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Research Paper

A short proof of the existence of the solution to elliptic boundary problem

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Abstract

There are several methods for proving the existence of the solution to the elliptic boundary problem $Lu = f$ in D, $u|_S = 0$, $(*)$. Here L is an elliptic operator of second order, f is a given function, and uniqueness of the solution to problem (*) is assumed. The known methods for proving the existence of the solution to (*) include variational methods, integral equation methods, method of upper and lower solutions. In this paper a method based on functional analysis is proposed. This method is conceptually simple. It requires some a priori estimates and a continuation in a parameter method, which is well-known.

Keywords: *Dynamical systems method (DSM); Homeomorphism; Nonlinear equations; Surjectivity*

1. Introduction

Consider the boundary problem

$$Lu = f \quad \text{in} \quad D, \tag{1}$$

$$u = 0 \quad \text{on} \quad S, \tag{2}$$

where $D \subset \mathbb{R}^3$ is a bounded domain with a C^2-smooth boundary S, L is an elliptic operator,

$$Lu = -\partial_i(a_{ij}(x)\partial_j u) + q(x)u. \tag{3}$$

Here and below $\partial_i = \frac{\partial}{\partial x_i}$, over the repeated indices summation is understood, $1 \le i, j \le 3$, $a_{ij}(x) = a_{ji}(x)$, $\Im a_{ij}(x) = 0$,

$$c_0|\xi|^2 \le a_{ij}(x)\xi_i\overline{\xi_j} \le c_1|\xi|^2, \quad \forall x \in D, \tag{4}$$

where $c_0, c_1 > 0$ are constants independent of x and $|\xi|^2 = \sum_{j=1}^3 |\xi_j|^2$. We assume that $q(x)$ is a real-valued bounded function, $|\nabla a_{ij}(x)| \le c$. By $c > 0$ various estimation constants are denoted. In this paper the Hilbert space $H := H^0 := L^2(D)$, the Sobolev space H_0^1, the closure of $C_0^\infty(D)$ in the norm of the Sobolev space $H^1 = H^1(D)$, and the Sobolev space $H_0^2 := H^2(D) \cap H_0^1$ are used.

We assume that problem (1)-(2) has no more than one solution. This, for example, is the case if

$$(Lu, u) \ge c_2(u, u), \quad \forall u \in D(L), \tag{5}$$

where $c_2 > 0$ is a constant, and $D(L) = H_0^2$. Let us denote the norm in the Sobolev space H^ℓ by the symbol $||\cdot||_\ell$,

$$||\cdot||_\ell = \Big(\int_D (|u|^2 + |\partial u|^2 + |\partial^2 u|^2)dx\Big)^{1/2}. \tag{6}$$

By $|\partial u|^2$ the sum of the squares of the derivatives of the first order is denoted, and $|\partial^2 u|^2$ is understood similarly.

There is an enormous literature on elliptic boundary problems (see [1], [2], [3], [5], [6], [7]) to name just a few books. Several methods were suggested to study problem (1) - (2): Hilbert space method, based on the Riesz theorem about bounded linear functionals ([5], [6]), integral equations of the potential theory ([7]), method of lower and upper solutions ([2]).

The goal of this paper is to suggest a method for a proof of the existence of the solution to problem (1) - (2), based on functional analysis. This method is simple, short, and does not require too much of a background knowledge from the reader.

The background material, that is used in our proof, includes the notions of closed (and closable) linear unbounded operators and symmetric operators (see [4]), second basic inequality (see [1], [2], [5], [6])

$$||Lu||_0 \geq c_3||u||_2, \quad \forall u \in D(L), \tag{7}$$

and the definition and basic properties of the mollification operator, see, for example, [1].

Let us outline the ideas of our proof.

We prove that

1) the range $R(L)$ is a closed subspace of H^0;
2) L is closed in H^0;
3) $R(L)^\perp = \{0\}$.

This implies that $R(L) = H$, that is, problem (1) - (2) has a solution. Uniqueness of the solution follows trivially from the assumption (6).

Let us summarize the (well-known) result.

Theorem 1.1. *Assume that S is C^2- smooth, inequalities (4), (6) hold, and q is a real-valued bounded function. Then problem (1) - (2) has a solution in H_0^2 for any $f \in H^0$, and this solution is unique.*

Remark 1.1. *We are not trying to formulate the result in its maximal generality. For example, one may consider by the same method elliptic operators which are non-self-adjoint in the sense of Lagrange. In Section 2 Remark 2.2 addresses this question.*

In Section 2 proofs are given.

2. Proof

It follows from (6) that

$$||Lu|| \geq c_2||u||_2 \geq c_2||u|| \quad \forall u \in D(L), \quad ||u|| := ||u||_0. \tag{8}$$

Therefore, if $Lu = 0$ then $u = 0$. This proves the *uniqueness* of the solution.

To prove the *existence* of the solution it is sufficient to prove that the range of L, denoted by $R(L)$, is closed and its orthogonal complement is just the zero element of H^0. Indeed, one has

$$H = \overline{R(L)} \bigoplus R(L)^\perp, \tag{9}$$

where $R(L)^\perp$ denotes the orthogonal complement in $H = H^0$. Therefore, if

$$R(L) = \overline{R(L)}, \tag{10}$$

and

$$R(L)^\perp = \{0\}, \tag{11}$$

then

$$R(L) = H, \tag{12}$$

and Theorem 1.1 is proved.

The closedness of $R(L)$ follows from inequality (7). Indeed, if $Lu_n \xrightarrow[H^0]{} f$ then, by (8), $u_n \xrightarrow[H_0^2]{} u$, so $u \in D(L)$, and $Lu = f$. A more detailed argument goes as follows.

Let $v \in D(L) = H_0^2$ be arbitrary. Then

$$(f, v) \xleftarrow[n\to\infty]{} (Lu_n, v) = (u_n, Lv) \xrightarrow[n\to\infty]{} (u, Lv), \quad \forall v \in D(L). \tag{13}$$

Inequality (7) implies that $u \in H_0^2 = D(L)$. Therefore, formula (13) implies $Lu = f$. This argument proves that $R(L)$ is a closed subspace of H^0 and the operator L is closed on $D(L)$.

Let us now prove that $R(L)^\perp = \{0\}$. Assume the contrary. Then there is an element $h \in H^0$ such that

$$(Lu, h) = 0, \quad \forall u \in D(L) = H_0^2. \tag{14}$$

We want to derive from (14) that $h = 0$. To do this, first assume that $L = -\Delta$, where Δ is the Laplacian. Take an arbitrary point $x \in D$, choose $\epsilon > 0$ so that the distance $d(x, S)$ from x to S is larger than ϵ, and set $u = w_\epsilon(|x-y|)$, where $w_\epsilon(|x|)$ is a mollification kernel (see, for example, [1], p.5). This implies that $w_\epsilon(|x|) \in C_0^\infty(D) \subset D(L)$, and

$$\lim_{\epsilon\downarrow 0} || \int_D w_\epsilon(|x-y|)h(y)dy - h(x)|| = \lim_{\epsilon\downarrow 0} ||w_\epsilon * h - h|| = 0, \tag{15}$$

where $w_\epsilon * h$ denotes the convolution. Then equation (14) yields

$$-\int_D \Delta_y w_\epsilon(|x-y|)h(y)dy = -\Delta_x w_\epsilon * h = 0, \quad x \in D. \tag{16}$$

Multiply (16) by $\eta_\epsilon := w_\epsilon * h$, integrate over D, and then integrate by parts, taking into account that $\eta_\epsilon = 0$ on S if $\text{dist}(x, S) > \epsilon$. The result is

$$\int_D \nabla\eta_\epsilon(x) \cdot \overline{\nabla\eta_\epsilon(x)}dx = 0. \tag{17}$$

From (17), (4) and (6) it follows that $\nabla\eta_\epsilon = 0$ in D, so $\eta_\epsilon = const$ in D. Since this constant vanishes at the boundary S, it is equal to zero. Thus

$$\eta_\epsilon(x) = w_\epsilon * h = 0 \quad \text{in} \quad D. \tag{18}$$

Let $\epsilon \downarrow 0$ in (18) and get $h = 0$ in D. Thus, $R(\Delta)^\perp = \{0\}$, so $R(-\Delta) = H^0 = L^2(D)$.

Let us now prove that $R(L) = H^0$ for the operator (3). This is proved by a continuation in a parameter. Define $L_s = L_0 + s(L - L_0)$, $0 \le s \le 1$, $L_0 = -\Delta$, $L_1 = L$. We prove that $R(L_s) = H^0$ for all $0 \le s \le 1$, and the map $L_s : H_0^2 \to H^0$ is an isomorphism. For $s = 0$ this was proved above.

Consider equation (1) with $L = L_s$ and apply the operator L_0^{-1} to this equation. The result is

$$u + sL_0^{-1}(L - L_0)u = L_0^{-1}f. \tag{19}$$

This equation is in the space H_0^2. The norm of the operator $sL_0^{-1}(L - L_0)$ in H_0^2 is less than one if s is sufficiently small. Indeed, inequality similar to (7) holds for L_s for any $0 \le s \le 1$ with the same constant c_3, because this constant depends only on the bounds on the coefficients of L_s and these bounds can be chosen independent of $s \in [0, 1]$. Thus,

$$||L_s u||_0 \ge c_3||u||_2, \quad \forall u \in H_0^2, \quad 0 \le s \le 1. \tag{20}$$

Therefore,

$$||L_0^{-1}(L - L_0)u||_2 \le \frac{1}{c_3}||(L - L_0)u||_0 \le c_3'||u||_2, \qquad \forall u \in H_0^2, \tag{21}$$

because $||(L - L_0)u||_0 \leq c||u||_2$, where $c > 0$ is a constant independent of s. This constant depends only on the bounds on the coefficients of L. Consequently, if $sc_3' < 1$, that is, if $s < (c_3')^{-1}$, then equation (19) is uniquely solvable in H_0^2 for any $f \in H^0$, and $R(L_s) = H^0$.

Let $s_0 = \frac{1}{2}(c_3')^{-1}$. Define $L_{s_0+s'} = L_{s_0} + s'(L - L_{s_0})$. Applying the same argument and using the fact that $||L_{s_0}^{-1}||_{H^0 \to H_0^2}$ does not depend on s_0, one gets

$$||L_{s_0}^{-1}(L - L_{s_0})u||_2 \leq c_3'||u||_2. \tag{22}$$

Therefore, for $s' < (c_3)^{-1}$, one has

$$||s'L_{s_0}^{-1}(L - L_{s_0})|| < 1. \tag{23}$$

Consequently, repeating the above argument finitely many times one reaches the operator L and gets both conclusions: $R(L) = H^0$ and L is an isomorphism of H_0^2 onto H^0.

Theorem 1.1 is proved.

Remark 2.1. *The method of continuation in a parameter goes back to [8].*

Remark 2.2. *Consider the operator $L_1 = L + L'$, where L' is an arbitrary first order differential operator and L is the same as in Section 2. The operator L_1 is not necessarily symmetric. Problem* (1) - (2) *is equivalent to the operator equation*

$$u + Au = L^{-1}f \quad in \quad H^0, \tag{24}$$

where

$$A = L^{-1}L' \tag{25}$$

is a compact operator in H^0. This follows from the Sobolev embedding theorem ([1], [2]).

Therefore, the Fredholm alternative holds for equation (24)*. So, if the homogeneous version of the equation* (24) *has only the trivial solution (zero solution) then equation* (24) *is solvable for any f, and its solution $u \in H_0^2$.*

Remark 2.3. *If L is symmetric on $D(L) = H_0^2(D)$, then Theorem 1.1 shows that L is self-adjoint on $D(L)$. Indeed, the definition of the adjoint operator L^* says that $w \in D(L^*)$ if $(Lu, w) = (u, w^*)$ for all $u \in D(L)$. By Theorem 1.1 there exists $z \in D(L)$ such that $Lz = w^*$. Thus, $(Lu, w) = (u, Lz) = (Lu, z)$. Thus, $w = z$. Consequently, $D(L) = D(L^*)$, and $L = L^*$, as claimed.*

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