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*Research Paper*

# A new explicit formula for the Bernoulli and Genocchi numbers in terms of the Stirling numbers

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## Abstract

In the paper, the authors concisely review some explicit formulas and establish a new explicit formula for the Bernoulli and Genocchi numbers in terms of the Stirling numbers of the second kind.

***Keywords***: *explicit formula; Bernoulli number; Genocchi number; Stirling number of the second kind*
***MSC***: *Primary 11B68; Secondary 11B73*

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## 1. Introduction and main results

It is well known that the Bernoulli numbers $B_n$ for $n \ge 0$ may be defined by the power series expansion

$$\frac{x}{e^x-1}=\sum_{n=0}^{\infty}B_n\frac{x^n}{n!}=1-\frac{x}{2}+\sum_{k=1}^{\infty}B_{2k}\frac{x^{2k}}{(2k)!}, \quad |x|<2\pi, \tag{1}$$

that Euler polynomials $E_n(x)$ are defined by

$$\frac{2e^{xt}}{e^t+1}=\sum_{n=0}^{\infty}E_n(x)\frac{t^n}{n!}, \tag{2}$$

that the Genocchi numbers $G_n$ for $n \in \mathbb{N}$ are given by the generating function

$$\frac{2t}{e^t+1}=\sum_{n=1}^{\infty}G_n\frac{t^n}{n!}, \tag{3}$$

and that the Stirling numbers of the second kind which may be generated by

$$\frac{(e^x-1)^k}{k!}=\sum_{n=k}^{\infty}S(n,k)\frac{x^n}{n!},\quad k\in\mathbb{N} \tag{4}$$

and may be computed by

$$S(k,m)=\frac{1}{m!}\sum_{\ell=1}^{m}(-1)^{m-\ell}\binom{m}{\ell}\ell^k,\quad 1\le m\le k. \tag{5}$$

By the way, the Stirling number of the second kind $S(n,k)$ may be interpreted combinatorially as the number of ways of partitioning a set of $n$ elements into $k$ nonempty subsets.

The Bernoulli numbers $B_n$ for $n\in\{0\}\cup\mathbb{N}$ satisfy

$$B_0=1,\quad B_1=-\frac{1}{2},\quad B_{2n+2}\ne 0,\quad B_{2n+3}=0. \tag{6}$$

For $n\in\mathbb{N}$, the Genocchi numbers meet $G_{2n+1}=0$. The first few Genocchi numbers $G_n$ are listed in Table 1.1. The Genocchi numbers $G_{2n}$ may be represented in terms of the Bernoulli numbers $B_{2n}$ and Euler polynomials $E_{2n-1}(0)$ as

**Table 1.1:** The first few Genocchi numbers $G_n$

| $n$ | 1 | 2 | 4 | 6 | 8 | 10 | 12 | 14 | 16 | 18 |
|---|---|---|---|---|---|---|---|---|---|---|
| $G_n$ | 1 | −1 | 1 | −3 | 17 | −155 | 2073 | −38227 | 929569 | −28820618 |

$$G_{2n}=2(1-2^{2n})B_{2n}=2nE_{2n-1}(0),\quad n\in\mathbb{N}. \tag{7}$$

See [1, p. 49]. As a result, we have

$$G_n=2(1-2^n)B_n,\quad n\in\mathbb{N}. \tag{8}$$

The first formula for the Bernoulli numbers $B_n$ listed in [2] is

$$B_n=\sum_{k=0}^{n}\frac{1}{k+1}\sum_{j=0}^{k}(-1)^j\binom{k}{j}j^n,\quad n\ge 0, \tag{9}$$

which is a special case of the general formula [13, (2.5)]. The formula (9) is equivalent to

$$B_n=\sum_{k=0}^{n}(-1)^k\frac{k!}{k+1}S(n,k),\quad n\in\{0\}\cup\mathbb{N}, \tag{10}$$

which was listed in [3, p. 536] and [4, p. 560]. Recently, four alternative proofs of the formula (10) were provided in [7, 16]. A generalization of the formula (10) was supplied in [6]. In all, we may collect at least seven alternative proofs for the formula (9) or (10) in [2, 4, 7, 13, 14, 16] and closely related references therein.

In [2, p. 48, (11)], it was deduced that

$$B_n=\sum_{j=0}^{n}(-1)^j\binom{n+1}{j+1}\frac{n!}{(n+j)!}\sum_{k=0}^{j}(-1)^{j-k}\binom{j}{k}k^{n+j},\quad n\ge 0, \tag{11}$$

which may be rearranged as

$$B_n=\sum_{i=0}^{n}(-1)^i\frac{\binom{n+1}{i+1}}{\binom{n+i}{i}}S(n+i,i),\quad n\ge 0. \tag{12}$$

The formula (12) was rediscovered in the paper [8]. On 21 January 2014, the authors searched out that the formula (12) was also derived in [12, p. 59] and [17, p. 140].

In [11, p. 1128, Corollary], among other things, it was found that

$$B_{2k}=\frac{1}{2}-\frac{1}{2k+1}-2k\sum_{i=1}^{k-1}\frac{A_{2(k-i)}}{2(k-i)+1} \tag{13}$$

for $k\in\mathbb{N}$, where $A_m$ is defined by

$$\sum_{m=1}^{n}m^k=\sum_{m=0}^{k+1}A_mn^m.$$

In [15, Theorem 1.4], among other things, it was presented that

$$B_{2k}=\frac{(-1)^{k-1}k}{2^{2(k-1)}(2^{2k}-1)}\sum_{i=0}^{k-1}\sum_{\ell=0}^{k-i-1}(-1)^{i+\ell}\binom{2k}{\ell}(k-i-\ell)^{2k-1}, \quad k\in\mathbb{N}. \tag{14}$$

In [10, Theorem 3.1], it was obtained that

$$B_{2k}=1+\sum_{m=1}^{2k-1}\frac{S(2k+1,m+1)S(2k,2k-m)}{\binom{2k}{m}}-\frac{2k}{2k+1}\sum_{m=1}^{2k}\frac{S(2k,m)S(2k+1,2k-m+1)}{\binom{2k}{m-1}}, \quad k\in\mathbb{N}. \tag{15}$$

The aim of this paper is to find the following new explicit formula for the Bernoulli numbers $B_k$, or say, the Genocchi numbers $G_k$, in terms of the Stirling numbers of the second kind $S(k,m)$.

**Theorem 1.1** *For all $k\in\mathbb{N}$, the Genocchi numbers $G_k$ may be computed by*

$$G_k=2(1-2^k)B_k=(-1)^kk\sum_{m=1}^{k}(-1)^m\frac{(m-1)!}{2^{m-1}}S(k,m). \tag{16}$$

## 2. Proof of Theorem 1.1

Differentiating on both sides of the equation (3) and employing Leibniz identity for differentiation give

$$\left(\frac{2t}{e^t+1}\right)^{(k)}=2\left[t\left(\frac{1}{e^t+1}\right)^{(k)}+k\left(\frac{1}{e^t+1}\right)^{(k-1)}\right]=\sum_{n=k}^{\infty}G_n\frac{t^{n-k}}{(n-k)!}.$$

In [9, Theorem 2.1] and [18, Theorem 3.1], it was obtained that, when $\lambda>0$ and $t\ne-\frac{\ln\lambda}{\alpha}$ or when $\lambda<0$ and $t\in\mathbb{R}$,

$$\left(\frac{1}{\lambda e^{\alpha t}-1}\right)^{(k)}=(-1)^k\alpha^k\sum_{m=1}^{k+1}(m-1)!S(k+1,m)\left(\frac{1}{\lambda e^{\alpha t}-1}\right)^m. \tag{17}$$

Specially, when $\lambda=-1$ and $\alpha=1$, the identity (17) becomes

$$\left(\frac{1}{e^t+1}\right)^{(k)}=(-1)^{k+1}\sum_{m=1}^{k+1}(-1)^m(m-1)!S(k+1,m)\left(\frac{1}{e^t+1}\right)^m. \tag{18}$$

Consequently, it follows that

$$\begin{aligned}G_k&=\lim_{t\to0}\sum_{n=k}^{\infty}G_n\frac{t^{n-k}}{(n-k)!}=2k\lim_{t\to0}\left(\frac{1}{e^t+1}\right)^{(k-1)}\\&=2k(-1)^k\sum_{m=1}^{k}(-1)^m(m-1)!S(k,m)\lim_{t\to0}\left(\frac{1}{e^t+1}\right)^m\\&=(-1)^kk\sum_{m=1}^{k}(-1)^m\frac{(m-1)!}{2^{m-1}}S(k,m).\end{aligned}$$

The proof of Theorem 1.1 is complete.

**Remark 2.1** *This paper is a slightly modified version of the preprint [5].*

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