*Global Journal of Mathematical Analysis, 2 (4) (2014) 286-289*
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*www.sciencepubco.com/index.php/GJMA*
*doi: 10.14419/gjma.v2i4.3497*
*Research Paper*

# Commutativity results and continuity of Jordan homomorphism on Banach algebras

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**Abstract**

In this paper we give conditions which entailing commutativity of Banach algebra $\mathcal{A}$ and then we show that under special hypotheses, each Jordan homomorphism $\varphi$ between Banach algebras $\mathcal{A}$ and $\mathcal{B}$ is continuous.

***Keywords***: *Jordan homomorphism, ring homomorphism, commutative.*

## 1. Introduction

Let $\mathcal{A}$ and $\mathcal{B}$ be Banach algebras and $\varphi : \mathcal{A} \longrightarrow \mathcal{B}$ be a linear map. Then $\varphi$ is called Jordan homomorphism if

$$\varphi(ab + ba) = \varphi(a)\varphi(b) + \varphi(b)\varphi(a) \qquad (a, b \in \mathcal{A}),$$

or equivalently, $\varphi(a^2) = \varphi(a)^2$ for all $a \in \mathcal{A}$, [4]. Moreover, if $\varphi$ is multiplicative, that is,

$$\varphi(ab) = \varphi(a)\varphi(b) \qquad (a, b \in \mathcal{A}),$$

then $\varphi$ is called ring homomorphism.

It is obvious that ring homomorphisms are Jordan, but the converse is false, in general. In fact, the converse is true under a certain conditions. For example, each Jordan homomorphism from a commutative Banach algebra $\mathcal{A}$ into $\mathbb{C}$ is a ring homomorphism.

In [5], Zelazko proved that each Jordan homomorphism of Banach algebra $\mathcal{A}$ into a semisimple commutative Banach algebra $\mathcal{B}$ is ring homomorphism. See also [6] for another characterization of this result.

Le Page [1] has shown that a complex unital Banach algebra $\mathcal{A}$ is necessarily commutative if it satisfies the following condition,

$$\|ab\| \leq \|ba\|, \qquad (a, b \in \mathcal{A}).$$

It is known that the Le Page's inequality does not imply commutativity in the non-unital case. A counter-example has been given in [2]. Also it has shown that the Banach algebra $\mathcal{A}$ is commutative, if for all $a \in \mathcal{A}$, $\|a\|^2 \leq \|a^2\|$, see [1] for example.

In this paper we investigate some conditions which entailing commutativity of Banach algebra $\mathcal{A}$ and then we give a sufficient condition that each Jordan homomorphism $\varphi : \mathcal{A} \longrightarrow \mathcal{B}$ to be ring homomorphism.

## 2. Main Result

For Banach algebra $\mathcal{A}$, we denote $J_n(\mathcal{A}) = \{a \in \mathcal{A} : \ a^n = a\}$, and the Banach algebra $\mathcal{A}$ is said to be idempotent if $J_2(\mathcal{A}) = \mathcal{A}$.

**Proposition 2.1** *Every idempotent Banach algebra $\mathcal{A}$ is commutative.*

Proof. Let $a, b$ be arbitrary elements of $\mathcal{A}$. Then

$$a + b = (a + b)^2 = a^2 + b^2 + ab + ba = a + b + ab + ba.$$

Hence, $ab = -ba$. Thus,

$$ab = (ab)^2 = (-ba)^2 = (ba)^2 = ba.$$

Therefore $ab = ba$, and $\mathcal{A}$ is commutative.

**Corollary 2.2** *Let $\mathcal{A}$ be a unital Banach algebra such that $(ab)^2 = a^2b^2$, for all $a, b \in \mathcal{A}$. Then $\mathcal{A}$ is commutative.*

Proof. Let $e$ be a unit element of $\mathcal{A}$, then for all $a, b \in \mathcal{A}$,

$$(a(b + e))^2 = a^2(b + e)^2,$$

which proves that

$$(ab)^2 + aba + a^2b + a^2 = a^2b^2 + 2a^2b + a^2.$$

Thus, $aba = a^2b$, for all $a, b \in \mathcal{A}$. Replacing $a$ by $a + e$ in the last equality, we get

$$(a + e)b(a + e) = (a + e)^2b.$$

Therefore we conclude that $ab = ba$, as required.

The next example shows that the hypothesis that $\mathcal{A}$ is unital in above corollary is essential.

**Example 2.3** *Let $\mathcal{A}$ be a unital Banach algebra and let $\mathcal{B}$ be the Banach algebra of all $2 \times 2$ matrices having $[a \ \ b]$ in the first line and $[0 \ \ 0]$ in the second line, for all $a, b \in \mathcal{A}$. Then $\mathcal{B}$ is not unital, but it is obvious to check that $(xy)^2 = x^2y^2$, for all $x, y \in \mathcal{B}$. However, $\mathcal{B}$ is not commutative.*

**Theorem 2.4** *Let $\mathcal{A}$ be a Banach algebra such that $J_3(\mathcal{A}) = \mathcal{A}$, then $\mathcal{A}$ is commutative.*

Proof. Let $a, b \in \mathcal{A}$ be arbitrary elements. Then

$$\begin{aligned}
(a^2ba^2 - ba^2)^2 &= (a^2ba^2 - ba^2)(a^2ba^2 - ba^2) \\
&= (a^2ba^2)(a^2ba^2) - (a^2ba^2)(ba^2) - (ba^2)(a^2ba^2) + (ba^2)(ba^2) \\
&= (a^2ba^4ba^2) - (a^2ba^2ba^2) - (ba^4ba^2) + (ba^2ba^2) \\
&= (a^2ba^2ba^2) - (a^2ba^2ba^2) - (ba^2ba^2) + (ba^2ba^2) \\
&= 0.
\end{aligned}$$

Thus, $(a^2ba^2 - ba^2)^2 = 0$. Similarly, we deduce $(a^2ba^2 - a^2b)^2 = 0$. So

$$(a^2ba^2 - ba^2)^3 = (a^2ba^2 - a^2b)^3 = 0.$$

Therefore by assumption we have

$$a^2ba^2 - ba^2 = a^2ba^2 - a^2b = 0.$$

Thus,

$$a^2b = ba^2, \qquad (a, b \in \mathcal{A}).$$

Now by the above equation we get

$$\begin{aligned}
ab = (ab)^3 = a(ba)^2b = (ba)^2ab = (bab)(a^2b) \\
= (bab)(ba^2) = b(ab^2)a^2 = b(b^2a)a^2 = b^3a^3 = ba.
\end{aligned}$$

Therefore $ab = ba$, and the proof is complete.

The set $Z(\mathcal{A}) = \{a \in \mathcal{A} : \ ab = ba \ (b \in \mathcal{A})\}$ is called the center of $\mathcal{A}$. Clearly, $\mathcal{A}$ is commutative if and only if $Z(\mathcal{A}) = \mathcal{A}$.

**Theorem 2.5** *Let $\mathcal{A}$ be a Banach algebra such that $a + a^2 \in Z(\mathcal{A})$ for all $a \in \mathcal{A}$. Then $\mathcal{A}$ is commutative.*

Proof. Let $a, b \in \mathcal{A}$. Then by assumption we get

$$(a + b) + (a + b)^2 \in Z(\mathcal{A}),$$

which shows that $ab + ba \in Z(\mathcal{A})$. Therefore

$$(ab + ba)a = a(ab + ba),$$

and so for all $a, b \in \mathcal{A}$

$$a^2b = ba^2. \tag{1}$$

On the other hand we have

$$(a + a^2)b = b(a + a^2). \tag{2}$$

By (1) and (2), we get $ab = ba$ and the proof is complete.

The proof of the following Lemma contained in [4].

**Lemma 2.6** *Let $\mathcal{A}$ and $\mathcal{B}$ be Banach algebras and $\varphi : \mathcal{A} \longrightarrow \mathcal{B}$ be a Jordan homomorphism. Then for all $a, b \in \mathcal{A}$,*

$$\varphi(aba) = \varphi(a)\varphi(b)\varphi(a).$$

**Theorem 2.7** *Let $\varphi : \mathcal{A} \longrightarrow \mathcal{B}$ be a Jordan homomorphism. If $J_2(\mathcal{A}) = \mathcal{A}$, then $\varphi$ is ring homomorphism.*

Proof. Let $a, b \in \mathcal{A}$. Since $\mathcal{A}$ is idempotent, we get $ab + ba = 0$. So

$$\varphi(a)\varphi(b) + \varphi(b)\varphi(a) = \varphi(ab + ba) = 0.$$

Thus,

$$\varphi(a)\varphi(b) = -\varphi(b)\varphi(a). \tag{3}$$

Since $\varphi$ is Jordan, by above Lemma we get

$$\varphi(aba) = \varphi(a)\varphi(b)\varphi(a) = -\varphi(b)\varphi^2(a) = -\varphi(b)\varphi(a^2) = -\varphi(b)\varphi(a). \tag{4}$$

By proposition 2.1, $\mathcal{A}$ is commutative, so

$$\varphi(aba) = \varphi(a^2b) = \varphi(ab). \tag{5}$$

Thus, (4) and (5) implies

$$\varphi(ab) = -\varphi(b)\varphi(a). \tag{6}$$

By (3) and (6) we deduce

$$\varphi(ab) = \varphi(a)\varphi(b),$$

for all $a, b \in \mathcal{A}$. Therefore $\varphi$ is ring homomorphism.

**Theorem 2.8** *Let $\varphi : \mathcal{A} \longrightarrow \mathcal{B}$ be a linear map. If $J_3(\mathcal{A}) = \mathcal{A}$, then $\varphi(a) = 0$, for all $a \in \mathcal{A}$.*

Proof. By Theorem 2.1, $\mathcal{A}$ is commutative, so for all $a, b \in \mathcal{A}$, we have

$$\varphi(a + b) = \varphi((a + b)^3) = \varphi(a^3 + b^3 + 3ab^2 + 3a^2b) = \varphi(a + b) + 3\varphi(ab^2 + a^2b).$$

Thus,

$$\varphi(ab^2 + a^2b) = 0. \tag{7}$$

Replacing $b$ by $b+c$ in (7), we have

$$\varphi(ab^2 + ac^2 + 2abc + a^2b + a^2c) = 0. \tag{8}$$

Combing (7) and (8), we get

$$\varphi(abc) = 0. \tag{9}$$

Take $a = b = c$ in (9), then $\varphi(a^3) = 0$, and so $\varphi(a) = 0$, as required.

It is well-known that every multiplicative linear functional $\varphi$ on Banach algebra $\mathcal{A}$ is continuous and $\|\varphi\| \leq 1$, see [1] for example.

Now we have the following.

**Proposition 2.9** *Let $\varphi : \mathcal{A} \longrightarrow \mathbb{C}$ be a Jordan homomorphism. Then $\varphi$ is continuous and $\|\varphi\| \leq 1$.*

Proof. Suppose that there exist $a \in \mathcal{A}$ with $\|a\| < 1$ and $|\varphi(a)| > 1$. Take $b = a/\varphi(a)$. Then $\|b\| < 1$ and $\varphi(b) = 1$, which is contradiction by Theorem 6 of [6], therefore for all $a \in \mathcal{A}$ with $\|a\| < 1$, $|\varphi(a)| \leq 1$. This complete the proof.

In [3], Draghia proved that every Jordan homomorphism from Banach algebra $\mathcal{A}$ onto a semisimple Banach algebra $\mathcal{B}$ is continuous.

The next result, which is a extension of above proposition, prove Draghia's Theorem without surjectivity.

**Theorem 2.10** *Let $\varphi : \mathcal{A} \longrightarrow \mathcal{B}$ be a Jordan homomorphism. If $\mathcal{B}$ is semisimple, then $\varphi$ is continuous.*

Proof. Let $\psi : \mathcal{B} \longrightarrow \mathbb{C}$ be a Jordan homomorphism. Then $\psi$ is bounded by above proposition, and

$$\psi \circ \varphi(a^2) = \psi(\varphi(a^2)) = \psi(\varphi(a)^2) = \psi(\varphi(a))^2 = \psi \circ \varphi(a)^2.$$

Therefore $\psi \circ \varphi$ is a Jordan homomorphism from $\mathcal{A}$ into $\mathbb{C}$, so it is continuous by above proposition. Now suppose that $(a_n)$ be a sequence in $\mathcal{A}$ such that $\lim_n a_n = a$ and $\lim_n \varphi(a_n) = b$. Then

$$\psi(b) = \psi(\lim_n \varphi(a_n)) = \lim_n \psi \circ \varphi(a_n) = \psi \circ \varphi(a),$$

thus, $\psi(b - \varphi(a)) = 0$. Since $\mathcal{B}$ is semisimple, we get $\varphi(a) = b$. Therefore $\varphi$ is continuous by the close graph Theorem.

# References


[1] F. F. Bonsall and J. Duncan, Complete normed algebra, *Springer-Verlag, New York*, (1973).

[2] O. H. Cheikh and M. Oudadess, "On a commutativity question in Banach algebras", *Gulf J. Sci. Res*, Vol. 2 (1988), pp. 173-179.

[3] D. D. Draghia, "Continuity of Jordan homomorphisms of Banach algebras", *Extracta Mathematicae*, Vol. 10 (1995), pp. 58-59.

[4] T. Palmer, Banach algebras and the general theory $C^*$-algebras, Vol I. *Cambridge: Univ Press*, (1994).

[5] W. Zelazko, "A characterization of multiplicative linear functionals in complex Banach algebras", *Studia Math*, Vol. 30 (1968), pp. 83-85.

[6] A. Zivari-kazempour, "A characterization of Jordan homomorphism on Banach algebras", *Chinese J. Math*, Vol. 2014, (2014), 3 pages.