126
Research Paper

Generalized Fibonacci-Like Polynomials and Some Identities

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Abstract

The Fibonacci polynomials and Lucas polynomials are famous for possessing wonderful and amazing properties and identities. In this paper, Generalized Fibonacci-Like Polynomials are introduced and defined by $m_n(x) = xm_{n-1}(x) + m_{n-2}(x),\ \mathrm{n} \geq 2.$ with $m_0(x) = 2s$ and $m_1(x) = 1+s$, where s is integer. Further, some basic identities are generated and derived by standard methods.

Keywords: *Fibonacci Polynomials, Lucas Polynomials, Generalized Fibonacci-Like Polynomials, Binet's formula.*

1. Introduction

The Fibonacci polynomials and Lucas polynomials are famous for possessing wonderful and amazing properties and identities. It is well-known that the Fibonacci polynomials and Lucas polynomials are closely related and widely investigated. Fibonacci polynomials appear in different frameworks. These polynomials are of great importance in the study of many subjects such as algebra, geometry, combinatorics, approximation theory, statistics and number theory itself. Moreover these polynomials have been applied in every branch of mathematics. Fibonacci polynomials are special cases of Chebyshev polynomials and have been studied on a more advanced level by many mathematicians.

Basin, S. L. [1] show that Q matrix generates a set of Fibonacci Polynomials is defined by the recurrence formula

$$f_{n+1}(x) = xf_n(x) + f_{n-1}(x),\ n \geq 2 \text{ with } f_0(x) = 0,\ f_1(x) = 1. \tag{1.1}$$

The Lucas Polynomials is defined by the recurrence formula

$$l_{n+1}(x) = xl_n(x) + l_{n-1}(x),\ n \geq 2 \text{ with } l_0(x) = 2,\ l_1(x) = x. \tag{1.2}$$

Generating function of Fibonacci polynomial is

$$\sum_{n=0}^{\infty} f_n(x)t^n = t\left(1 - xt - t^2\right)^{-1} \tag{1.3}$$

Generating function of Lucas polynomial is

$$\sum_{n=0}^{\infty} l_n(x)t^n = (2 - xt)\left(1 - xt - t^2\right)^{-1} \tag{1.4}$$

Explicit sum formula for (1.1) is given by

$$f_n(x) = \sum_{k=0}^{\left[\frac{n-1}{2}\right]} \binom{n-k-1}{k} x^{n-1-2k}. \tag{1.5}$$

where $\binom{n}{k}$ a binomial coefficient and [x] is define as the greatest integer less than or equal to x.

Explicit sum formula for (1.3) is given by

$$l_n(x)=\sum_{k=0}^{\left[\frac{n}{2}\right]}\frac{n}{n-k}\binom{n-k}{k}x^{n-2k}. \tag{1.6}$$

where $\binom{n}{k}$ a binomial coefficient and [x] is defined as the greatest integer less than or equal to x.

Fibonacci-Like polynomials [11] is defined by the recurrence relation:

$s_n(x)= xs_{n-1}(x)+s_{n-2}(x), n\geq 2$ with initial terms $s_0(x)= 2$ *and* $s_1(x)=2x$. (1.7)

Generalized Fibonacci-Lucas Polynomials [12] is defined by recurrence relation.

$b_n(x)= xb_{n-1}(x)+b_{n-2}(x)$, $n\geq 2$ With initial conditions $b_0(x)= 2b$ and $b_1(x)= s$, (1.8)

where b and s are integers.

The Fibonacci and Lucas polynomials possess many fascinating properties which have been studied in [2] to [12]. In this paper, Fibonacci-Like Polynomials are introducing with some basic identities and derived by standard method.

2. Generalized Fibonacci-Like Polynomials:

Generalized Fibonacci-Like Polynomials $m_n(x)$ are defined by recurrence relation

$m_n(x)=xm_{n-1}(x)+m_{n-2}(x)$, $n\geq 2$. with $m_0(x)=2s$ and $m_1(x)=1+s$, where s is integer. (2.1)

The first few terms of generalized Fibonacci-Like Polynomials are as follows:

$m_0(x)=2s$,

$m_1(x)=1+s$,

$m_2(x)=(1+s)x+2s$,

$m_3(x)=(1+s)x^2+2sx+(1+s)$ and so on.

If x=1, then $m_n(1)$ is generalized Fibonacci-Like sequence.

The characteristic equation of recurrence relation (2.1) is $\lambda^2-x\lambda-1=0$

$$\alpha=\frac{x+\sqrt{x^2+4}}{2} \quad \textit{and} \quad \beta=\frac{x-\sqrt{x^2+4}}{2} \tag{2.2}$$

Also, $\alpha\beta=-1$, Binet's formula of Generalized Fibonacci-Like sequence is defined by

$$m_n(x)=A\alpha^n+B\beta^n$$

$$m_n(x)=A\left(\frac{x+\sqrt{x^2+4}}{2}\right)^n+B\left(\frac{x+\sqrt{x^2+4}}{2}\right)^n$$

Here, $A=\dfrac{2s(x-\beta)}{\alpha-\beta}$ *and* $B=\dfrac{2s(\alpha-x)}{\alpha-\beta}$. (2.3)

Also, $AB=\dfrac{4s^2}{(\alpha-\beta)^2}$, $A+B=m_0(x)=2s$, $\alpha-\beta=\sqrt{x^2+4}$, and $\alpha^2+\beta^2=x^2+5$. (2.4)

Generating function of Generalized Fibonacci-Like Polynomials is

$$\sum_{n=0}^{\infty}m_n(x)t^n=\frac{2s(1-xt)+(1+s)t}{(1-xt-t^2)}. \tag{2.5}$$

Now Hyper geometric representation of generating function

$$\sum_{n=0}^{\infty}m_n(x)t^n=\left[2s(1-xt)+(1+s)t\right]\left(1-xt-t^2\right)^{-1},$$

$$=\left[2s(1-xt)+(1+s)t\right]\left[1-(x+t)t\right]^{-1},$$

$$\sum_{n=0}^{\infty} m_n(x)t^n = \left[2s(1-xt)+(1+s)t\right]\sum_{n=0}^{\infty}(x+t)^n t^n$$

$$= \left[2s(1-xt)+(1+s)t\right]\sum_{n=0}^{\infty}.t^n \sum_{k=0}^{n}\binom{n}{k} x^{n-k}.t^k$$

$$= \left[2s(1-xt)+(1+s)t\right]\sum_{n=0}^{\infty}.\sum_{k=0}^{n}\frac{n!}{k!(n-k)!} x^{n-k}.t^{k+n}$$

$$= \left[2s(1-xt)+(1+s)t\right]\sum_{n=0}^{\infty}.\sum_{k=0}^{\infty}\frac{(n+k)!}{k!n!} x^{n}.t^{2k+n}$$

$$= \left[2s(1-xt)+(1+s)t\right]\sum_{n=0}^{\infty}.\frac{(xt)^n}{n!}\sum_{k=0}^{\infty}\frac{(n+k)!}{n!k!}.t^{2k}$$

$$= \left[2s(1-xt)+(1+s)t\right]e^{xt}\sum_{k=0}^{\infty}\frac{(n+k)}{n!}.\frac{(t^2)^k}{k!}$$

$$= \left[2s(1-xt)+(1+s)t\right] e^{xt}\sum_{k=0}^{\infty}\frac{\Gamma(n+k+1)}{\Gamma(n+k)}\frac{(t^2)^k}{k!}.$$

$$= \left[2s(1-xt)+(1+s)t\right]e^{xt}\sum_{k=0}^{\infty}(n+1)_k \frac{(1)_k}{(1)_k}.\frac{(t^2)^k}{k!}$$

$$= \left[2s(1-xt)+(1+s)t\right]e^{xt}\,{}_2F_1\left[n+1:1;1;\, t^2\right],$$

$$\sum_{n=0}^{\infty} m_n(x)t^n = \left[2s(1-xt)+(1+s)t\right]e^{xt}\,{}_2F_1\left[n+1:1;1;\, t^2\right] \tag{2.6}$$

3. Some Identities of Generalized Fibonacci-Like Polynomials

In this section, we present some identities like Catalan's; Cassini's; d'Ocagne's identities etc. by Binet's formula or explicit sum formula or generating function.

Theorem 3.1: Prove that

$$m_{n+1}(x)-m_{n-1}(x) = xm_n(x),\ n\geq 1. \tag{3.1}$$

Proof: By Generating function of Fibonacci-Like polynomials, we have

$$\sum_{n=0}^{\infty} m_n(x)t^n = \left[2s(1-xt)+(1+s)t\right]\left(1-xt-t^2\right)^{-1}$$

Differentiating both sides with respect to t, we get

$$\sum_{n=0}^{\infty} nm_n(x)t^{n-1} = \left[2s(1-xt)+(1+s)t\right](x+2t)\left(1-xt-t^2\right)^{-2} + (1+s-2sx)\left(1-xt-t^2\right)^{-1}$$

$$\left(1-xt-t^2\right)\sum_{n=0}^{\infty} nm_n(x)t^{n-1} = \left[2s(1-xt)+(1+s)t\right](x+2t)\left(1-xt-t^2\right)^{-1} + (1+s-2sx)$$

$$\left(1-xt-t^2\right)\sum_{n=0}^{\infty} nm_n(x)t^{n-1} = (x+2t)\sum_{n=0}^{\infty} m_n(x)t^n + (1+s-2sx)$$

$$\sum_{n=0}^{\infty} nm_n(x)t^{n-1} - \sum_{n=0}^{\infty} xnm_n t^n - \sum_{n=0}^{\infty} nm_n(x)t^{n+1} = \sum_{n=0}^{\infty} xm_n(x)t^n - \sum_{n=0}^{\infty} 2m_n t^{n+1} + (1+s-2sx)$$

Now equating the coefficient of t^n on both sides we get,

$$(n+1)m_{n+1}(x) - nxm_n(x) - (n-1)m_{n-1}(x) = xm_n(x) + 2m_{n-1}(x),$$

$$(n+1)m_{n+1}(x) - (n+1)m_{n-1}(x) = (n+1)xm_n(x),$$

$$m_{n+1}(x) - m_{n-1}(x) = xm_n(x)$$

This is required results.

Theorem 3.2: Prove that

$$m_n'(x) = xm_{n-1}'(x) + m_{n-1}(x) + m_{n-2}'(x), \quad n \geq 2. \tag{3.2}$$

Proof: By generating function of Fibonacci-Like polynomials, we have

$$\sum_{n=0}^{\infty} m_n(x)t^n = \left[2s(1-xt)+(1+s)t\right]\left(1-xt-t^2\right)^{-1}$$

Differentiating both sides with respect to x, we get

$$\sum_{n=0}^{\infty} m_n(x)t^n = \left[2s(1-xt)+(1+s)t\right]\left(1-xt-t^2\right)^{-1},$$

$$\sum_{n=0}^{\infty} m_n'(x)t^n = \left[2s(1-xt)+(1+s)t\right](-1)\left(1-xt-t^2\right)^{-2}(-t)+\left(1-xt-t^2\right)^{-1}(-2st)$$

$$\left(1-xt-t^2\right)\sum_{n=0}^{\infty} m_n'(x)t^n = \left[2s(1-xt)+(1+s)t\right](-1)\left(1-xt-t^2\right)^{-1}(-t)-2st$$

$$\left(1-xt-t^2\right)\sum_{n=0}^{\infty} m_n'(x)t^n = t\sum_{n=0}^{\infty} m_n(x)t^n - 2st,$$

$$\sum_{n=0}^{\infty} m_n'(x)t^n - \sum_{n=0}^{\infty} xm_n'(x)t^{n+1} - \sum_{n=0}^{\infty} m_n'(x)t^{n+2} = t\sum_{n=0}^{\infty} m_n(x)t^{n+1} - 2st,$$

Now equating the coefficient of t^n, we get

$$m_n'(x) - xm_{n-1}'(x) - m_{n-2}'(x) = m_{n-1}(x).$$

$$m_n'(x) = xm_{n-1}'(x) + m_{n-1}(x) + m_{n-2}'(x).$$

Theorem 3.3: Prove that

$$m_{n+1}'(x) = xm_n'(x) + m_n(x) + m_{n-1}'(x), n \geq 1. \tag{3.3}$$

Proof: By (3.1), we have

$$m_{n+1}(x) - m_{n-1}(x) = xm_n(x), n \geq 1.$$

By differentiating with respect to x, we get

$$m_{n+1}'(x) - m_{n-1}'(x) = xm_n'(x) + m_n(x),$$

$$m_{n+1}'(x) = xm_n'(x) + m_n(x) + m_{n-1}'(x).$$

Theorem 3.4: Prove that

$$nm_n(x) = xm_n'(x) - 2m_{n-1}'(x)., n \geq 1 \text{ and } xm_{n+1}'(x) = (n+1)m_{n+1}(x) - 2m_n'(x), n \geq 1.$$

Proof: By generating function of Fibonacci-Like polynomials, we have

$$\sum_{n=0}^{\infty} m_n(x)t^n = \left[2s(1-xt)+(1+s)t\right]\left(1-xt-t^2\right)^{-1}.$$

Differentiating both sides with respect to t, we get

$$\sum_{n=0}^{\infty} nm_n(x)t^{n-1} = (1+s-2sx)\left(1-xt-t^2\right)^{-1} + \left[2s(1-xt)+(1+s)t\right](x+2t)\left(1-xt-t^2\right)^{-2} \quad . \tag{3.4}$$

Differentiating both sides with respect to x, we get

$$\sum_{n=0}^{\infty} m_n'(x)t^n = (-2st)\left(1-xt-t^2\right)^{-1} + \left[2s(1-xt)+(1+s)t\right].t\left(1-xt-t^2\right)^{-2}$$

$$\sum_{n=0}^{\infty} m_n'(x)t^{n-1} = (-2s)\left(1-xt-t^2\right)^{-1} + \left[2s(1-xt)+(1+s)t\right]\left(1-xt-t^2\right)^{-2}$$

$$\sum_{n=0}^{\infty} m_n'(x)t^{n-1} + 2s\left(1-xt-t^2\right)^{-1} = \left[2s(1-xt)+(1+s)t\right]\left(1-xt-t^2\right)^{-2} \tag{3.5}$$

Using (3.5) in (3.4), we get

$$\sum_{n=0}^{\infty} nm_n(x)t^{n-1} = (1+s-2sx)(1-xt-t^2)^{-1} + (x+2t)\left\{\sum_{n=0}^{\infty} m_n'(x)t^{n-1} + 2s(1-xt-t^2)^{-1}\right\},$$

$$\sum_{n=0}^{\infty} nm_n(x)t^{n-1} = (1+s-2sx)(1-xt-t^2)^{-1} + (x+2t)\sum_{n=0}^{\infty} m_n'(x)t^{n-1} + 2s(x+2t)(1-xt-t^2)^{-1}$$

$$\sum_{n=0}^{\infty} nm_n(x)t^{n-1} = (1+s-2sx)(1-xt-t^2)^{-1} + x\sum_{n=0}^{\infty} m_n'(x)t^{n-1} + \sum_{n=0}^{\infty} 2m_n'(x)t^n + 2s(x+2t)(1-xt-t^2)^{-1}$$

Equating the coefficient of t^{n-1} on both sides, we get

$$nm_n(x) = xm_n'(x) + 2m_{n-1}'(x). \tag{3.6}$$

Again equating the coefficient of t^n, we get

$$(n+1)m_{n+1}(x) = xm_{n+1}'(x) + 2m_n'(x),$$

$$xm_{n+1}'(x) = (n+1)m_{n+1}(x) - 2m_n'(x).. \tag{3.7}$$

Theorem 3.5: Prove that

$$(n+1)m_n(x) = m_{n+1}'(x) + m_{n-1}'(x)., \ n \ge 1.$$

Proof: By (3.1), we have

$$m_{n+1}(x) - m_{n-1}(x) = xm_n(x), \ n \ge 1.$$

By differentiating with respect to x, we get

$$m_{n+1}'(x) - m_{n-1}'(x) = xm_n'(x) + m_n(x),$$

$$xm_n'(x) + m_n(x) = m_{n+1}'(x) - m_{n-1}'(x). \tag{3.8}$$

Using equation (3.5) in equation (3.8) we get

$$nm_n(x) - 2m_{n-1}'(x) + m_n(x) = m_{n+1}'(x) - m_{n-1}'(x),$$

$$nm_n(x) + m_n(x) = m_{n+1}'(x) + 2m_{n-1}'(x) - m_{n-1}'(x),$$

$$(n+1)m_n(x) = m_{n+1}'(x) + m_{n-1}'(x). \tag{3.9}$$

Theorem 3.6: Prove that

$$xm_n'(x) = 2m_{n+1}'(x) - (n+2)m_n(x), \ n \ge 0.$$

Proof: Using equation (3.7) in equation (3.9), we get

$$(n+1)m_n(x) = m_{n+1}'(x) + \frac{1}{2}\left[nm_n(x) - xm_n'(x)\right],$$

$$2(n+1)m_n(x) = 2m_{n+1}'(x) + \left[nm_n(x) - xm_n'(x)\right],$$

$$xm_n'(x) = 2m_{n+1}'(x) + nm_n(x) - (2n+2)m_n(x),$$

$$xm_n'(x) = 2m_{n+1}'(x) + (n-2n-2)m_n(x),$$

$$xm_n'(x) = 2m_{n+1}'(x) - (n+2)m_n(x). \tag{3.10}$$

Theorem 3.7: $(n+1)xm_n'(x) = nm_{n+1}'(x) - (n+2)m_{n-1}'(x), \ n \ge 1.$

Proof: Using equation (2.3) we get

$$(n+1)\{m_{n+1}'(x) - xm_n'(x) - m_{n-1}'(x)\} = m_{n+1}'(x) + m_{n-1}'(x)$$

$$(n+1)m_{n+1}'(x) - (n+1)xm_n'(x) - (n+1)m_{n-1}'(x) = m_{n+1}'(x) + m_{n-1}'(x),$$

$$(n+1)m_{n+1}'(x) - (n+1)m_{n-1}'(x) - m_{n+1}'(x) - m_{n-1}'(x) = (n+1)xm_n'(x),$$

$$nm_{n+1}'(x) - (n+2)m_{n-1}'(x) = (n+1)xm_n'(x).$$

$$(n+1)xm_n'(x) = nm_{n+1}'(x) - (n+2)m_{n-1}'(x). \tag{3.11}$$

Theorem 3.8 (Explicit sum formula): For Generalized Fibonacci-Like polynomials are given by

$$m_n(x) = 2s\sum_{k=0}^{\left[\frac{n}{2}\right]}\binom{n-k}{k}x^{n-2k} \ . \tag{3.12}$$

Proof: By generating function (2.5), we have

$$\sum_{n=0}^{\infty} m_n(x)t^n = \left[2s(1-xt)+(1+s)t\right]\left(1-xt-t^2\right)^{-1}$$

$$=\left[2s(1-xt)+(1+s)t\right]\left(1-xt-t^2\right)^{-1}$$

$$=\left[2s(1-xt)+(1+s)t\right]\sum_{n=0}^{\infty}(x+t)^n t^n$$

$$=\left[2s(1-xt)+(1+s)t\right]\sum_{n=0}^{\infty}.t^n\sum_{m=0}^{n}\binom{n}{k}x^{n-k}.t^k$$

$$=\left[2s(1-xt)+(1+s)t\right]\sum_{n=0}^{\infty}.\sum_{m=0}^{n}\frac{n!}{k!(n-k)!}x^{n-k}.t^{k+n}$$

$$=\left[2s(1-xt)+(1+s)t\right]\sum_{n=0}^{\infty}.\sum_{m=0}^{n}\frac{(n+k)!}{k!n!}x^{n}.t^{2k+n}$$

$$=\left[2s(1-xt)+(1+s)t\right]\sum_{n=0}^{\infty}.\sum_{m=0}^{\infty}\frac{(n+k)!}{k!n!}x^{n}.t^{2k-n} \ ,$$

$$m_n(x) = 2s\sum_{k=0}^{\left[\frac{n}{2}\right]}\frac{(n-k)!}{k!n-2k!}x^{n-2k}.$$

Equating coefficients of t^n on both sides, we get required explicit formula.

Theorem 3.9 For positive integer $n \geq 0$, prove that

$$m_n(x) = 2sx^n \,{}_2F_1\left(\frac{-n}{2} : \frac{-n+1}{2}; -n; \frac{-4}{x^2}\right). \tag{3.13}$$

Proof: By explicit sum formula (3.13), it follows that

$$m_n(x) = 2s\sum_{k=0}^{\left[\frac{n}{2}\right]}\binom{n-k}{k}x^{n-2k},$$

$$m_n(x) = 2sx^n\sum_{k=0}^{\left[\frac{n}{2}\right]}\frac{(n-k)!}{k!n-2k!}x^{-2k},$$

$$m_n(x) = 2sx^n\sum_{k=0}^{\left[\frac{n}{2}\right]}\frac{(-1)^k(1)_n(-n)_{2k}}{(-n)_k(-1)^{2k}(1)_n}\frac{x^{-2k}}{k!},$$

$$m_n(x) = 2sx^n\sum_{k=0}^{\left[\frac{n}{2}\right]}\frac{(-1)^k 2^{2k}\left(\frac{-n}{2}\right)_k\left(\frac{-n+1}{2}\right)_k\frac{x^{-2k}}{k!}}{(-n)_k(-1)^{2k}},$$

$$m_n(x) = 2sx^n\sum_{k=0}^{\left[\frac{n}{2}\right]}\frac{(-1)^k 2^{2k}\left(\frac{-n}{2}\right)_k\left(\frac{-n+1}{2}\right)_k\left(\frac{-4}{x^2}\right)}{(-n)_k\,k!}.$$

Hence,

$$m_n(x) = 2sx^n \,{}_2F_1\left(\frac{-n}{2}, \frac{-n+1}{2}; -n, \frac{-4}{x^2}\right).$$

Theorem 3.10: For positive integer $n \geq 0$, prove that

$$\sum_{n=0}^{\infty}(c)_n m_n(x)\frac{t^n}{n!} = 2s(1-xt)^{-c} \,{}_3F_1\left(\frac{c}{2}, \frac{c+1}{2}, n+1, \frac{n+1}{2}, \frac{n+2}{2}, \frac{t^2}{(1-xt)^2}\right). \tag{3.14}$$

Proof: Multiplying both sides of (3.13) by $(c)_n \frac{t^n}{n!}$ and summing between the limit n=0 to $n=\infty$, we obtain

$$\sum_{n=0}^{\infty}(c)_n m_n(x)t^n = 2s\sum_{n=0}^{\infty}\sum_{k=0}^{\frac{n}{2}}\frac{n-k!}{k!n-2k!}(c)_n x^{n-2k}\frac{t^n}{n},$$

$$= 2s\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\frac{n+k!}{k!n!n+2k!}(c)_{n+2k} x^n t^{n+2k}$$

$$= 2s\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\frac{n+k!}{k!n!(n+2k)!}(c+2k)_n (c)_{2k}(xt)^n t^{2k}$$

$$= 2s\left\{\sum_{n=0}^{\infty}(c+2k)_n \frac{(xt)^n}{n!}\right\}\sum_{n=0}^{\infty}\frac{n+k!}{k!n+2k!}(c)_{2k} t^{2k}$$

$$= 2s(1-xt)^{-(c+2k)}\sum_{k=0}^{\infty}\frac{n+k!}{k!n+2k!}(c)_{2k} t^{2k}$$

$$= 2s\sum_{k=0}^{\infty}(1-xt)^{-(c+2k)}\frac{n+k!}{k!n+2k!}(c)_{2k} t^{2k}$$

$$= 2s(1-xt)^{-c}\sum_{k=0}^{\infty}\frac{n+k!}{k!n+2k!}(c)_{2k}\left[\frac{t^2}{(1-xt)}\right]^k$$

$$= 2s(1-xt)^{-c}\sum_{k=0}^{\infty}\frac{n+k!}{k!n+2k!}(2)^{2k}\left(\frac{c}{2}\right)_k\left(\frac{c+1}{2}\right)_k\left[\frac{t^2}{(1-xt)^2}\right]^k$$

$$= 2s(1-xt)^{-c}\sum_{k=0}^{\infty}\frac{\frac{n+k!}{n!}}{k!n+2k!}(2)^{2k}\left(\frac{c}{2}\right)_k\left(\frac{c+1}{2}\right)_k\left[\frac{t^2}{(1-xt)^2}\right]^k$$

$$= 2s(1-xt)^{-c}\sum_{k=0}^{\infty}\frac{(n+1)_k}{(n+1)_{2k}}(2)^{2k}\left(\frac{c}{2}\right)_k\left(\frac{c+1}{2}\right)_k\left[\frac{t^2}{(1-xt)^2}\right]^k$$

$$= 2s(1-xt)^{-c}\sum_{k=0}^{\infty}\frac{(n+1)_k}{2^{2k}\left(\frac{n+1}{2}\right)_k\left(\frac{n+2}{2}\right)_k}(2)^{2k}\left(\frac{c}{2}\right)_k\left(\frac{c+1}{2}\right)_k\left[\frac{t^2}{(1-xt)^2}\right]^k$$

$$= 2s(1-xt)^{-c}\sum_{s=0}^{\infty}\frac{\left(\frac{c}{2}\right)_k\left(\frac{c+1}{2}\right)_k(n+1)_k}{\left(\frac{n+1}{2}\right)_k\left(\frac{n+2}{2}\right)_k}\left[\frac{t^2}{(1-xt)^2}\right]^k.$$

Hence, $\sum_{n=0}^{\infty}(c)_n m_n(x)\frac{t^n}{n!} = 2s(1-xt)^{-c}\,{}_3F_1\left(\frac{c}{2},\frac{c+1}{2},n+1,\frac{n+1}{2},\frac{n+2}{2},\frac{t^2}{(1-xt)^2}\right).$

Theorem 3.11 (Explicit sum formula): For Generalized Fibonacci-Like polynomials

$$m_{n+1}(x) = (1+s-2sx)\sum_{k=0}^{\left[\frac{n}{2}\right]}\binom{n-k}{k}x^{n-2k}. \tag{3.15}$$

Proof: Generating function (2.5), we have

$$\sum_{n=0}^{\infty}m_n(x)t^n = \left[2s(1-xt)+(1+s)t\right]\left(1-xt-t^2\right)^{-1},$$

$$= \left[2s(1-xt)+(1+s)t\right]\left(1-xt-t^2\right)^{-1}$$

$$= \left[2s(1-xt)+(1+s)t\right]\sum_{n=0}^{\infty}(x+t)^n t^n,$$

$$\sum_{n=0}^{\infty} m_n(x)t^n = \left[2s(1-xt)+(1+s)t\right]\sum_{n=0}^{\infty}.t^n \sum_{m=0}^{n}\binom{n}{k}x^{n-k}t^k ,$$

$$= \left[2s(1-xt)+(1+s)t\right]\sum_{n=0}^{\infty}.\sum_{m=0}^{n}\frac{n!}{k!(n-k)!}x^{n-k}.t^{k+n}$$

$$= \left[2s(1-xt)+(1+s)t\right]\sum_{n=0}^{\infty}.\sum_{m=0}^{n}\frac{(n+k)!}{k!n!}x^{n}.t^{2k+n}$$

$$= \left[2s(1-xt)+(1+s)t\right]\sum_{n=0}^{\infty}.\sum_{m=0}^{\infty}\frac{(n+k)!}{k!n!}x^{n}.t^{2k-n} .$$

Equating coefficients of t^n on both sides, we get required explicit formula

$$m_n(x) = 2s\sum_{k=0}^{\left[\frac{n}{2}\right]}\frac{(n-k)!}{k!n-2k!}x^{n-2k}.$$

Equating coefficients of t^{n+1} on both sides, we get required explicit formula

$$m_{n+1}(x) = (1+s-2s)\sum_{k=0}^{\left[\frac{n}{2}\right]}\frac{(n-k)!}{k!n-2k!}x^{n-2k}.$$

Theorem 3.12 (Catalan's Identity): Let $m_n(x)$ be the n^th term of generalized Fibonacci-Like polynomials, then

$$m_n^2(x)-m_{n+r}(x)m_{n-r}(x) = \frac{(-1)^{n-r}}{1-5s^2}\left[(1+s)m_r(x)-2sm_{r+1}(x)\right], n>r\geq 1. \tag{3.16}$$

Proof. Using Binet's formula (2.5), we have

$$m_n^2(x)-m_{n+r}(x)m_{n-r}(x)$$
$$=\left(A\alpha^n+B\beta^n\right)^2-\left(A\alpha^{n+r}+B\beta^{n+r}\right)\left(A\alpha^{n-r}+B\beta^{n-r}\right)$$
$$=AB(\alpha\beta)^n\left(2-\alpha^r\beta^{-r}-\alpha^{-r}\beta^r\right)$$
$$=AB(-1)^{n-r}\left(\alpha^r-\beta^r\right)^2$$
$$=\frac{4s^2}{(\alpha-\beta)^2}(-1)^{n-r}\left(\alpha^r-\beta^r\right)^2$$
$$=4s^2(-1)^{n-r}\left(\frac{\alpha^r-\beta^r}{\alpha-\beta}\right)^2$$

$Since\ \frac{\alpha^r-\beta^r}{\alpha-\beta} = \frac{(1+s)m_r(x)-2sm_{r+1}(x)}{(1+s)^2-2s(1+s)-4s^2} = \frac{(1+s)m_r(x)-2sm_{r+1}(x)}{1-5s^2}$, we obtain

$$m_n^2(x)-m_{n+r}(x)m_{n-r}(x) = \frac{(-1)^{n-r}}{1-5s^2}\left[(1+s)m_r(x)-2sm_{r+1}(x)\right], n>r\geq 1.$$

Corollary 3.13 (Cassini's Identity): Let $m_n(x)$ be the n^th term of generalized Fibonacci-Like polynomials, then

$$m_n^2(x)-m_{n+1}(x)m_{n-1}(x) = (-1)^{n-1}(1-5s^2), n\geq 1 \tag{3.17}$$

Proof: If r = 1 in the Catalan's identity, then obtained required result.

Theorem 3.14 (d'Ocagne's Identity): Let $m_n(x)$ be the n^th term of generalized Fibonacci-Like polynomials, then

$$m_p(x)m_{n+1}(x)-m_{p+1}(x)m_n(x) = (-1)^n\left[\frac{(1+s)m_{p-n}(x)-2sm_{p-n+1}}{1-5s^2}\right],\ p\geq 1, n\geq 0,\ p>n. \tag{3.18}$$

Proof. Using the Binet's formula (2.5), we have

$$m_p(x)m_{n+1}(x)-m_{p+1}(x)m_n(x)=\left(A\alpha^p+B\beta^p\right)\left(A\alpha^{n+1}+B\beta^{n+1}\right)-\left(A\alpha^{p+1}+B\beta^{p+1}\right)\left(A\alpha^n+B\beta^n\right),$$

$$=AB\left(\alpha^p\beta^{n+1}+\alpha^{n+1}\beta^p-\alpha^n\beta^{p+1}-\alpha^{p+1}\beta^n\right)$$

$$=AB(\alpha\beta)^n\left[\beta\left(\alpha^{p-n}-\beta^{p-n}\right)-\alpha\left(\alpha^{p-n}-\beta^{p-n}\right)\right]$$

$$=AB(-q)^n(\alpha-\beta)\left(\alpha^{p-n}-\beta^{p-n}\right),$$

$$=\frac{4s^2}{(\alpha-\beta)^2}(-1)^n(\alpha-\beta)\left(\alpha^{p-n}-\beta^{p-n}\right),$$

$$=4s^2(-1)^n\left(\frac{\alpha^{p-n}-\beta^{p-n}}{\alpha-\beta}\right).$$

Using subsequent results of Binet's formula, we get

$$Since\frac{\alpha^{p-n}-\beta^{p-n}}{\alpha-\beta}=\frac{(1+s)m_{p-n}(x)-2sm_{p-n+1}}{(1+s)^2-2s(1+s)-4s^2}=\frac{(1+s)m_{p-n}(x)-2sm_{p-n+1}}{1-5s^2},$$ we obtain

$$m_p(x)m_{n+1}(x)-m_{p+1}(x)m_n(x)=(-1)^n\left[\frac{(1+s)m_{p-n}(x)-2sm_{p-n+1}}{1-5s^2}\right],\ p\geq 1, n\geq 0,\ p>n.$$

Theorem 3.15 (Generalized Identity): Let $m_n(x)$ be the nth term of generalized Fibonacci-Like polynomials, then

$$m_p(x)m_n(x)-m_{p-r}(x)m_{n+r}(x)=(-1)^{p-r}\left[(1+s)m_r-2sm_{r-1}\right]\left[(1+s)m_{n-p+r}-2sm_{n-p+r}\right], n>m\geq r\geq 1.$$

Proof. Using the Binet's formula (2.5), we have

$$m_p(x)m_{n+1}(x)-m_{p+1}(x)m(x)_n=\left(A\alpha^p+B\beta^p\right)\left(A\alpha^n+B\beta^n\right)-\left(A\alpha^{p-r}+B\beta^{p-r}\right)\left(A\alpha^{n+r}+B\beta^{n+r}\right)$$

$$=AB\left(\alpha^r-\beta^r\right)\left[\frac{\alpha^p\beta^n}{\alpha^r}-\frac{\alpha^n\beta^p}{\beta^r}\right]$$

$$=AB\frac{\left(\alpha^r-\beta^r\right)}{(\alpha\beta)^r}\left(\alpha^p\beta^{n+r}-\alpha^{n+r}\beta^p\right)$$

$$=AB\left(\alpha^r-\beta^r\right)\left[\frac{\alpha^p\beta^n}{\alpha^r}-\frac{\alpha^n\beta^p}{\beta^r}\right]$$

$$=AB(-1)^{-r}\left(\alpha^r-\beta^r\right)\left(\alpha^p\beta^{n+r}-\alpha^{n+r}\beta^p\right)$$

$$=AB(-1)^{-r}\alpha^p\beta^p\left(\alpha^r-\beta^r\right)\left(\beta^{n-p+r}-\alpha^{n-p+r}\right)$$

$$=-AB(-1)^{-r}\alpha^p\beta^p\left(\alpha^r-\beta^r\right)\left(\alpha^{n-p+r}-\beta^{n-p+r}\right).$$

Using subsequent results of Binet's formula, we get

$$Since\frac{\alpha^r-\beta^r}{\alpha-\beta}=\frac{1}{\left(1-5s^2\right)}\left[(1+s)m_r-2sm_{r+1}\right].$$

$$\frac{\alpha^{n-p+r}-\beta^{n-p+r}}{\alpha-\beta}=\frac{(1+s)m_{n-p+r}-2sm_{n-p+r+1}}{1-5s^2},$$

$$m_p(x)m_n(x)-m_{p-r}(x)m_{n+r}(x)=(-1)^{p-r}\left[(1+s)m_r-2sm_{r-1}\right]\left[(1+s)m_{n-p+r}-2sm_{n-p+r}\right], n>m\geq r\geq 1 \tag{3.19}$$

The identity (3.15) provides Catalan's identity, Cassini and d'Ocagne and other identities:

4. Conclusion

In this paper, Generalized Fibonacci-Like Polynomials are introduced. Some basic identities are generated and derived by standard methods.

Acknowledgement

The authors are thankful to the reviewers for their constructive suggestions and comments for improving the exposition of the original version.

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