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Research paper

The examination of the quotient of numerical semigroup with RF-matrices

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Abstract

In this paper, we study quotients of a numerical semigroups with RF (Row-Factorization) matrices. We prove a formula for the Frobenious number of quotients of some families of numerical semigroups. Moreover, we examine half of the numerical semigroups, pseudo-symmetric numerical semigroups.

Keywords: *Quotient of A Numerical Semigroup; Pseudo-Frobenious Number; RF (Row Factorization) Matrices.*

1. Introduction

A generalization of the linear Diophantine Frobenius problem can be stated as follows. Let $n_1, \ldots, n_p$ and f be positive integers with $\gcd\{n_1, \ldots, n_p\} = 1$. Find a formula for the largest multiple of f not belonging to $\langle n_1, \ldots, n_p \rangle$.This problem is equivalent to the computation of the Frobenius number of the semigroup $\frac{\langle n_1,\ldots,n_p\rangle}{f}$, and it still remains open for $p = 2$.Semigroup of the form $\frac{T}{f}$ also occur in a natural way in the scope of proportionally modular numerical semigroup. In [1], it is shown that a numerical semigroup is proportionally modular if and only if it is the quotient of an embedding dimension two numerical semigroup. This result is later sharpened in [2] where it is shown that it suffices to take numerical semigroups generated by an integer and this integer plus one . So far we have no general formula for the largest multiple of an integer not belonging to $\langle a, a + 1\rangle$, with a an integer greater than two. Since numerical semigroups with embedding dimension two are symmetric, we wondered which is the class of all numerical semigroups that are quotients of symmetric numerical semigroups. Surprisingly, this class covers the set of all numerical semigroups as shown in [3]. What is more amazing is that it suffices to divide by two. The same does not hold for pseudo-symmetric numerical semigroups, and we need to divide by four to obtain the whole set of numerical semigroups as quotients of pseudo-symmetric numerical semigroups (see [4]). As for other families of numerical semigroups, for instance, we still do not know how to decide if a numerical semigroup is the quotient of a numerical semigroup with embedding dimension three.The Notion of quotient of a numerical semigroup was introduced by in [5] in order to solve diophantine inequalities. Several Authors studied such quotients; for instance J.C.Rosales and P.A.Garaa Sanchez proved in [6,7] that every numerical semigroup is one half of infinitely many symmetric semigroups.Moscariello introdued the RF(Row-Factorization) matrices whichis very useful in the classification of almost symmetric numerical semigroup in 2016 [8].In this study, we presnt the quotient of a numerical semigroups.

2. The quotient of numerical semigroup

Definition.2.1: *Let* $\mathbb{N}$ *is the set of non-negative integers and* $T \in \mathbb{N}$. *If T is closed under the addition in* $\mathbb{N}$ *and* $0 \in T$ *and* $\mathbb{N} \setminus T$ *is finite then This is a numerical semigroup, for all* $t_1, t_2, \ldots, t_n \in T$ *it is denoted by*

$T = \langle t_1, t_2, \ldots, t_n \rangle = \{ \sum_{i=1}^{n} t_i x_i : x_i \in \mathbb{N} \}$

And the following is correct

$(t_1, t_2, \ldots, t_n) = 1 \Leftrightarrow \mathbb{N} \setminus T$ is finite.

Example.2.2: *Let* $T == \langle 3,5\rangle = \{3x_1 + 3x_2 : x_1, x_2 \in \mathbb{N}\} = \{0,3,5,6,8,9,10, \rightarrow\}$

i) $0 \in T$,

ii) For all $t_1, t_2 \in T$, there is $x_1, x_2, y_1, y_2 \in \mathbb{N}$ such that $t_1 = 3x_1 + 5x_2$, $t_2 = 3y_1 + 5y_2$ and $x + y = 3(x_1 + x_2) + 5(x_1 + x_2) \in T$,

iii) $\mathbb{N}\backslash T = \{1,2,4,7\}$ is finite so T is a numerical semigroup.

Definition2.3: *Let T be a numerical semigroup. The largest integer that is not in T is called the frobenius number of T and denoted by F(T),*

$F(T) = \max(\mathbb{N}\backslash T)$

Or

$F(T) = \max\{x \in \mathbb{Z} : x \notin T\}$.

Definition 2.4: *The positive elements that is not in T and is denoted by G(T). The elements of gaps is called genus of T and g(T)=* $|G(T)|$.

Proposition 2.5: *Let T be a numerical semigroup and let f be a positive integer .Set*

$\frac{T}{f} = \{x \in \mathbb{N} \mid fx \in T\}$.

1) $\frac{T}{f}$ is a numerical semigroup.
2) $T \subseteq \frac{T}{f}$.
3) $\frac{T}{f} = \mathbb{N}$ if and only if $f \in T$.

The semigroup $\frac{T}{f}$ is called the quotient of T by f. Accordingly we say that $\frac{T}{2}$ is one half of T

and that $\frac{\boldsymbol{T}}{\boldsymbol{4}}$ is one fourth of T.

Proposition 2.6: *Let T be a numerical semigroup and let f a positive integer. Then*

$FG\left(\frac{T}{f}\right) = \left\{\frac{h}{f} \mid h \in FG(T) \text{ and } h \equiv 0 \bmod f\right\}$.

Proof: The integer h belongs to $FG\left(\frac{T}{f}\right)$ if and only if $h \notin \frac{T}{f}$ and $kh \in \frac{T}{f}$ for every integer k greater than one. This is equivalent to $fh \notin T$ and $kfh \in T$ for for any integer k than one.

Corollary 2.7. Let T be a numerical semigroup and let f be a positive integer.Then

$f \in FG(T)$ if and only if $\frac{T}{f} = \langle 2,3\rangle$.

Proof: Observe that $FG((\langle 2,3\rangle) = 1$. Then use Proposition 2.6.

As we know, one of the best ways to describe a numerical semigroup is by means of the Apery set of any of its nonzero elements.

Note that if T is a numerical semigroup, $n \in T$ and $f|n$, *then* $\frac{n}{f} \in \frac{T}{f}$. We describe $Ap\left(\frac{T}{f},\frac{n}{f}\right)$ in terms of $Ap(T,n)$.

Proposition 2.8: *Let T be a numerical semigroup. Let n be a nonzero element of T and let f be a divisor of n.*

Then ,

$AP\left(\frac{T}{f},\frac{n}{f}\right) = \left\{\frac{v}{f} \mid v \in Ap(T,n) \text{ and } v \equiv 0 \bmod f\right\}$.

Proof: The idea of the proof is analogous to the proof of Proposition 2.6.

Corollary 2.9. Let T be a numerical semigroup. Let n be a nonzero element of T and let f be a divisor of n.

Assume that Ap (T,n) $= \{0, k_1 n + 1, \dots \dots, k_{n-1} n + n - 1\}$. Then

1) $Ap\left(\frac{T}{f},\frac{n}{f}\right) = \left\{0, k_f \frac{n}{f} + 1, \dots \dots, k_{\left(\frac{n}{f}-1\right)f} \frac{n}{f} + \frac{n}{f} - 1\right\}$,
2) $g\left(\frac{T}{f}\right) = k_f + k_{2f} + \cdots . + k_{\left(\frac{n}{d}-1\right)f}$
3) $F\left(\frac{T}{f}\right) = max\left\{k_f \frac{n}{f} + 1, \dots \dots . ., k_{\left(\frac{n}{f}-1\right)f} \frac{n}{f} + \frac{n}{f} - 1\right\} - \frac{n}{f}$.

Example 2.10: *Let T be a numerical semigroup and n be a nonzero element of T and let f be a divisor of n.*

T= $\langle 5,7,9\rangle = \{0,5,7,9,10,12,14,\rightarrow\}$ and for $f = 2$,

$\frac{T}{2} = \{x \in \mathbb{N} : \ 2x \in T\} = \{0,5,6,7,8, \rightarrow\}$.

We will use this formula to find the Apery set, $n_1 = 5$, $n_2 = 7$, $n_3 = 9$

For $n_1 = 5$ Ap= $\left(\frac{T}{f}, \frac{n}{f}\right) = \left\{0, k_2 \frac{5}{2} + 1, \dots \dots, k_{\left(\frac{5}{2}-1\right)2} \frac{5}{2} + \frac{5}{2} - 1\right\} = \{0,6,7,8,9\}$

$k_2 \frac{5}{2} + 1 = 2.\frac{5}{2} + 1 = 6$

$k_{\left(\frac{5}{2}-1\right)2} \frac{5}{2} + \frac{5}{2} - 1 = \frac{3}{2} .2. \frac{5}{2} + \frac{3}{2} = \frac{15}{2} + \frac{3}{2} = \frac{18}{2} = 9$

$F\left(\frac{T}{2}\right) = max\left\{k_2 \frac{5}{2} + 1, \dots \dots, k_{\left(\frac{5}{2}-1\right)2} \frac{5}{2} + \frac{5}{2} - 1\right\} - \frac{5}{2}$

$F\left(\frac{T}{2}\right) = \{0,6,7,8,9\}$

$F\left(\frac{T}{2}\right) = 9$

3. RF-matrices the quotient of numerical semigroup

For a numerical semigroup T, set

$2T = \{2t \mid t \in T\}$.

This set is a submonoid of ℕ. Moreover , $2\langle n_1, \dots, n_p\rangle = \langle 2n_1, \dots, 2n_p\rangle$.

Theorem 3.1. Let $T = \langle n_1, \dots, n_p\rangle \ with \ PF(T) = \{f_1, \dots, f_l\}$. Let f be an odd integer such that $f - f_i - f_j \in T \ for \ all$

$i, j \in \{1, \dots, l\}$. Then

$L = \langle 2n_1, 2n_2, \dots, 2n_p, f - 2f_1, \dots, f - 2f_t\rangle$

Is a symmetric numerical semigroup with Frobenius number $f \ and \ T = \frac{L}{2}$. Moreover,

$L = 2T \cup (\{f - 2f_1, \dots, f - 2f_l\} + 2T)$.

Proof . See [9].

Definition 3.2: PF(T) is the set of Pseudo-Frobenius number of T,

$PF(T) = \left\{x \notin T \mid x + t \in T, for \ every \ t \in T \backslash \{0\}\right\}$

$= \{x \notin T \mid x + n_i \ \in T, for \ every \ i = 1,2, \dots, e\}$.

Corollary 3.3. Let T be a numerical semigroup. Then there exist infinitely many symmetric numerical semigroups L such that $T = \frac{L}{2}$.

Proof: Assume that $PF(T) = \{f_1, \dots, f_l\}. Choose \ an \ odd \ integer \ f$ greater than or equal to $3F(T) + 1$.

For $i, j \in \{1, \dots, l\}, f - f_i - f_j \geq 3F(T) + 1 - F(T) - F(T) = F(T) + 1$. Thus $f - f_i - f_j \in T$. From Theorem 3.1, we know that

$L_f = 2T \cup (\{f - 2f_1, \dots, f - 2f_l\} + 2T)$ is a symmetric numerical semigroup with Frobenius number f and such that $T = \frac{L_f}{2}$.

The proof now follows by observing that we can choose infinitely many odd numbers greater than or equal to $3F(T) + 1$, and that for each of them we obtain a different L_f .

Definition 3.4: If $f = PF(T), then \ f + n_i \in T$, for every $i = 1, \dots \dots, e$, hence there exist $a_{i1}, a_{i2}, \dots \dots, a_{ie} \in \mathbb{N}$ such that

$f + n_t = \sum_{j=1}^{e} a_{ij} \, n_j$.

Nevertheless , $a_{ii} > 0$ would imply $f \in T$; $thus \ a_{ii} = 0$. Thus, for every i,there exist $a_{i1}, a_{i2}, \dots, a_{ie} \in \mathbb{N}$ such that

$f = \sum_{j=1}^{e} a_{ij} \, n_j \ and \ a_{ii} = -1$.

Let $T = \langle n_1, n_2, \dots, n_e\rangle$ be a numerical semigroup and $f \in PF(T)$.

We say that $A = (a_{ij}) \in M_e(\mathbb{Z})$ is an RF-Matrix (Row-Factorization matrix) for $f\ if\ a_{ii} = -1$ for every $i = 1,2, \dots, e, a_{ij} \in \mathbb{N}\ if$

$i \neq j\ and\ for\ every\ i = 1,2, \dots, e$

$\sum_{j=1}^{e} a_{ij}\, n_j = f.$

İf T is almost symmetric and $f \in PF(T) \backslash \{F(T)\}$, there exists an RF-Matrix for both f and $F(T) - f$. İn general , this matrix is not unique.

Example 3.5: Let $T = \langle 6,7,9,10 \rangle = \{0,6,7,9,10,12,13,14,15,16,17, \rightarrow\}$,

$F(T) = 11$,

$G(T) = \mathbb{N} \backslash T = \{1,2,3,4,5,8,11\}$,

$n_1 = 6\ , n_2 = 7\ , n_3 = 9\ , n_4 = 10.$

Let take $f = 8 \in PF(T)$ and let try to write RF-Matrices of f .Firstly ,we find the numbers $a_{12}, a_{13}, a_{14} \in \mathbb{N}$ such that ,

$8 = a_{11}6 + a_{12}7 + a_{13}9 + a_{14}10$,

Where $a_{11} = -1$ from the equality,

$8 = (-1).6 + 2.7 + 0.9 + 0.10$,

We find $a_{12} = 2\ , a_{13} = 0\ , a_{14} = 0.$

Hence ,first row of the RF-Matrices is found for $f = 8PF(T)$.

In a similar manner ,the numbers are found $a_{21} = 1\ , a_{23} = 1\ , a_{24} = 0$ such that

$8 = 1.6 + (-1).7 + 1.9 + 0.10$,

Where $a_{21} = -1$

This gives the second row of the matrix.Go on in this way ,RF-Matrix of $f = 8 \in PF(T)$

$$\begin{pmatrix} -1 & 2 & 0 & 0 \\ 1 & -1 & 1 & 0 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 2 & -1 \end{pmatrix}.$$

Also ,if we consider ,

$8 = 3.6 + 0.7 + 0.9 + (-1).10$.

Then the RF-Matrices of f turns to

$$\begin{pmatrix} -1 & 2 & 0 & 0 \\ 1 & -1 & 1 & 0 \\ 0 & 1 & -1 & 1 \\ 3 & 0 & 0 & -1 \end{pmatrix}.$$

This proves that RF-Matrices are not unique.

Lemma 3.6: *Let T be a numerical semigroup with even Frobenius number. Then*

$F\left(\frac{T}{2}\right) = \frac{F(T)}{2}.$

Proof: Follows by Proposition 2.6., by taking into account the Frobenius number of a numerical semigroup is the maximum of the fundamental gaps.

Example 3.7: Let T be a numerical semigroup with even Frobenius number. $T = \langle 3,5 \rangle = \{0,3,5,6, \rightarrow\}$

The set of gaps of T $G(T) = \mathbb{N} \setminus T = \{1,2,4\}$ and the set of Frobenius numbers of T. $F(T) = max(\mathbb{N} \backslash T) = 4.$

Now let's find half of T with the even Frobenius number of T;

$T/2 = \{x \in \mathbb{N} : 2x \in T\} = \{0,3,4,5,6, \rightarrow\}$

Let's find the set of gaps of half of T;

$G\left(\frac{T}{2}\right) = (\mathbb{N}\backslash T/2) = \{1,2\}$ and $F\left(\frac{T}{2}\right) = max\left(G\left(\frac{T}{2}\right)\right) = 2$

$F\left(\frac{T}{2}\right) = 2\ and\ \frac{F(T)}{2} = \frac{4}{2} = 2$

$F\left(\frac{T}{2}\right) = \frac{F(T)}{2}$

The set of pseudo-Frobienus numbers of T;

$PF(T) = \{f \notin T \mid f + n_i \ \in T, \forall_i = 1,2\}$

$G(T) = \mathbb{N}\backslash T = \{1,2,4\}\ n_1 = 3\ and\ n_2 = 5$

For $f = 1\ .f + n_1 = 1 + 3 = 4 \notin T \Rightarrow f = 1 \notin PF(T)$

For $f = 2\ .f + n_1 = 2 + 3 = 6\ \in T \Rightarrow f = 2 \in PF(T)$

$f + n_2 = 2 + 5 = 7\ \in T \Rightarrow f = 2 \in PF(T)$

For $f = 4. f + n_1 = 4 + 3 = 7 \in T \Rightarrow f = 4 \in PF(T)$

$f + n_2 = 4 + 5 = 9 \in T\ \Rightarrow f = 4 \in PF(T)$

PF(T)= {2,4}.

$Let's\ get\ f = 2 \in PF(T)\ and\ a_{11} = -1$

$2 = a_{11}.3 + a_{12}5$

$2 = (-1).3 + (1).5$

$2 = -3 + 5$

From the equation $a_{11} = -1\ , a_{12} = 1$

And still $a_{22} = -1$

$2 = a_{21}.3 + a_{22}.5$

$2 = \left(\frac{7}{3}\right).3 + (-1).5$

$2 = 7 + (-5)$

From the equation $f = 2 \in PF(T)'s\ RF - matrix.$

$$\begin{pmatrix} -1 & 1 \\ \frac{7}{3} & -1 \end{pmatrix}$$

Similarly $f = 4 \in pf(T)$ .Let's find the RF-Matrix of PF(T).

$4 = a_{11}.3 +\ a_{12}.5\ a_{11} = -1\ , a_{12}\ = \frac{7}{5}$

$4 = (-1).3 + \frac{7}{5}\ .5$

$4 =\ -3 + 7$

$4 =\ a_{21}.3 +\ a_{22}\ .5\ a_{21} = 3\ , a_{22}\ = -1$

$4 = 3.3 + (-1).5$

$4 = 9 + (-5)$

Then the RF-Matrix of $f = 4 \in PF(T)$.

$\begin{pmatrix} -1 & \frac{7}{5} \\ 3 & -1 \end{pmatrix}$

Lemma 3.8: *Let T be a symmetric numerical semigroup. Let* $B = \left\{F(T) + 2k \mid k \in \left\{1, \dots, \frac{F(T)-1}{2}\right\}\right\}$ *.Then*

$L = 2T \cup B \cup \{2F(T) + 1, \rightarrow\}$

Is a pseudo-symmetric numerical semigroup with Frobenius number $2F(T)$ and such that $T = \frac{L}{2}$.

Proof: Since T is symmetric, $F(T)$ is odd. Notice that B is the set of odd integers belonging to the set

$\{F(T) + 2, \dots, 2F(T) - 1\}$ and that $\#B = \frac{F(T)-1}{2}$.

We start by proving that L is a numerical semigroup. In one hand ,it is obvious that the sum of two elements of 2T is an element of 2T and that the result of adding any nonnegative integer to any element in $\{2F(T) + 1, \rightarrow\}$ remains in $\{2F(T) + 1, \rightarrow\}$. On the other hand the sum of elements of B is an element of $\{2F(T) + 1, \rightarrow\}$ .Finally the sum of an element of 2T with an element of B is an element of $B \cup \{2F(T) + 1, \rightarrow\}$. Notice also that since $\{2F(T) + 1, \rightarrow\} \subseteq L$, we have that $\mathbb{N} \backslash L$ is finite.
Now let us prove that $2F(T)\ is\ the\ Frobenius\ number\ of\ L$. As $\{2F(T) + 1, \rightarrow\} \subseteq L$, we only have to show that $2F(T) \notin L$.
But this holds since 2F(T) is an even number and $2F(T) \notin 2T$.Next we will see that L is a pseudo-symmetric numerical semigroup.
And since 2F(T) is the Frobenius number of L, , it suffices to prove that $n(L) = F(T)$. As T is symmetric, we have $n(T) = \frac{F(T)+1}{2}$.
Hence $\#\{x \in 2T \mid x \leq 2F(T)\} = \#\{x \in T \mid x \leq F(T)\} = n(T) = \frac{F(T)+1}{2}$ .Therefore $n(L) = \frac{F(T)+1}{2} + \#\, B = \frac{F(T)+1}{2} + \frac{F(T)-1}{2} = F(T)$.
Finally we prove that $T = \frac{L}{2}$ .We have $x \in \frac{L}{2}$ if and only if $2x \in L$ .Since the elements of B are odd, we obtain $2x \in L$ if and only if $2x \in 2T \cup \{2F(T) + 1, \rightarrow\}$ .If $2x \in 2T$,then trivially $x \in T$. İf $2x \geq 2F(T) + 1, then\ x \geq F(T) + 1$ and thus $x \in T$ .Therefore $x \in \frac{L}{2}$ if and only if $x \in T$.As a consequence of this lemma and Corollary 3.3., we obtain the following.

Theorem 3.9: *Every numerical semigroup is one fourth of infinitely many pseudo-symmetric numerical semigroups.*

Lemma 3.10: *Let T be a pseudo-symmetric numerical semigroup. Let* $B = \left\{F(T) + 2K - 1 \mid k \in \left\{1, \dots, \frac{F(T)}{2}\right\}\right\}$.

Then

$T = 2T \cup B \cup \{2F(T) + 1, \rightarrow\}$

Is a pseudo-symmetric numerical semigroup with Frobenius number 2F(T) and such that $T = \frac{L}{2}$.
Proof: Since T is pseudo-symmetric, F(T) is even. Notice that B is the set of odd integers belonging to the set $\{F(T) + 1, \dots, 2F(T) - 1\}\ and\ that\ \#\, B = \frac{F(T)}{2}$. The proof that L is a numerical semigroup with Frobenius number 2F(T) and that $T = \frac{L}{2}$ is similar to the one performed in Lemma 3.8.
Let us see that L is pseudo-symmetric. The fact that 2F(T) is the Frobenius number of L , it suffices to prove that $n(L) = F(T)$.
Since T is a pseudo-symmetric numerical semigroup and that $n(T) = \frac{F(T)}{2}$.
Hence $\#\{x \in 2T \mid x \leq 2F(T)\} = \#\{x \in T \mid x \leq F(T)\} = n(T) = \frac{F(T)}{2}$. Therefore $n(L) = \frac{F(T)}{2} + \#\, B = \frac{F(T)}{2} + \frac{F(T)}{2} = F(T)$.

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