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Research paper

Generalized Hyers-Ulam-Rassisa Stability of An Additive (β_1, β_2)-Functional Inequalities With n- Variables In Complex Banach Space

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Abstract

In this paper we study to solve the additive (β_1, β_2)-functional inequality with $n-$variables and their Hyers-Ulam stability. First are investigated in complex Banach spaces with a fixed point method and last are investigated in complex Banach spaces with a direct method. I will show that the solutions of the additive (β_1, β_2)-functional inequality are additive mapping. Then $Hyers-Ulam$ stability of these equation are given and proven. These are the main results of this paper .

Keywords: *Additive (β_1, β_2)-Functional Inequality; Fixed Point Method; Direct Method; Banach Space; $Hyers-Ulam$ Stability.*
Mathematics Subject Classification : $46S10$, $39B62$, $39B52$, $47H10$,

1. Introduction

Let $\mathbf{X}$ and $\mathbf{Y}$ be normed spaces on the same field $\mathbb{K}$, and $f : \mathbf{X} \to \mathbf{Y}$. We use the notation $\|\cdot\|$ for all the norms on both $\mathbf{X}$ and $\mathbf{Y}$. In this paper, we investisgate additive (β_1, β_2)-functional inequality when $\mathbf{X}$ is real or complex normed space and $\mathbf{Y}$ a complex Banach space. We solve and prove the Hyers-Ulam stability of forllowing additive (β_1, β_2)-functional inequality.

$$\begin{aligned}
&\left\|2f\left(\frac{x_1+x_2}{2}+\frac{x_3+x_4+\ldots+x_k}{4}\right)-f(x_1)-f\left(x_2+\frac{x_3+x_4+\ldots+x_k}{2}\right)\right\|_{\mathbf{Y}}\\
&\leq\left\|\beta_1\left(f\left(x_1+x_2+\frac{x_3+x_4+\ldots+x_k}{2}\right)+f\left(x_1-x_2-\frac{x_3+x_4+\ldots+x_k}{2}\right)-2f(x_1)\right)\right\|_{\mathbf{Y}}\\
&+\left\|\beta_2\left(f\left(x_1+x_2+\frac{x_3+x_4+\ldots+x_k}{2}\right)-f(x_1)-f\left(x_2+\frac{x_3+x_4+\ldots+x_k}{2}\right)\right)\right\|_{\mathbf{Y}} \qquad (1)
\end{aligned}$$

In which β_1, β_2 are fixed nonzero complex numbers with $\sqrt{2}\left|\beta_1\right|+\left|\beta_2\right|<1$.
Note that in the preliminaries we just recap some of the most essential properties for the above problem and for the specific problem, please see the document. The Hyers-Ulam stability was first investigated for

functional equation of Ulam in [28] concerning the stability of group homomorphisms.

The functional equation

$$f(x+y) = f(x) + f(y)$$

is called the Cauchy equation. In particular, every solution of the Cauchy equation is said to be an additive mapping.
The Hyers [13] gave first affirmative partial answer to the equation of Ulam in Banach spaces. After that, Hyers'Theorem was generalized by Aoki[1] additive mappings and by Rassias [25] for linear mappings considering an unbouned Cauchy diffrence. Ageneralization of the Rassias theorem was obtained by Găvruta [10] by replacing an unbounded Cauchy difference with a general control function in the spirit of Rassias' approach.
The stability of quadratic functional equation was proved by Skof [27] for mappings $f : X \to Y$, where X is a normed space and Y is a Banach space. Park [24], [25] defined additive γ -functional inequalities and proved the HyersUlam stability of the additive γ -functional inequalities in Banach spaces and nonArchimedean Banach spaces. The stability problems of various functional equations have been extensively investigated by a number of authors on the world. We recall a fundamental result in fixed point theory. Recently, in [3], [4], [21], [22], [24], [25] the authors studied the Hyers-Ulam stability for the following functional inequalities

$$\left\|f\left(\frac{x+y}{2}+z\right)-f\left(\frac{x+y}{2}\right)-f(z)\right\| \leq \left\|f\left(\frac{x+y}{2^2}+\frac{z}{2}\right)-\frac{1}{2}f\left(\frac{x+y}{2}\right)-\frac{1}{2}f(z)\right\| \tag{2}$$

$$\left\|f\left(\frac{x+y}{2^2}+\frac{z}{2}\right)-\frac{1}{2}f\left(\frac{x+y}{2}\right)-\frac{1}{2}f(z)\right\| \leq \left\|f\left(\frac{x+y}{2}+z\right)-f\left(\frac{x+y}{2}\right)-f(z)\right\| \tag{3}$$

$$\left\|f(x+y)-f(x)-f(y)\right\| \leq \left\|\rho\left(2f\left(\frac{x+y}{2}\right)-f(x)-f(y)\right)\right\| \tag{4}$$

$$\left\|2f\left(\frac{x+y}{2}\right)-f(x)-f(y)\right\| \leq \left\|\rho\left(f(x+y)-f(x)-f(y)\right)\right\| \tag{5}$$

and

$$\begin{aligned}&\left\|f\left(\frac{x+y}{2}+z\right)+f\left(\frac{x+y}{2}-z\right)-2f\left(\frac{x+y}{2}\right)-2f(z)\right\|\\ \leq\ &\left\|\beta\left(2f\left(\frac{x+y}{2^2}+\frac{z}{2}\right)+2f\left(\frac{x+y}{2^2}-\frac{z}{2}\right)-f\left(\frac{x+y}{2}\right)-f(z)\right)\right\|\end{aligned} \tag{6}$$

$$\begin{aligned}&\left\|2f\left(\frac{x+y}{2^2}+\frac{z}{2}\right)+2f\left(\frac{x+y}{2^2}-\frac{z}{2}\right)-f\left(\frac{x+y}{2}\right)-f(z)\right\|\\ \leq\ &\left\|\beta\left(f\left(\frac{x+y}{2}+z\right)+f\left(\frac{x+y}{2}-z\right)-2f\left(\frac{x+y}{2}\right)-2f(z)\right)\right\|\end{aligned} \tag{7}$$

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$$\left\|f(x+y)-f(x)-f(y)\right\| \leq \left\|\beta_1\left(f(x+y)+f(x-y)-2f(x)\right)\right\| + \left\|\beta_2\left(2f\left(\frac{x+y}{2}\right)-f(x)-f(y)\right)\right\| \tag{8}$$

in complex Banach spaces

In this paper, we solve and proved the Hyers-Ulam stability for (β_1,β_2)-functional inequalities (1), ie the (β_1,β_2) -functional inequalities with three variables. Under suitable assumptions on spaces X and Y, we will prove that the mappings satisfy the (β_1,β_2)-functional inequatility (1). Thus, the results in this paper are generalization of those in [3], [4], [14], [21] for (β_1,β_2)-functional inequatilies with three variables. The paper is organized as followns: In section preliminaries we remind some basic notations in [3], [7] such as complete generalized metric space and Solutions of the inequalities.
Section 3: In this section, I use the method of the fixed to prove the Hyers-Ulam stability of the addive (β_1,β_2)- functional inequalities (1) when X is a real or complete normed space and Y complex Banach space.
Section 4: In this section, I use the method of directly determining the solution for (1) when X is a real or complete normed space and Y complex Banach space.

2. preliminaries

2.1. Complete Generalized Metric space And Solutions of the Iinequalities

Theorem 1. Let (X,d) be a complete generalized metric space and let $J : X \to X$ is a strictly contractive mapping with Lipschitz constant $L < 1$. Then for each given element $x \in X$, either

$$d\left(J^n,J^{n+1}\right)=\infty$$

for all nonegative integers n or there exists a positive integer n_0 such that

1. $d\left(J^n,J^{n+1}\right)<\infty,\ \forall n \geq n_0$;
2. The sequence $\left\{J^n x\right\}$ converges to a fixed point y^* of J;
3. y^* is the unique fixed point of J in the set $Y=\left\{y \in X | d\left(J^n,J^{n+1}\right)<\infty\right\}$;
4. $d\left(y,y^*\right) \leq \frac{1}{1-l} d\left(y,Jy\right)\ \forall y \in Y$

2.2. Solutions of the Inequalities.

The functional equation

$$f(x+y)=f(x)+f(y)$$

is called the cauchuy equation. In particular, every solution of the Cauchuy equation is said to be an *additive mapping*.

3. Establish the Solution of the Additive (β_1,β_2)-Function Inequalities Using a Fixed Point Method

Now, we first study the solutions of (1). Note that for these inequalities, when **X** is a real or complete normed space and **Y** complex Banach space.

Lemma 2. A mapping $f:\mathbf{X}\to\mathbf{Y}$ satisfies $f\left(0\right)=0$ and

$$\left\|2f\left(\frac{x_1+x_2}{2}+\frac{x_3+x_4+\ldots+x_k}{4}\right)-f\left(x_1\right)-f\left(x_2+\frac{x_3+x_4+\ldots+x_k}{2}\right)\right\|_{\mathbf{Y}}$$
$$\leq\left\|\beta_1\left(f\left(x_1+x_2+\frac{x_3+x_4+\ldots+x_k}{2}\right)+f\left(x_1-x_2-\frac{x_3+x_4+\ldots+x_k}{2}\right)-2f\left(x_1\right)\right)\right\|_{\mathbf{Y}}$$
$$+\left\|\beta_2\left(f\left(x_1+x_2+\frac{x_3+x_4+\ldots+x_k}{2}\right)-f\left(x_1\right)-f\left(x_2+\frac{x_3+x_4+\ldots+x_k}{2}\right)\right)\right\|_{\mathbf{Y}} \tag{9}$$

for all $x_j\in\mathbf{X}$, $j=1\to n$, then $f:\mathbf{X}\to\mathbf{Y}$ is additive

Proof. Assume that $f:\mathbf{X}\to\mathbf{Y}$ satisfies (9)
Replacing $\left(x_1,\ldots,x_n\right)$ by $\left(x,0,0\ldots,0\right)$ in (9), we get

$$\left\|2f\left(\frac{x}{2}\right)-f\left(x\right)\right\|_{\mathbf{Y}}\leq 0$$

and so $2f\left(\frac{x}{2}\right)=f\left(x\right)$ for all $x\in\mathbf{X}$.
Thus

$$f\left(\frac{x}{2}\right)=\frac{1}{2}f\left(x\right) \tag{10}$$

for all $x\in\mathbf{X}$ It follows from (9) and (10) that

$$\left\|f\left(x_1+x_2+\frac{x_3+\ldots+x_n}{2}\right)-f\left(x_1\right)-f\left(x_2+\frac{x_3+\ldots+x_n}{2}\right)\right\|_{\mathbf{Y}}$$
$$=\left\|2f\left(\frac{x_1+x_2}{2}+\frac{x_3+x_4+\ldots+x_k}{4}\right)-f\left(x_1\right)-f\left(x_2+\frac{x_3+x_4+\ldots+x_k}{2}\right)\right\|_{\mathbf{Y}}$$
$$\leq\left\|\beta_1\left(f\left(x_1+x_2+\frac{x_3+x_4+\ldots+x_k}{2}\right)+f\left(x_1-x_2-\frac{x_3+x_4+\ldots+x_k}{2}\right)-2f\left(x_1\right)\right)\right\|_{\mathbf{Y}}$$
$$+\left\|\beta_2\left(f\left(x_1+x_2+\frac{x_3+x_4+\ldots+x_k}{2}\right)-f\left(x_1\right)-f\left(x_2+\frac{x_3+x_4+\ldots+x_k}{2}\right)\right)\right\|_{\mathbf{Y}} \tag{11}$$

for all $x_j\in\mathbf{X}$, $j=1\to n$ and so

$$\left(1-\left|\beta_2\right|\right)\left\|f\left(x_1+x_2+\frac{x_3+\ldots+x_n}{2}\right)-f\left(x_1\right)-f\left(x_2+\frac{x_3+\ldots+x_n}{2}\right)\right\|_{\mathbf{Y}}$$
$$\leq\left\|\beta_1\left(f\left(x_1+x_2+\frac{x_3+x_4+\ldots+x_k}{2}\right)+f\left(x_1-x_2-\frac{x_3+x_4+\ldots+x_k}{2}\right)-2f\left(x_1\right)\right)\right\|_{\mathbf{Y}} \tag{12}$$

Next we letting $u=x_1+x_2+\frac{x_3+\ldots+x_n}{2}, v=x_1-x_2-\frac{x_3+x_4+\ldots+x_k}{2}$ in (12), we get

$$\left(1-\left|\beta_2\right|\right)\left\|f\left(u\right)-f\left(\frac{u+v}{2}\right)-f\left(\frac{u-v}{2}\right)\right\|_{\mathbf{Y}}$$
$$\leq\left|\beta_1\right|\left\|f\left(u\right)+f\left(v\right)-2f\left(\frac{u+v}{2}\right)\right\|_{\mathbf{Y}} \tag{13}$$

for all $u,v\in\mathbf{X}$
and so

$$\frac{1}{2}\left(1-\left|\beta_2\right|\right)\left\|f\left(u+v\right)+f\left(u-v\right)-2f\left(u\right)\right\|_{\mathbf{Y}}$$
$$\leq\left|\beta_1\right|\left\|f\left(u+v\right)-f\left(u\right)-f\left(v\right)\right\|_{\mathbf{Y}} \tag{14}$$

for all $u,v\in\mathbf{X}$ It follows from (12) and (14) that

$$\frac{1}{2}\left(1-\left|\beta_2\right|\right)^2\left\|f(x_1+x_2+\frac{x_3+\ldots+x_n}{2})-f\left(x_1\right)-f(x_2+\frac{x_3+\ldots+x_n}{2})\right\|_{\mathbf{Y}}$$
$$\leq\left|\beta_1\right|^2\left\|f(x_1+x_2+\frac{x_3+\ldots+x_n}{2})-f\left(x_1\right)-f(x_2+\frac{x_3+\ldots+x_n}{2})\right\|_{\mathbf{Y}} \tag{15}$$

Since $\sqrt{2}\left|\beta_1\right|+\left|\beta_2\right|<1$
and so

$$f(x_1+x_2+\frac{x_3+\ldots+x_n}{2})=f\left(x_1\right)+f(x_2+\frac{x_3+\ldots+x_n}{2})$$

. for all $x_j\in\mathbf{X}, j=1\to n$. Thus f is additive. □

Theorem 3. Suppose $\varphi:\mathbf{X}^n\to\left[0,\infty\right)$ be a function such that there exists an $L<1$ with

$$\varphi\left(x_1,x_2,\ldots,x_n\right)\leq 2L\varphi\left(\frac{x_1}{2},\frac{x_2}{2},\ldots,\frac{x_n}{2}\right) \tag{16}$$

for all $x,y,z\in\mathbf{X}$. If $f:\mathbf{X}\to\mathbf{Y}$ be a mapping satisfy $f\left(0\right)=0$ and

$$\left\|2f(\frac{x_1+x_2}{2}+\frac{x_3+x_4+\ldots+x_k}{4})-f\left(x_1\right)-f\left(x_2+\frac{x_3+x_4+\ldots+x_k}{2}\right)\right\|_{\mathbf{Y}}$$
$$\leq\left\|\beta_1\left(f\left(x_1+x_2+\frac{x_3+x_4+\ldots+x_k}{2}\right)+f\left(x_1-x_2-\frac{x_3+x_4+\ldots+x_k}{2}\right)-2f\left(x_1\right)\right)\right\|_{\mathbf{Y}}$$
$$+\left\|\beta_2\left(f\left(x_1+x_2+\frac{x_3+x_4+\ldots+x_k}{2}\right)-f\left(x_1\right)-f\left(x_2+\frac{x_3+x_4+\ldots+x_k}{2}\right)\right)\right\|_{\mathbf{Y}}$$
$$+\varphi\left(x_1,x_2,\ldots,x_n\right) \tag{17}$$

for all $x_j\in\mathbf{X}, j=1\to n$.
Then there exists a unique mapping $\psi:\mathbf{X}\to\mathbf{Y}$ such that

$$\left\|f\left(x\right)-\psi\left(x\right)\right\|_{\mathbb{Y}}\leq\frac{1}{\left(1-L\right)}\varphi\left(x,0,\ldots,0\right) \tag{18}$$

for all $x\in\mathbf{X}$

Proof. Replacing $(x_1,x_2,...,x_n)$ by $(x,0,...,0)$ in (17), we get

$$\left\|2f\left(\frac{x}{2}\right)-f(x)\right\|_{\mathbb{Y}} \leq \varphi(x,0,...,0) \tag{19}$$

for all $x \in \mathbf{X}$.
Consider the set

$$\mathbb{S} := \left\{h : \mathbf{X} \to \mathbf{Y}, h(0) = 0\right\}$$

and introduce the generalized metric on $\mathbb{S}$:

$$d\left(g,h\right) := inf\left\{\lambda \in \mathbb{R} : \left\|g(x)-h(x)\right\| \leq \lambda\varphi(x,0,...,0), \forall x \in \mathbf{X}\right\},$$

where, as usual, $inf\phi = +\infty$. It is easy to show that $\left(\mathbb{S},d\right)$ is complete $\left(see[16]\right)$ Now we cosider the linear mapping $J : \mathbb{S} \to \mathbb{S}$ such that

$$Jg(x) := 2g\left(\frac{x}{2}\right)$$

for all $x \in \mathbf{X}$. Let $g,h \in \mathbb{S}$ be given such that $d(g,h) = \varepsilon$. Then

$$\left\|g(x)-h(x)\right\| \leq \varepsilon\varphi(x,0,...,0)$$

for all $x \in \mathbf{X}$.
Hence

$$\begin{aligned}\left\|Jg(x)-Jh(x)\right\| &= \left\|2g\left(\frac{x}{2}\right)-2hf\left(\frac{x}{2}\right)\right\| \leq 2\varepsilon\varphi\left(\frac{x}{2},0,...,0\right)\\ &\leq 2\varepsilon\frac{L}{2}\varphi(x,0,...,0) \leq L\varepsilon\varphi(x,0,...,0)\end{aligned}$$

for all $x \in \mathbf{X}$. So $d(g,h) = \varepsilon$ implies that $d\left(Jg,Jh\right) \leq L\cdot\varepsilon$. This means that

$$d\left(Jg,Jh\right) \leq Ld\left(g,h\right)$$

for all $g,h \in \mathbb{S}$ It folows from (19) that

$$d\left(f,Jf\right) \leq 1.$$

By Theorem 1, there exists a mapping $\psi : \mathbf{X} \to \mathbf{Y}$ satisfying the fllowing:

1. ψ is a fixed point of J, ie.,

$$\psi(x) = 2\psi\left(\frac{x}{2}\right) \tag{20}$$

for all $x \in \mathbf{X}$. The mapping ψ is a unique fixed point J in the set

$$\mathbb{M} = \left\{g \in \mathbb{S} : d\left(f,g\right) < \infty\right\}$$

This implies that ψ is a unique mapping satisfying (20) such that there exists a $\lambda \in \left(0,\infty\right)$ satisfying

$$\left\|f(x)-\psi(x)\right\| \leq \lambda\varphi(x,0,...,0)$$

for all $x \in \mathbf{X}$

2. $d\Big(J^l f,\psi\Big)\to 0$ as $l\to\infty$. This implies equality

$$\lim_{l\to\infty} 2^n f\left(\frac{x}{2^n}\right)=\psi(x)$$

for all $x\in\mathbb{X}$
3. $d\Big(f,\psi\Big)\leq\frac{1}{1-L}d\Big(f,Jf\Big)$. which implies

$$\Big\|f(x)-\psi(x)\Big\|\leq\frac{1}{1-L}\varphi(x,0,...,0)$$

for all $x\in X$. It follows (16) and (17) that

$$\begin{aligned}
&\left\|2f\Big(\frac{x_1+x_2}{2}+\frac{x_3+x_4+...+x_k}{4}\Big)-f(x_1)-f\Big(x_2+\frac{x_3+x_4+...+x_k}{2}\Big)\right\|_{\mathbf{Y}}\\
&=\lim_{n\to\infty}2^n\left\|2f\big(\frac{x_1+x_2}{2^{n+1}}+\frac{x_3+x_4+...+x_n}{2^{n+2}}\big)-f\big(\frac{x_1}{2^n}\big)-f\big(\frac{x_2}{2^n}+\frac{x_3+x_4+...+x_n}{2^{n+1}}\big)\right\|_{\mathbf{Y}}\\
&\leq\lim_{n\to\infty}2^n|\beta_1|\left\|f\big(\frac{x_1+x_2}{2^n}+\frac{x_3+x_4+...+x_n}{2^{n+1}}\big)-f\big(\frac{x_1-x_2}{2^n}-\frac{x_3+x_4+...+x_n}{2^{n+1}}\big)-2f\big(\frac{x_1}{2n}\big)\right\|_{\mathbf{Y}}\\
&+\lim_{n\to\infty}2^n|\beta_2|\left\|f\big(\frac{x_1+x_2}{2^n}+\frac{x_3+x_4+...+x_n}{2^{n+1}}\big)-f\big(\frac{x_1}{2^n}\big)-f\big(\frac{x_2}{2^n}+\frac{x_3+x_4+...+x_n}{2^{n+1}}\big)\right\|_{\mathbf{Y}}\\
&+\lim_{n\to\infty}2^n\varphi\big(\frac{x_1}{2^n},\frac{x_2}{2^n},...,\frac{x_n}{2^n}\big)\\
&=\left\|\beta_1\Big(\psi\Big(x_1+x_2+\frac{x_3+x_4+...+x_k}{2}\Big)+\psi\Big(x_1-x_2-\frac{x_3+x_4+...+x_k}{2}\Big)-2\psi(x_1)\Big)\right\|_{\mathbf{Y}}\\
&+\left\|\beta_2\Big(\psi\Big(x_1+x_2+\frac{x_3+x_4+...+x_k}{2}\Big)-\psi(x_1)-\psi\Big(x_2+\frac{x_3+x_4+...+x_k}{2}\Big)\Big)\right\|_{\mathbf{Y}}
\end{aligned}\tag{21}$$

for all $x_j\in\mathbf{X}$, $j=1\to n$. So

$$\begin{aligned}
&\left\|2\psi\Big(\frac{x_1+x_2}{2}+\frac{x_3+x_4+...+x_k}{4}\Big)-\psi(x_1)-\psi\Big(x_2+\frac{x_3+x_4+...+x_k}{2}\Big)\right\|_{\mathbf{Y}}\\
&\leq\left\|\beta_1\Big(\psi\Big(x_1+x_2+\frac{x_3+x_4+...+x_k}{2}\Big)+\psi\Big(x_1-x_2-\frac{x_3+x_4+...+x_k}{2}\Big)-2\psi(x_1)\Big)\right\|_{\mathbf{Y}}\\
&+\left\|\beta_2\Big(\psi\Big(x_1+x_2+\frac{x_3+x_4+...+x_k}{2}\Big)-\psi(x_1)-\psi\Big(x_2+\frac{x_3+x_4+...+x_k}{2}\Big)\Big)\right\|_{\mathbf{Y}}
\end{aligned}$$

for all $x_j\in\mathbf{X}$, $j=1\to n$. By Lemma 2, the mapping $\psi:\mathbf{X}\to\mathbf{Y}$ is additive. Ei

$$\psi\Big(x_1+x_2+\frac{x_3+x_4+...+x_k}{2}\Big)-\psi(x_1)-\psi\Big(x_2+\frac{x_3+x_4+...+x_k}{2}\Big)=0$$

□

Theorem 4. Suppose $\varphi : \mathbf{X}^n \to [0,\infty)$ be a function such that there exists an $L<1$ with

$$\varphi\left(\frac{x_1}{2},\frac{x_2}{2},\dots,\frac{x_n}{2}\right) \le \frac{L}{2}\varphi\left(x_1,x_2,\dots,x_n\right) \tag{22}$$

for all $x,y,z \in \mathbf{X}$. If $f:\mathbf{X}\to\mathbf{Y}$ be a mapping satisfy $f(0)=0$ and

$$\begin{aligned}
&\left\|2f\left(\frac{x_1+x_2}{2}+\frac{x_3+x_4+\dots+x_k}{4}\right)-f(x_1)-f\left(x_2+\frac{x_3+x_4+\dots+x_k}{2}\right)\right\|_{\mathbf{Y}}\\
&\le \left\|\beta_1\left(f\left(x_1+x_2+\frac{x_3+x_4+\dots+x_k}{2}\right)+f\left(x_1-x_2-\frac{x_3+x_4+\dots+x_k}{2}\right)-2f(x_1)\right)\right\|_{\mathbf{Y}}\\
&+\left\|\beta_2\left(f\left(x_1+x_2+\frac{x_3+x_4+\dots+x_k}{2}\right)-f(x_1)-f\left(x_2+\frac{x_3+x_4+\dots+x_k}{2}\right)\right)\right\|_{\mathbf{Y}}\\
&+\varphi\left(x_1,x_2,\dots,x_n\right)
\end{aligned} \tag{23}$$

for all $x_j \in \mathbf{X}$, $j=1\to n$.
Then there exists a unique mapping $\psi:\mathbf{X}\to\mathbf{Y}$ such that

$$\left\|f(x)-\psi(x)\right\|_{\mathbf{Y}} \le \frac{L}{(1-L)}\varphi(x,0,\dots,0) \tag{24}$$

for all $x\in\mathbf{X}$.

Proof. Replacing $(x_1,x_2,\dots,x_n)$ by $(x,0,\dots,0)$ in (23), we get

$$\left\|2f\left(\frac{x}{2}\right)-f(x)\right\|_{\mathbf{Y}} \le \varphi(x,0,\dots,0) \tag{25}$$

for all $x\in\mathbf{X}$.
So

$$\left\|f(x)-\frac{1}{2}f(2x)\right\|_{\mathbf{Y}} \le \frac{1}{2}\varphi(2x,0,\dots,0) \tag{26}$$

for all $x\in\mathbf{X}$.
Suppose $(\mathbb{S},d)$ be the generalized metric space defined in the proof of Theeorem 3.2 Now we cosider the linear mapping $J:\mathbb{S}\to\mathbb{S}$ such that

$$Jg(x) := \frac{1}{2}g\left(2x\right)$$

for all $x\in\mathbf{X}$. That It follows from (26)

$$\left\|f(x)-\frac{1}{2}f(2x)\right\|_{\mathbf{Y}} \le \frac{1}{2}\varphi(2x,0,\dots,0) \le L\varphi(x,0,\dots,0)$$

The rest of the proof is similar to proof of Theorem 3. □

From proving the theorems we have consequences:

Corollary 1. Let $r > 1$ and θ be nonnegative real numbers and let $f : \mathbf{X} \to \mathbf{Y}$ be a mapping satisfy $f(0) = 0$ and

$$\begin{aligned}
&\left\|2f\left(\frac{x_1+x_2}{2}+\frac{x_3+x_4+\ldots+x_k}{4}\right)-f(x_1)-f\left(x_2+\frac{x_3+x_4+\ldots+x_k}{2}\right)\right\|_{\mathbf{Y}}\\
&\quad\le\left\|\beta_1\left(f\left(x_1+x_2+\frac{x_3+x_4+\ldots+x_k}{2}\right)+f\left(x_1-x_2-\frac{x_3+x_4+\ldots+x_k}{2}\right)-2f(x_1)\right)\right\|_{\mathbf{Y}}\\
&\quad+\left\|\beta_2\left(f\left(x_1+x_2+\frac{x_3+x_4+\ldots+x_k}{2}\right)-f(x_1)-f\left(x_2+\frac{x_3+x_4+\ldots+x_k}{2}\right)\right)\right\|_{\mathbf{Y}}\\
&\quad+\theta\left(\left\|x_1\right\|^r+\left\|x_2\right\|^r+\ldots+\left\|x_k\right\|^r\right)
\end{aligned}\tag{27}$$

for all $x_j \in \mathbf{X}$ for all $j = 1 \to k$.
Then there exists a unique mapping $\psi : \mathbf{X} \to \mathbf{Y}$ such that

$$\left\|f(x)-\psi(x)\right\|_{\mathbf{Y}}\le\frac{2^r\theta}{2^r-2}\left\|x\right\|_{\mathbf{X}}^r\tag{28}$$

for all $x \in \mathbf{X}$.

Corollary 2. Let $r < 1$ and θ be nonnegative real numbers and let $f : \mathbf{X} \to \mathbf{Y}$ be a mapping satisfy $f(0) = 0$ and

$$\begin{aligned}
&\left\|2f\left(\frac{x_1+x_2}{2}+\frac{x_3+x_4+\ldots+x_k}{4}\right)-f(x_1)-f\left(x_2+\frac{x_3+x_4+\ldots+x_k}{2}\right)\right\|_{\mathbf{Y}}\\
&\quad\le\left\|\beta_1\left(f\left(x_1+x_2+\frac{x_3+x_4+\ldots+x_k}{2}\right)+f\left(x_1-x_2-\frac{x_3+x_4+\ldots+x_k}{2}\right)-2f(x_1)\right)\right\|_{\mathbf{Y}}\\
&\quad+\left\|\beta_2\left(f\left(x_1+x_2+\frac{x_3+x_4+\ldots+x_k}{2}\right)-f(x_1)-f\left(x_2+\frac{x_3+x_4+\ldots+x_k}{2}\right)\right)\right\|_{\mathbf{Y}}\\
&\quad+\theta\left(\left\|x_1\right\|^r+\left\|x_2\right\|^r+\ldots+\left\|x_k\right\|^r\right)
\end{aligned}\tag{29}$$

for all $x_j \in \mathbf{X}$ for all $j = 1 \to k$.
Then there exists a unique mapping $\psi : \mathbf{X} \to \mathbf{Y}$ such that

$$\left\|f(x)-\psi(x)\right\|_{\mathbf{Y}}\le\frac{2^r\theta}{2-2^r}\left\|x\right\|_{\mathbf{X}}^r\tag{30}$$

for all $x \in \mathbf{X}$.

4. Establish the Solution of the Additive (β_1, β_2)-Function Inequalities Using a Direct Method

Next, we study the solutions of (1) . Note that for these inequalities, when $\mathbf{X}$ be a real or complete normed space and $\mathbf{Y}$ complex Banach space.

Theorem 5. Suppose $\varphi : \mathbf{X}^n \to [0,\infty)$ be a function and let $f : \mathbf{X} \to \mathbf{Y}$ be a mapping such that

$$\phi(x_1,x_2,\ldots,x_n) := \sum_{j=1}^{\infty} 2^j \varphi\left(\frac{x_1}{2^j},\frac{x_2}{2^j},\ldots,\frac{x_n}{2^j}\right) < \infty\tag{31}$$

and let $f : \mathbf{X} \to \mathbf{Y}$ be a mapping $f(0) = 0$ and

$$\left\|2f\left(\frac{x_1+x_2}{2}+\frac{x_3+x_4+\ldots+x_k}{4}\right)-f(x_1)-f\left(x_2+\frac{x_3+x_4+\ldots+x_k}{2}\right)\right\|_{\mathbf{Y}}$$
$$\leq \left\|\beta_1\left(f\left(x_1+x_2+\frac{x_3+x_4+\ldots+x_k}{2}\right)+f\left(x_1-x_2-\frac{x_3+x_4+\ldots+x_k}{2}\right)-2f(x_1)\right)\right\|_{\mathbf{Y}}$$
$$+\left\|\beta_2\left(f\left(x_1+x_2+\frac{x_3+x_4+\ldots+x_k}{2}\right)-f(x_1)-f\left(x_2+\frac{x_3+x_4+\ldots+x_k}{2}\right)\right)\right\|_{\mathbf{Y}}$$
$$+\varphi(x_1,x_2,\ldots,x_n) \tag{32}$$

for all $x_j \in \mathbf{X}, j = 1 \to n$.
Then there exists a unique mapping $\psi : \mathbf{X} \to \mathbf{Y}$ such that

$$\left\|f(x)-\psi(x)\right\|_{\mathbf{Y}} \leq \varphi(x,0,\ldots,0) \tag{33}$$

for all $x \in \mathbf{X}$.

Proof. Replacing $(x_1,x_2,\ldots,x_n)$ by $(x,0,\ldots,0)$ in (32), we get

$$\left\|2f\left(\frac{x}{2}\right)-f(x)\right\|_{\mathbf{Y}} \leq \varphi(x,x,0,\ldots,0) \tag{34}$$

for all $x \in X$
. Hence

$$\left\|2^l f\left(\frac{x}{2^l}\right)-2^m f\left(\frac{x}{2^m}\right)\right\|_{\mathbf{Y}}$$
$$\leq \sum_{j=l}^{m-1}\left\|2^j f\left(\frac{x}{2^j}\right)-2^{j+1} f\left(\frac{x}{2^{j+1}}\right)\right\|_{\mathbf{Y}}$$
$$\leq \sum_{j=l}^{m-1} 2^j \varphi\left(\frac{x}{2^{j+1}},0,\ldots,0\right) \tag{35}$$

for all nonnegative integers m and l with $m > l$ and all $x \in \mathbf{X}$. It follows from (35) that the sequence $\left\{2^n f\left(\frac{x}{2^n}\right)\right\}$ is a Cauchy sequence for all $x \in \mathbf{X}$. Since $\mathbf{Y}$ is complete, the sequence $\left\{2^n f\left(\frac{x}{2^n}\right)\right\}$ coverges. So one can define the mapping $\psi : \mathbf{X} \to \mathbf{Y}$ by

$$\psi(x) := \lim_{n\to\infty} \frac{1}{2^n} f\left(2^n x\right) \tag{36}$$

for all $x \in \mathbf{X}$. Moreover, letting $l = 0$ and passing the limit $m \to \infty$ in (36), we get (33) It follows from (31) and (32) that

$$\begin{aligned}
&\left\|2f\Big(\frac{x_1+x_2}{2}+\frac{x_3+x_4+\ldots+x_k}{4}\Big)-f(x_1)-f\Big(x_2+\frac{x_3+x_4+\ldots+x_k}{2}\Big)\right\|_{\mathbf{Y}}\\
&=\lim_{n\to\infty}2^n\left\|2f\big(\frac{x_1+x_2}{2^{n+1}}+\frac{x_3+x_4+\ldots+x_n}{2^{n+2}}\big)-f\big(\frac{x_1}{2^n}\big)-f\big(\frac{x_2}{2^n}+\frac{x_3+x_4+\ldots+x_n}{2^{n+1}}\big)\right\|_{\mathbf{Y}}\\
&\leq\lim_{n\to\infty}2^n|\beta_1|\left\|f\big(\frac{x_1+x_2}{2^n}+\frac{x_3+x_4+\ldots+x_n}{2^{n+1}}\big)-f\big(\frac{x_1-x_2}{2^n}-\frac{x_3+x_4+\ldots+x_n}{2^{n+1}}\big)-2f\big(\frac{x_1}{2n}\big)\right\|_{\mathbf{Y}}\\
&+\lim_{n\to\infty}2^n|\beta_2|\left\|f\big(\frac{x_1+x_2}{2^n}+\frac{x_3+x_4+\ldots+x_n}{2^{n+1}}\big)-f\big(\frac{x_1}{2^n}\big)-f\big(\frac{x_2}{2^n}+\frac{x_3+x_4+\ldots+x_n}{2^{n+1}}\big)\right\|_{\mathbf{Y}}\\
&+\lim_{n\to\infty}2^n\varphi\big(\frac{x_1}{2^n},\frac{x_2}{2^n},\ldots,\frac{x_n}{2^n}\big)\\
&=\left\|\beta_1\Big(\psi\Big(x_1+x_2+\frac{x_3+x_4+\ldots+x_k}{2}\Big)+\psi\Big(x_1-x_2-\frac{x_3+x_4+\ldots+x_k}{2}\Big)-2\psi(x_1)\Big)\right\|_{\mathbf{Y}}\\
&+\left\|\beta_2\Big(\psi\Big(x_1+x_2+\frac{x_3+x_4+\ldots+x_k}{2}\Big)-\psi(x_1)-\psi\Big(x_2+\frac{x_3+x_4+\ldots+x_k}{2}\Big)\Big)\right\|_{\mathbf{Y}}
\end{aligned}\tag{37}$$

for all $x_j\in\mathbf{X}$, $j=1\to n$. So

$$\begin{aligned}
&\left\|2\psi\Big(\frac{x_1+x_2}{2}+\frac{x_3+x_4+\ldots+x_k}{4}\Big)-\psi(x_1)-\psi\Big(x_2+\frac{x_3+x_4+\ldots+x_k}{2}\Big)\right\|_{\mathbf{Y}}\\
&\leq\left\|\beta_1\Big(\psi\Big(x_1+x_2+\frac{x_3+x_4+\ldots+x_k}{2}\Big)+\psi\Big(x_1-x_2-\frac{x_3+x_4+\ldots+x_k}{2}\Big)-2\psi(x_1)\Big)\right\|_{\mathbf{Y}}\\
&+\left\|\beta_2\Big(\psi\Big(x_1+x_2+\frac{x_3+x_4+\ldots+x_k}{2}\Big)-\psi(x_1)-\psi\Big(x_2+\frac{x_3+x_4+\ldots+x_k}{2}\Big)\Big)\right\|_{\mathbf{Y}}
\end{aligned}$$

for all $x_j\in\mathbf{X}$, $j=1\to n$. By Lemma 2, the mapping $\psi:\mathbf{X}\to\mathbf{Y}$ is additive. Ei

$$\psi\Big(x_1+x_2+\frac{x_3+x_4+\ldots+x_k}{2}\Big)-\psi(x_1)-\psi\Big(x_2+\frac{x_3+x_4+\ldots+x_k}{2}\Big)=0$$

Now, let $\psi':\mathbb{X}\to\mathbb{Y}$ be another additive mapping satisfying (33). Then we have

$$\begin{aligned}
\left\|\psi(x)-\psi'(x)\right\|&=\left\|2^q\psi\big(\frac{x}{2^q}\big)-2^q\psi'\big(\frac{x}{2^q}\big)\right\|_{\mathbb{Y}}\\
&\leq\left\|2^q\psi\big(\frac{x}{2^q}\big)-2^qf\big(\frac{x}{2^q}\big)\right\|+\left\|2^q\psi'\big(\frac{x}{2^q}\big)-2^qf\big(\frac{x}{2^q}\big)\right\|_{\mathbb{Y}}\\
&\leq 2^{q+1}\phi\Big(\frac{x}{2^q},0,\ldots,0\Big)
\end{aligned}$$

which tends to zero as $q\to\infty$ for all $x\in\mathbf{X}$. So we can conclude that $\psi(x)=\psi'(x)$ for all $x\in\mathbf{X}$. This proves the uniqueness of ψ. □

Theorem 6. Suppose $\varphi:\mathbf{X}^n\to[0,\infty)$ be a function and let $f:\mathbf{X}\to\mathbf{Y}$ be a mapping such that

$$\phi(x_1,x_2,\ldots,x_n):=\sum_{j=1}^{\infty}\frac{1}{2^j}\varphi\Big(2^jx_1,2^jx_2,\ldots,2^jx_n\Big)<\infty\tag{38}$$

and let $f: \mathbf{X} \to \mathbf{Y}$ be a mapping $f(0) = 0$ and

$$\begin{aligned}
&\left\|2f\Big(\frac{x_1+x_2}{2}+\frac{x_3+x_4+\ldots+x_k}{4}\Big)-f(x_1)-f\Big(x_2+\frac{x_3+x_4+\ldots+x_k}{2}\Big)\right\|_{\mathbf{Y}}\\
&\le \left\|\beta_1\Big(f\Big(x_1+x_2+\frac{x_3+x_4+\ldots+x_k}{2}\Big)+f\Big(x_1-x_2-\frac{x_3+x_4+\ldots+x_k}{2}\Big)-2f(x_1)\Big)\right\|_{\mathbf{Y}}\\
&+\left\|\beta_2\Big(f\Big(x_1+x_2+\frac{x_3+x_4+\ldots+x_k}{2}\Big)-f(x_1)-f\Big(x_2+\frac{x_3+x_4+\ldots+x_k}{2}\Big)\Big)\right\|_{\mathbf{Y}}\\
&+\varphi(x_1,x_2,\ldots,x_n) \qquad (39)
\end{aligned}$$

for all $x_j \in \mathbf{X}$, $j = 1 \to n$.
Then there exists a unique mapping $\psi: \mathbf{X} \to \mathbf{Y}$ such that

$$\left\|f(x)-\psi(x)\right\|_{\mathbf{Y}} \le \varphi(x,0,\ldots,0) \qquad (40)$$

for all $x \in \mathbf{X}$.

Proof. Replacing $(x_1,x_2,\ldots,x_n)$ by $(x,0,\ldots,0)$ in (39), we get

$$\left\|2f\Big(\frac{x}{2}\Big)-f(x)\right\|_{\mathbf{Y}} \le \varphi(x,0,\ldots,0) \qquad (41)$$

for all $x \in X$. So

$$\left\|f(x)-\frac{1}{2}f(2x)\right\|_{\mathbf{Y}} \le \frac{1}{2}\varphi(2x,0,\ldots,0) \qquad (42)$$

for all $x \in X$. Hence

$$\begin{aligned}
&\left\|\frac{1}{2^l}f\left(2^l x\right)-\frac{1}{2^m}f\left(2^m x\right)\right\|_{\mathbb{Y}}\\
&\qquad\le \sum_{j=l}^{m-1}\left\|\frac{1}{2^j}f\left(2^j x\right)-\frac{1}{2^{j+1}}f\left(2^{j+1}x\right)\right\|_{\mathbb{Y}}\\
&\qquad\le \sum_{j=l}^{m-1}\frac{1}{2^{j+1}}\varphi(2^j x,0,\ldots,0x) \qquad (43)
\end{aligned}$$

for all nonnegative integers m and l with $m > l$ and all $x \in \mathbf{X}$. It follows from (42) that the sequence $\left\{\frac{1}{2^n}f\left(2^n x\right)\right\}$ is a Cauchy sequence for all $x \in \mathbf{X}$. Since $\mathbf{Y}$ is complete, the sequence $\left\{\frac{1}{2^n}f\left(2^n x\right)\right\}$ coverges. So one can define the mapping $\psi: \mathbf{X} \to \mathbf{Y}$ by

$$\psi(x) := \lim_{n\to\infty}\frac{1}{2^n}f\left(2^n x\right) \qquad (44)$$

Moreover, letting $l = 0$ and passing the limit $m \to \infty$ in (42), we get (40).
The rest of the proof is similar to the proof of theorem 5. □

From proving the theorems we have consequences:

Corollary 3. Let $r > 1$ and θ be nonnegative real numbers and let $f : \mathbf{X} \to \mathbf{Y}$ be a mapping satisfy $f(0) = 0$ and

$$\begin{aligned}&\left\|2f\Big(\frac{x_1+x_2}{2}+\frac{x_3+x_4+\ldots+x_k}{4}\Big)-f(x_1)-f\Big(x_2+\frac{x_3+x_4+\ldots+x_k}{2}\Big)\right\|_{\mathbf{Y}}\\ &\le\left\|\beta_1\Big(f\Big(x_1+x_2+\frac{x_3+x_4+\ldots+x_k}{2}\Big)+f\Big(x_1-x_2-\frac{x_3+x_4+\ldots+x_k}{2}\Big)-2f(x_1)\Big)\right\|_{\mathbf{Y}}\\ &+\left\|\beta_2\Big(f\Big(x_1+x_2+\frac{x_3+x_4+\ldots+x_k}{2}\Big)-f(x_1)-f\Big(x_2+\frac{x_3+x_4+\ldots+x_k}{2}\Big)\Big)\right\|_{\mathbf{Y}}\\ &+\theta\Big(\|x_1\|^r+\|x_2\|^r+\ldots+\|x_n\|^r\Big) \end{aligned} \tag{45}$$

for all $x_j \in \mathbf{X}, j = 1 \to k$.
Then there exists a unique mapping $\psi : \mathbf{X} \to \mathbf{Y}$ such that

$$\|f(x)-\psi(x)\|_{\mathbf{Y}} \le \frac{2^r\theta}{2^r-2}\|x\|^r_{\mathbf{X}} \tag{46}$$

Corollary 4. Let $r < 1$ and θ be nonnegative real numbers and let $f : \mathbf{X} \to \mathbf{Y}$ be a mapping satisfy $f(0) = 0$ and

$$\begin{aligned}&\left\|2f\Big(\frac{x_1+x_2}{2}+\frac{x_3+x_4+\ldots+x_k}{4}\Big)-f(x_1)-f\Big(x_2+\frac{x_3+x_4+\ldots+x_k}{2}\Big)\right\|_{\mathbf{Y}}\\ &\le\left\|\beta_1\Big(f\Big(x_1+x_2+\frac{x_3+x_4+\ldots+x_k}{2}\Big)+f\Big(x_1-x_2-\frac{x_3+x_4+\ldots+x_k}{2}\Big)-2f(x_1)\Big)\right\|_{\mathbf{Y}}\\ &+\left\|\beta_2\Big(f\Big(x_1+x_2+\frac{x_3+x_4+\ldots+x_k}{2}\Big)-f(x_1)-f\Big(x_2+\frac{x_3+x_4+\ldots+x_k}{2}\Big)\Big)\right\|_{\mathbf{Y}}\\ &\theta\Big(\|x_1\|^r+\|x_2\|^r+\ldots+\|x_n\|^r\Big) \end{aligned} \tag{47}$$

for all $x_j \in \mathbb{X}, j = 1 \to k$.
Then there exists a unique mapping $\psi : \mathbf{X} \to \mathbf{Y}$ such that

$$\|f(x)-\psi(x)\|_{\mathbb{Y}} \le \frac{2^r\theta}{2-2^r}\|x\|^r_{\mathbf{X}} \tag{48}$$

for all $x \in \mathbf{X}$.

5. Conclusion

In this paper, I have shown that the solutions of the (β_1, β_2)-functional inequalities are additive mappings. The Hyers-Ulam stability for these given from theorems. These are the main results of the paper , which are the generalization of the results [3], [4], [14], [21].

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