Global Journal of Mathematical Analysis, 3 (1) (2015) 18-25
www.sciencepubco.com/index.php/GJMA
©Science Publishing Corporation
doi: 10.14419/gjma.v3i1.3996
Research Paper

Boundedness in Lebesgue spaces of Riesz potentials on commutative hypergroups

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Abstract

In the present paper we consider Riesz potentials on commutative hypergroups and prove the boundedness of these potentials from $L^p(K,\lambda)$ to $L^q(K,\lambda)$. We also prove the inequality from $L^1(K,\lambda)$ to weak $L^q(K,\lambda)$ for Riesz potentials on commutative hypergroups.

Keywords: *Hardy-Littlevood maximal function, hypergroup, Riesz potential.*

1. Introduction

For $0 < \alpha < n$, the operator

$$R_\alpha f(x) = \int\limits_{R^n} |x-y|^\alpha f(y)dy$$

is called a classical Riesz potential.

By the classical Hardy-Littlewood-Sobolev theorem, if $1 < p < \infty$ and $\alpha p < n$, then $R_\alpha f$ is an operator of strong type (p,q), where $\dfrac{1}{q} = \dfrac{1}{p} - \dfrac{\alpha}{n}$. If $p = 1$, then $R_\alpha f$ is an operator of weak type $(1,q)$, where $\dfrac{1}{q} = 1 - \dfrac{\alpha}{n}$ (see [16], [24]).

The Hardy-Littlewood-Sobolev theorem is an important result in fractional integral theory and potential theory. There are a lot of generalizations and analogues of this theorem. The Hardy-Littlewood-Sobolev theorem was proved for Riesz potentials associated to doubling measures in [18] and nondoubling measures in [19], [5]. In [3] and [11], generalized potential-type integral operators were considered and (p,q) properties of these operators were proved. In [21], [22], [13], [12] the Hardy-Littlewood-Sobolev theorem was extended to Orlicz and Musielak-Orlicz spaces for generalized Riesz potentials. The analogues of Hardy-Littlewood-Sobolev theorem on the different hypergroups were considered in [8], [9], [10], [4], [14]. The strong type (p,q)inequality for Riesz potentials on commutative hypergroups was given in [15] with the condition $\lambda B(e,r) = Cr^N$.

In this paper, we prove the boundedness of Riesz potentials on commutative hypergroups from $L^p(K,\lambda)$ to $L^q(K,\lambda)$ in the case of upper Ahlfors N-regular measure. We also prove the inequality from $L^1(K,\lambda)$ to weak $L^q(K,\lambda)$ for Riesz potentials on commutative hypergroups.

2. Preliminaries

Let K be a set. A function $\rho : K \times K \to [0, \infty)$ is called quasi-metric if:

1. $\rho(x, y) = 0 \Leftrightarrow x = y$;
2. $\rho(x, y) = \rho(y, x)$;
3. there exists a constant $c \geq 1$ such that for every $x, y, z \in K$

$$\rho(x, y) \leq c(\rho(x, z) + \rho(z, y)).$$

Let all balls $B(x, r) = \{y \in K : \rho(x, y) < r\}$ be λ-measurable and assume that the measure λ fulfils the doubling condition

$$0 < \lambda B(x, 2r) \leq D\lambda B(x, r) < \infty. \tag{1}$$

A space (K, ρ, λ) which satisfies all conditions mentioned above is called a space of homogeneous type (see [2]).

In the theory of locally compact groups there arise certain spaces which, though not groups, have some of the structure of groups. Often, the structure can be expressed in terms of an abstract convolution of measures on the space.

A hypergroup $(K, *)$ consists of a locally compact Hausdorff space K together with a bilinear, associative, weakly continuous convolution on the Banach space of all bounded regular Borel measures on K with the following properties:

1. For all $x, y \in K$, the convolution of the point measures $\delta_x * \delta_y$ is a probability measure with compact support.
2. The mapping: $K \times K \to \mathcal{C}(K)$, $(x, y) \mapsto supp(\delta_x * \delta_y)$ is continuous with respect to the Michael topology on the space $\mathcal{C}(K)$ of all nonvoid compact subsets of K, where this topology is generated by the sets

$$U_{V,W} = \{L \in \mathcal{C}(K) : L \cap V \neq \emptyset, L \subset W\}$$

   with V, W open in K.
3. There is an identity $e \in K$ with $\delta_e * \delta_x = \delta_x * \delta_e = \delta_x$ for all $x \in K$.
4. There is a continuous involution $*$ on K such that

$$(\delta_x * \delta_y)^* = \delta_{y^*} * \delta_{x^*}$$

   and $e \in supp(\delta_x * \delta_y) \Leftrightarrow x = y^*$ for $x, y \in K$ (see [17], [23], [1], [20], [7]).

A hypergroup K is called commutative if $\delta_x * \delta_y = \delta_y * \delta_x$ for all $x, y \in K$. It is well known that every commutative hypergroup K possesses a Haar measure which will be denoted by λ (see [23]). That is, for every Borel measurable function f on K,

$$\int_K f(\delta_x * \delta_y) d\lambda(y) = \int_K f(y) d\lambda(y) \;\; (x \in K).$$

Define the generalized translation operators T^x, $x \in K$, by

$$T^x f(y) = \int_K f d(\delta_x * \delta_y)$$

for all $y \in K$. If K is a commutative hypergroup, then $T^x f(y) = T^y f(x)$ and the convolution of two functions is defined by

$$(f * g)(x) = \int_K T^x f(y) g(y^*) d\lambda(y).$$

Let $(K, *)$ be a commutative hypergroup, with quasi-metric ρ, Haar measure λ and $N \in (0, \infty)$. We will say K is called upper Ahlfors N-regular by an identity, if there exists a constant $C > 0$, not depending $r > 0$, such that

$$\lambda B(e, r) \leq C r^N \tag{2}$$

Let $p > 0$. By $L^p(K,\lambda)$ denote a class of all λ-measurable functions $f : K \to (-\infty, +\infty)$ with $\|f\|_{L^p(K,\lambda)} = \left(\int\limits_K |f(x)|^p \, d\lambda(x)\right)^{\frac{1}{p}} < \infty$.
Let T be a linear operator from $L^p(K,\lambda)$ to $L^q(K,\lambda)$, where $p, q \in (0,\infty)$. T is said to be an operator of strong type (p,q) on (K,λ), if there exists a positive constant C such that

$$\|Tf\|_{L^p(K,\lambda)} \leq C_p \|f\|_{L^p(K,\lambda)},$$

If for arbitrary $\beta > 0$ and $f \in L^p(K,\lambda)$

$$\lambda\{x : Tf(x) > \beta\} \leq \left(\frac{C}{\beta}\|f\|_{L^1(K,\lambda)}\right)^q, \beta > 0$$

then T is called an operator of weak type (p,q) on (K,λ).
The notation $\chi_A(x)$ denotes the characteristic function of set A.
Let $(K,*)$ be a commutative hypergroup, with quasi-metric ρ, Haar measure λ, upper Ahlfors N-regular by an identity. Define Hardy-Littlewood maximal function

$$Mf(x) = \sup_{r>0} \frac{1}{\lambda B(e,r)} \left(|f| * \chi_{B(e,r)}\right)(x) \tag{3}$$

and Riesz potential

$$I_\alpha f(x) = \left(\rho(e,\cdot)^{\alpha-N} * f\right)(x), \; 0 < \alpha < N \tag{4}$$

on commutative hypergroup $(K,*)$ equipped with the quasi-metric ρ.

3. Main results

In this section we formulate the main results of this paper.

Theorem 3.1 *Let $(K,*)$ be a commutative hypergroup, with quasi-metric ρ and doubling Haar measure λ, upper Ahlfors N-regular by an identity and let $0 < \alpha < N$, $1 \leq p < \frac{N}{\alpha}$ If $f \in L^p(K)$, then the integral*

$$I_\alpha f(x) = \int\limits_X T^x \rho(e,y)^{\alpha-N} f(y^*) d\lambda(y)$$

is absolutely convergent for almost every $x \in K$.

Proof Let $f \in L^p(K,\lambda)$ and $1 \leq p < \frac{N}{\alpha}$. Write $I_\alpha f(x)$ in the form

$$I_\alpha f(x) = \int\limits_{B(e,1)} \rho(e,y)^{\alpha-N} T^x f(y^*) d\lambda(y)$$

$$+ \int\limits_{K\setminus B(e,1)} \rho(e,y)^{\alpha-N} T^x f(y^*) d\lambda(y) = J_1(x) + J_2(x).$$

Let us estimate $J_1(x)$. It is clear that

$$|J_1(x)| \leq \int\limits_K \rho(e,y)^{\alpha-N} \chi_{B(e,1)}(y) T^x |f(y^*)| d\lambda(y)$$

By Young's inequality

$$\|J_1(\cdot)\|_{L^p(K,\lambda)} \leq \|\rho(e,\cdot)^{\alpha-N}\chi_{B(e,1)}(\cdot)\|_{L^1(K,\lambda)} \|T^x f\|_{L^p(K,\lambda)}$$
$$\leq C\|\rho(e,\cdot)^{\alpha-N}\chi_{B(e,1)}(\cdot)\|_{L^1(K,\lambda)} \|f\|_{L^p(K,\lambda)}$$

and

$$\|\rho(e,\cdot)^{\alpha-N}\chi_{B(e,1)}(\cdot)\|_{L^1(K,\lambda)} = \int\limits_{B(e,1)} \rho(e,y)^{\alpha-N} d\lambda(y)$$

$$\leq \sum_{k=1}^{\infty} \int\limits_{2^{-k}\leq\rho(e,y)<2^{-k+1}} \rho(e,y)^{\alpha-N} d\lambda(y)$$

$$\leq \sum_{k=1}^{\infty} \left(2^{-k}\right)^{\alpha-N} \int\limits_{\rho(e,y)<2^{-k+1}} d\lambda(y)$$

$$\leq C \sum_{k=1}^{\infty} 2^{(N-\alpha)k} 2^{N(-k+1)} < C$$

Then

$$\|J_1(\cdot)\|_{L^p(K,\lambda)} \leq C\|f\|_{L^p(K,\lambda)},$$

e.g. $J_1(x)$ is absolutely convergent almost every $x \in K$.

By Hölder's inequality we have

$$|J_2(x)| \leq \int\limits_{K\setminus B(e,1)} \rho(e,y)^{\alpha-N} T^x|f(y^*)| d\lambda(y)$$

$$\leq \|T^x f(\cdot)\|_{L^p(K,\lambda)} \left(\int\limits_{K\setminus B(e,1)} \rho(e,y)^{(\alpha-N)p'} d\lambda(y) \right)^{\frac{1}{p'}}$$

$$\leq C\|f\|_{L^p(K,\lambda)} \left(\int\limits_{K\setminus B(e,1)} \rho(e,y)^{(\alpha-N)p'} d\lambda(y) \right)^{\frac{1}{p'}}$$

and

$$\int\limits_{K\setminus B(e,1)} \rho(e,y)^{(\alpha-N)p'} d\lambda(y)$$

$$\leq \sum_{k=0}^{\infty} \int\limits_{2^k<\rho(e,y)\leq 2^{k+1}} \rho(e,y)^{(\alpha-N)p'} d\lambda(y)$$

$$\leq \sum_{k=0}^{\infty} 2^{(N-\alpha)p'k} \int\limits_{\rho(e,y)\leq 2^{k+1}} d\lambda(y)$$

$$\leq \sum_{k=0}^{\infty} 2^{(N-\alpha)p'k} 2^{(k+1)N} < C$$

Hence for $1 \leq p < \frac{N}{\alpha}$

$$|J_2(x)| \leq C\|f\|_{L^p(K,\lambda)}$$

Thus for all functions $f \in L^p(K,\lambda)$, $1 \leq p < \frac{N}{\alpha}$ the fractional integrals $I_\alpha f(x)$ are absolutely convergent for almost every $x \in K$. The theorem is proved.

Theorem 3.2 *Let $(K,*)$ be a commutative hypergroup, with quasi-metric ρ and doubling Haar measure λ, upper Ahlfors N-regular by an identity. Assume $0 < \alpha < N$, $1 < p < \frac{N}{\alpha}$, $\frac{1}{p} - \frac{1}{q} = \frac{\alpha}{N}$ and Hardy-Littlewood maximal function (3) is an operator of strong type (p,p) on $(K,*)$. Then the Riesz potential (4) is an operator of strong type (p,q) on $(K,*)$.*

Proof Split $I_\alpha f(x)$ in the standard way

$$I_\alpha f(x) = \int\limits_{B(e,r)} \rho(e,y)^{\alpha-N} T^x f(y^*) d\lambda(y) + \int\limits_{K\setminus B(e,r)} \rho(e,y)^{\alpha-N} T^x f(y^*) d\lambda(y)$$
$$= U_1(x,r) + U_2(x,r).$$

Then for $U_1(x,r)$ we have the estimate

$$|U_1(x,r)| \le \int\limits_{B(e,r)} \rho(e,y)^{\alpha-N} T^x |f(y^*)| d\lambda(y).$$
$$\le \sum_{k=1}^{\infty} \int\limits_{2^{-k}r \le \rho(e,y) < 2^{-k+1}r} \rho(e,y)^{\alpha-N} T^x |f(y^*)| d\lambda(y)$$
$$\le \sum_{k=1}^{\infty} \left(2^{-k}r\right)^{\alpha-N} \int\limits_{\rho(e,y) < 2^{-k+1}r} T^x |f(y^*)| d\lambda(y)$$
$$= \sum_{k=1}^{\infty} \left(2^{-k}r\right)^{\alpha-N} \lambda B(e, 2^{-k+1}r) \frac{1}{\lambda B(e, 2^{-k+1}r)} \int\limits_{B(e,2^{-k+1}r)} T^x |f(y^*)| d\lambda(y)$$
$$\le C r^\alpha M f(x).$$

Therefore it follows that

$$|U_1(x,r)| \le C r^\alpha M f(x), \tag{5}$$

where $C > 0$ does not depend f, x and r.
Estimate $U_2(x,r)$. By Hölder's inequality we have

$$|U_2(x,r)| \le \left(\int\limits_{K\setminus B(e,r)} |T^x f(y^*)|^p d\lambda(y) \right)^{\frac{1}{p}} \left(\int\limits_{K\setminus B(e,r)} \rho(e,y)^{(\alpha-N)p'} d\lambda(y) \right)^{\frac{1}{p'}}.$$

Here

$$\left(\int\limits_{K\setminus B(e,r)} \rho(e,y)^{(\alpha-N)p'} d\lambda(y) \right)^{\frac{1}{p'}}$$
$$= \left(\sum_{k=0}^{\infty} \int\limits_{2^k r \le \rho(e,y) < 2^{k+1}r} \rho(e,y)^{(\alpha-N)p'} d\lambda(y) \right)^{\frac{1}{p'}}$$
$$\le \left(\sum_{k=0}^{\infty} \left(2^k r\right)^{(\alpha-N)p'} \int\limits_{\rho(e,y) < 2^{k+1}r} d\lambda(y) \right)^{\frac{1}{p'}}$$
$$\le C \left(\sum_{k=0}^{\infty} \left(2^k r\right)^{(\alpha-N)p'} \left(2^{k+1} r\right)^N \right)^{\frac{1}{p'}}$$
$$\le C r^{\alpha - N + \frac{N}{p'}}$$
$$= C r^{-\frac{N}{q}}.$$

Therefore

$$|U_2(x,r)| \le C r^{-\frac{N}{q}} \|f\|_{L^p(K,\lambda)} \tag{6}$$

From (5) and (6), we have

$$|I_\alpha f(x)| \le C\left(r^\alpha Mf(x) + r^{-\frac{N}{q}}\|f\|_{L^p(K,\lambda)}\right)$$

Minimum of the right-hand side is attained at $r = \left[\frac{\|f\|_{L^p(K,\lambda)}}{Mf(x)}\right]^{\frac{p}{N}}$. So

$$|I_\alpha f(x)| \le C\left(Mf(x)\right)^{\frac{p}{q}}\|f\|_{L^p(K,\lambda)}^{1-\frac{p}{q}}$$

Hence, by the boundedness of Hardy-Littlewood maximal function (3) on $L^p(K,\lambda)$ we have

$$\int\limits_K |I_\alpha f(x)|^q d\lambda(y) \le C\|f\|_{L^p(K,\lambda)}^{q-p}\int\limits_K (Mf(y))^p\, d\lambda(y) \le C\|f\|_{L^q(K,\lambda)}^q.$$

The theorem is proved.

Theorem 3.3 *Let $(K,*)$ be a commutative hypergroup, with quasi-metric ρ and doubling Haar measure λ, upper Ahlfors N-regular by an identity and let $0<\alpha<N$, $\frac{1}{q} = 1-\frac{\alpha}{N}$ and assume that the maximal operator M is an operator of weak type $(1,1)$ on $(K,*)$. Then the Riesz potential (4) is an operator of weak type $(1,q)$ on $(K,*)$.*

Proof Let $f\in L^1(K,\lambda)$. It is clear that

$$\lambda\{x\in K:\, |I_\alpha f(x)|>2\beta\}$$
$$\le \lambda\{x\in K:\, |U_1(x,r)|>\beta\} + \lambda\{x\in K:\, |U_2(x,r)|>\beta\},$$

where

$$U_1(x,r) = \int\limits_{B(e,r)} \rho(e,y)^{\alpha-N} T^x f(y^*) d\lambda(y),$$
$$U_2(x,r) = \int\limits_{K\setminus B(e,r)} \rho(e,y)^{\alpha-N} T^x f(y^*) d\lambda(y).$$

Further, from inequality (5) we derive that

$$\beta\lambda\{x\in K:\, |U_1(x,r)|>\beta\} = \beta \int\limits_{\{x\in K: |U_1(x,r)|>\beta\}} d\lambda(y)$$
$$\le \beta \int\limits_{\{x\in K: Cr^\alpha Mf(x)>\beta\}} d\lambda(y)$$
$$= \beta\lambda\left\{x\in K: Mf(x) > \frac{\beta}{Cr^\alpha}\right\}$$
$$\le \beta\frac{Cr^\alpha}{\beta}\int\limits_K |f(y)|d\lambda(y) = Cr^\alpha\|f\|_{L_1(K,\lambda)}$$

and

$$|U_2(x,r)| \le \int\limits_{K\setminus B(e,r)} \rho(e,r)^{\alpha-N}|T^x f(y^*)|d\lambda(y)$$
$$\le r^{\alpha-N}\int\limits_{K\setminus B(e,r)} |T^x f(y^*)|d\lambda(y)$$
$$\le Cr^{-\frac{N}{q}}\int\limits_K |f(y)|d\lambda(y) = Cr^{-\frac{N}{q}}\|f\|_{L_1(K,\lambda)}.$$

Thus, if $\beta = r^{-\frac{N}{q}}\|f\|_{L_1(K,\lambda)}$, then $|U_2(x,r)|\le\beta$, and, consequently, $\lambda\{x\in K:\, |U_2(x,r)|>\beta\}=0$. Thus

$$\lambda\{x\in K:\, |I_\alpha f(x)|>2\beta\}$$

$$\leq \frac{C}{\beta} r^{\alpha} \|f\|_{L_1(K,\lambda)}$$
$$= Cr^{\alpha+\frac{N}{q}} = Cr^{N} = C\beta^{-q} \|f\|^{q}_{L_1(K,\lambda)}$$
$$\leq \left(\frac{C}{\beta} \|f\|_{L_1(K,\lambda)}\right)^{q}.$$

The theorem is proved.

Acknowledgements

This work was supported by the Science Development Foundation under the President of the Republic of Azerbaijan Grant EIF-2012-2(6)-39/10/1. The author would like to express his thanks to Academician Akif Gadjiev for valuable remarks.

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