Global Journal of Mathematical Analysis, 3 (2) (2015) 77-80
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www.sciencepubco.com/index.php/GJMA
doi: 10.14419/gjma.v3i2.4605
Research Paper

Logarithmically complete monotonicity of a power-exponential function involving the logarithmic and psi functions

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Abstract

Let Γ and $\psi=\frac{\Gamma'}{\Gamma}$ be respectively the classical Euler gamma function and the psi function and let $\gamma=-\psi(1)=0.57721566\dots$ stand for the Euler-Mascheroni constant. In the paper, the authors simply confirm the logarithmically complete monotonicity of the power-exponential function $q(t)=t^{t[\psi(t)-\ln t]-\gamma}$ on the unit interval $(0,1)$, concisely deny that $q(t)$ is a Stieltjes function, surely point out fatal errors appeared in the paper [V. Krasniqi and A. Sh. Shabani, *On a conjecture of a logarithmically completely monotonic function*, Aust. J. Math. Anal. Appl. **11** (2014), no. 1, Art. 5, 5 pages; Available online at http://ajmaa.org/cgi-bin/paper.pl?string=v11n1/V11I1P5.tex], and partially solve a conjecture posed in the article [B.-N. Guo, Y.-J. Zhang, and F. Qi, *Refinements and sharpenings of some double inequalities for bounding the gamma function*, J. Inequal. Pure Appl. Math. **9** (2008), no. 1, Art. 17; Available online at http://www.emis.de/journals/JIPAM/article953.html].

Keywords: *Logarithmically complete monotonicity; Stieltjes function; Logarithmic function; Psi function; Power-exponential function; Conjecture*

1. Introduction

For bounding the classical Euler gamma function

$$\Gamma(z)=\int_0^\infty t^{z-1}e^{-t}\,\mathrm{d}\,t,\quad \Re(z)>0$$

and for bounding the ratio $\frac{\Gamma(x)}{\Gamma(y)}$, Guo, Zhang, and Qi established some inequalities in [10]. When comparing two inequalities, they created the function

$$q(t)=t^{t[\psi(t)-\ln t]-\gamma},\quad t>0 \tag{1.1}$$

in [10, Remark 8], where $\psi(x)=\frac{\Gamma'(x)}{\Gamma(x)}$ is called the psi function and $\gamma=0.577\dots$ stands for the Euler-Mascheroni constant. As said in [10, Remark 8], Guo, Zhang, and Qi demonstrated, by using the software MATHEMATICA to plot, the decreasing monotonicity of $q(t)$ on $(0,\infty)$. Then they conjectured that the function $q(t)$ is logarithmically completely monotonic on $(0,\infty)$. A function f is said to be logarithmically completely monotonic on an interval I if it is infinitely differentiable and satisfies

$$(-1)^k[\ln f(t)]^{(k)}\ge 0$$

on I for $k\in\mathbb{N}$, where $\mathbb{N}$ denotes the set of positive integers. See [2, 3, 6, 13, 14, 16, 17, 19, 20] and plenty of closely related references in [18]. For our own convenience, we denote the set of logarithmically completely monotonic functions on I by $\mathcal{L}[I]$.

Recall from [12, Chapter XIII], [20, Chapter 1], and [21, Chapter IV] that a function $f(x)$ is said to be completely monotonic on an interval I if and only if f has derivatives of all orders and satisfies

$$(-1)^{k-1}f^{(k-1)}(x)\ge 0$$

on I for $k\in\mathbb{N}$. The class of completely monotonic functions on $(0,\infty)$ may be characterized by [21, p. 161, Theorem 12b] which reads that a necessary and sufficient condition that $f(x)$ should be completely monotonic for $0<x<\infty$ is that

$$f(x)=\int_0^\infty e^{-xs}\,\mathrm{d}\,\mu(s),$$

where $\mu(s)$ is non-decreasing and the integral converges for $0<x<\infty$. We will use $\mathcal{C}[I]$ to denote the set of completely monotonic functions on I.

Also recall from [3, 20, 21] that if $f(x)$ can be represented as

$$f(x)=a+\int_0^\infty \frac{1}{x+s}\,\mathrm{d}\,\mu(s)$$

on $(0,\infty)$, where $a\ge 0$, the measure μ is nonnegative on $[0,\infty)$, and

$$\int_0^\infty \frac{1}{1+s}\,\mathrm{d}\,\mu(s)<\infty,$$

then $f(x)$ is said to be a Stieltjes function. The set of Stieltjes functions will be denoted by $\mathcal{S}$.

In [11], Krasniqi and Shabani claimed that they confirmed the above conjecture. However, because $-\ln t\in\mathcal{C}[(0,1)]$, but $-\ln t\notin\mathcal{C}[(0,\infty)]$, all the proofs of [11, Conjecture 2.1 and Theorem 2.2] are wrong.

Our main results in this paper may be stated as the following theorem.

Theorem 1.1. *The function $q(t)$ defined by (1.1) satisfies*

$$q(t)\in\mathcal{L}[(0,1)],\quad q(t)\notin\mathcal{S},\quad \lim_{t\to0^+}q(t)=\infty,\quad \lim_{t\to\infty}q(t)=0. \tag{1.2}$$

It is clear that Theorem 1.1 partially confirms the logarithmically complete monotonicity of the function $q(t)$ on $(0,\infty)$ and partially solves the above conjecture posed in [10, Remark 8].

2. Proof of theorem 1.1

In [3, 5, 6, 13, 14, 19], the inclusions

$$\mathcal{S}\setminus\{0\}\subset\mathcal{L}[(0,\infty)]\quad\text{and}\quad\mathcal{L}[I]\subset\mathcal{C}[I] \tag{2.1}$$

were discovered. For more information, see [18] and plenty of closely related references therein. In order to prove the first property in (1.2), from the first inclusion in (2.1), it suffices to show that the function $q(t)$ is a Stieltjes function. In [3], Berg remarked that $f\in\mathcal{S}$ and $f\ne0$ if and only if $\frac{1}{xf(x)}\in\mathcal{S}$. Therefore, if $q(t)\in\mathcal{S}$, then $\frac{1}{tq(t)}\in\mathcal{S}$, and so $\frac{1}{tq(t)}$ should be decreasing on $(0,\infty)$. However, an easy calculation leads to

$$1=\frac{1}{tq(t)}\bigg|_{t=1}<\frac{1}{tq(t)}\bigg|_{t=2}=2^{3(\gamma-1)+2\ln 2}=1.08518525\dots$$

As a result, the function $q(t)$ is not a Stieltjes function, that is, the second property in (1.2) is true.

In [1, pp. 374–375, Theorem 1] and [9, p. 105, Theorem 1], Alzer, Guo, and Qi proved that

$$\theta_\alpha(x)=x^\alpha[\ln x-\psi(x)]\in\mathcal{C}[(0,\infty)]$$

if and only if $\alpha \le 1$ and that

$$\lim_{x\to 0^+} \theta_1(x) = 1, \quad \lim_{x\to\infty} \theta_1(x) = \frac{1}{2}, \quad \lim_{x\to 0^+} \theta_\alpha(x) = \infty, \quad \lim_{x\to\infty} \theta_\alpha(x) = 0, \quad \alpha < 1. \tag{2.2}$$

It is clear that

$$\ln q(t) = \{t[\psi(t) - \ln t] - \gamma\} \ln t = (-\ln t)[\theta_1(t) + \gamma]. \tag{2.3}$$

Since $-\ln t \in \mathcal{C}[(0,1)]$ and the product or sum of finitely many completely monotonic functions is still a completely monotonic function on their common interval, it is easy to see that $\ln q(t) \in \mathcal{C}[(0,1)]$ and $q(t) \in \mathcal{L}[(0,1)]$.

The limits in (1.2) follows immediately from taking $t \to 0^+$ and $t \to \infty$ on both sides of the equation (2.3) and making use of the first two limits in (2.2).

The first limit in (1.2) may also be deduced directly from the argument

$$\begin{aligned}
\lim_{t\to 0^+} \ln q(t) &= \Big\{\lim_{t\to 0^+} [t\psi(t)] - \lim_{t\to 0^+} (t\ln t) - \gamma\Big\} \lim_{t\to 0^+} \ln t \\
&= \Big\{\lim_{t\to 0^+} [t\psi(t) + 1 - 1] - 0 - \gamma\Big\} \lim_{t\to 0^+} \ln t \\
&= \left\{\lim_{t\to 0^+} \left(t\left[\psi(t) + \frac{1}{t}\right]\right) - 1 - \gamma\right\} \lim_{t\to 0^+} \ln t \\
&= \left\{\lim_{t\to 0^+} [t\psi(t+1)] - 1 - \gamma\right\} \lim_{t\to 0^+} \ln t \\
&= -(1+\gamma) \lim_{t\to 0^+} \ln t \\
&= \infty.
\end{aligned}$$

The proof of Theorem 1.1 is complete.

3. Remarks

Remark 3.1. On the interval $(1,\infty)$, it is obvious that the function $\ln t$ is positive and increasing and $\theta_1(t) + \gamma \in \mathcal{C}[(1,\infty)]$. Therefore, we can not make clear the monotonicity of the functions $(\ln t)[\theta_1(t)+\gamma] = -\ln q(t)$ and $(-\ln t)[\theta_1(t)+\gamma] = \ln q(t)$, say nothing of the logarithmically complete monotonicity of $q(t)$, on $(1,\infty)$. This is the key difficulty to completely solve the above conjecture.

Remark 3.2. By a similar argument to the second property in (1.2), we can show that $1 - G(x) \notin \mathcal{S}$, where

$$G(x) = \left[1 - \frac{\ln x}{\ln(x+1)}\right] x \ln x, \quad x > 0.$$

However, Guo and Qi conjectured that $1 - G(x) \in \mathcal{C}[(0,\infty)]$ in [8] and its preprint [15]. Later, Berg and Pedersen verified this conjecture in [4].

4. An open problem

For $\beta \in \mathbb{R}$, let

$$h_\beta(t) = t^{t[\psi(t) - \ln t] - \beta}, \quad t > 0.$$

If $h_\beta(t) \in \mathcal{L}[(0,\infty)]$, then the first derivative of its logarithm

$$[\ln h_\beta(t)]' = \frac{t^2\psi'(t)\ln t - t(\ln t)^2 - 2t\ln t + t\psi(t) + t\psi(t)\ln t - b}{t}$$

should be non-positive on $(0,\infty)$, that is,

$$b \ge t^2(\ln t)\psi'(t) + t(1+\ln t)\psi(t) - (\ln t + 2)t\ln t \to -\frac{1}{2}, \quad t \to \infty.$$

This means that $\beta \ge -\frac{1}{2}$ is a necessary condition such that $h_\beta(t) \in \mathcal{L}[(0,\infty)]$.

Is the condition $\beta \ge -\frac{1}{2}$ sufficient for $h_\beta(t) \in \mathcal{L}[(0,\infty)]$?

Open Problem 4.1. *Determine the best constant* $-\frac{1}{2} \le \beta \le \gamma$ *such that* $h_\beta(t) \in \mathcal{L}[(0,\infty)]$.

Remark 4.1. This paper is a slightly revised version of the preprint [7].

Acknowledgements

The authors thank Professor Dr. Valmir Krasniqi for his comments on the original version of this paper through an e-mail on 2 March 2015.

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