Global Journal of Mathematical Analysis, 3 (3) (2015) 121-125
www.sciencepubco.com/index.php/GJMA
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doi: 10.14419/gjma.v3i3.4885
Research Paper

A Hilbert-type integral inequality with its best extension

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Abstract

By using the way of weight function and the technique of real analysis, a new Hilbert-type integral inequality with a kernel as $min\{x^{\lambda_1}, y^{\lambda_2}\}$ and its equivalent form are established. As application, the constant factor on the plane are the best value and its best extension form with some parameters and the reverse forms are also considered.

Keywords: *Weight Function, Hilbert-Type Integral Inequality, Best Extension, Reverse.*

1. Introduction

If $f(x), g(x) \geq 0, 0 < \int_0^\infty f^p(x)dx < \infty$ and $0 < \int_0^\infty g^q(x)dx < \infty$,then we have[1]

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{x+y} dxdy < \pi(\int_0^\infty f^2(x)dx \int_0^\infty g^q(y)dy)^{\frac{1}{2}}, \tag{1}$$

where the constant factor $\pi$is the best possible. we call (1) Hilbert's integral inequality.(1) is important in analysis and its application$[2-3]$. In recent years, by using the way of weight function,a number of extensions (1) were given by Yang et al.$[4-6]$. In 2010, Yang [7] gave a new inequality with a non-homogeneous kernel as follows:

$$\int_0^\infty \int_0^\infty min\{x, y\}^\lambda f(x)g(y)dxdy < \frac{rs}{\lambda}(\int_0^\infty x^{p(1+\frac{\lambda}{r})-1} f^p(x)dx)^{\frac{1}{p}}(\int_0^\infty y^{q(1+\frac{\lambda}{s})-1} g^q(y)dy)^{\frac{1}{q}}. \tag{2}$$

where the constant factor$\frac{rs}{\lambda}$ is the best possible.

In this paper, by introducing some parameters and using the way of weight function and the technic of real analysis and complex analysis,we give a new Hilbert-type integral inequality with a kernel as $k(x,y) := min\{x^{\lambda_1}, y^{\lambda_2}\}$ which is an extensions of (2). As application, the equivalent form and the reverse forms are obtained.

2. Some Lemmas

Lemma 2.1 *If* $\lambda_1, \lambda_2 > 0, v_1, v_2 > 0, v_1 + v_2 = 1, \eta(v_1, v_2) := \frac{1}{v_1 v_2}$,*define the following weight function*
$\omega_{\lambda_1\lambda_2}(v_1, v_2, y) := \int_0^\infty k(x,y)\frac{y^{-\lambda_2 v_2}}{x^{1+\lambda_1 v_1}}dx \quad (y > 0)$,

$\omega_{\lambda_1\lambda_2}(v_1,v_2,x) := \int_0^\infty k(x,y)\frac{x^{-\lambda_1 v_1}}{y^{1+\lambda_2 v_2}}dy \quad (x>0),$
then we have

$$\omega_{\lambda_1\lambda_2}(v_1,v_2,y) = \frac{1}{\lambda_1}\eta(v_1,v_2), \omega_{\lambda_1\lambda_2}(v_1,v_2,y) = \frac{1}{\lambda_2}\eta(v_1,v_2). \tag{3}$$

Proof *Setting* $u = x^{\lambda_1}/y^{\lambda_2}$, *we obtain*

$$\begin{aligned}\omega_{\lambda_1,\lambda_2}(v_1,v_2,y) &= \frac{1}{\lambda_1}\int_0^\infty min\{1,u\}u^{-v_1-1}du \\ &= \frac{1}{\lambda_1}[\int_0^1 u^{v_2-1}du + \int_1^\infty u^{-v_1-1}du] \\ &= \frac{1}{\lambda_1}(\frac{1}{v_2}+\frac{1}{v_1}) = \frac{1}{\lambda_1}\eta(v_1,v_2).\end{aligned}$$

Similarly, we can calculate that
$\omega_{\lambda_1\lambda_2}(v_1,v_2,y) = \frac{1}{\lambda_2}\eta(v_1,v_2).$

Lemma 2.2 *As the assumption of Lemma 2.1, if* $p>1, \frac{1}{p}+\frac{1}{q}=1, f(x)\geq 0$,*we have*

$$J := \int_0^\infty y^{-p\lambda_2 v_2 - 1}\left[\int_0^\infty k(x,y)f(x)dx\right]^p dy \leq \left[\frac{\eta(v_1,v_2)}{\lambda_1^{1/q}\lambda_2^{1/p}}\right]^p \int_0^\infty x^{p(1+\lambda_1 v_1)-1}f^p(x)dx. \tag{4}$$

Proof *By Höleder's inequality with weight*[8],*we obtain*

$$\begin{aligned}&\int_0^\infty f(x)k(x,y)dx = \int_0^\infty k(x,y)[\frac{x^{(1+\lambda_1 v_1)/q}}{y^{(1+\lambda_2 v_2)/p}}f(x)][\frac{y^{(1+\lambda_2 v_2)/p}}{x^{(1+\lambda_1 v_1)/q}}]dx \\ \leq\ & \{\int_0^\infty k(x,y)\frac{x^{(1+\lambda_1 v_1)(p-1)}}{y^{(1+\lambda_2 v_2)}}f^p(x)dx\}^{1/p}\{\int_0^\infty k(x,y)\frac{y^{(1+\lambda_2 v_2)(q-1)}}{x^{(1+\lambda_1 v_1)}}dx\}^{1/q}\end{aligned}$$

$$= y^{\frac{1}{p}+\lambda_2 v_2}[\omega_{\lambda_1\lambda_2}(v_1,v_2,y)]^{1/q}\{\int_0^\infty k(x,y)\frac{x^{(1+\lambda_1 v_1)(p-1)}}{y^{(1+\lambda_2 v_2)}}f^p(x)dx\}^{1/p}. \tag{5}$$

By the result of Lemma 2.1 and Fubini's theorem[9], *we obtain*

$$\begin{aligned}J &\leq [\frac{\eta(v_1,v_2)}{\lambda_1}]^{p-1}\int_0^\infty\int_0^\infty k(x,y)\frac{x^{(1+\lambda_1 v_1)(p-1)}}{y^{(1+\lambda_2 v_2)}}f^p(x)dxdy \\ &= [\frac{\eta(v_1,v_2)}{\lambda_1}]^{p-1}\int_0^\infty \omega_{\lambda_1,\lambda_2}(\alpha,v_1,v_2,x)x^{p(1+\lambda_1 v_1)-1}f^p(x)dx.\end{aligned}$$

Hence by the above results, we have (4). The lemma is proved.

3. Main results

Theorem 3.1 *As the assumption of Lemma 2.1, if* $p>1, \frac{1}{p}+\frac{1}{q}=1, f(x),g(y)\geq 0, 0<\int_0^\infty x^{p(1+\lambda_1 v_1)-1}f^p(x)dx < \infty, 0 < \int_0^\infty y^{q(1+\lambda_2 v_2)-1}g^q(y)dy < \infty$, *then we have two equivalent inequalities as*

$$I := \int_0^\infty\int_0^\infty k(x,y)f(x)g(y)dxdy < \frac{\eta(v_1,v_2,)}{\lambda_1^{1/q}\lambda_2^{1/p}}\left\{\int_0^\infty x^{p(1+\lambda_1 v_1)-1}f^p(x)dx\right\}^{\frac{1}{p}}\left\{\int_0^\infty y^{q(1+\lambda_2 v_2)-1}g^q(y)dy\right\}^{\frac{1}{q}}, \tag{6}$$

$$J = \int_0^\infty y^{-p\lambda_2 v_2-1}\left[\int_0^\infty k(x,y)f(x)dx\right]^p dy < [\frac{\eta(v_1,v_2)}{\lambda_1^{1/q}\lambda_2^{1/p}}]^p\int_0^\infty x^{p(1+\lambda_1 v_1)-1}f^p(x)dx. \tag{7}$$

Proof *We conform that the middle of (5) keeps the form of strict inequality. Otherwise,there exist constants A and B, such that they are not all zero and* [9]

$A\frac{x^{(1+\lambda_1 v_1)(p-1)}}{y^{(1+\lambda_2 v_2)}}f^p(x)=B\frac{y^{(1+\lambda_2 v_2)(q-1)}}{x^{(1+\lambda_1 v_1)}}g^q(y) \quad a.e.in(0,\infty)\times(0,\infty).$

It follows $Ax^{p(1+\lambda_1 v_1)}f^p(x)=By^{q(1+\lambda_2 v_2)}g^q(y) \quad a.e.in(0,\infty)\times(0,\infty)$. *Assuming that* $A\neq 0$, *there exists* $y>0$, *such that* $x^{p(1+\lambda_1 v_1)-1}f^p(x)=\frac{By^{q(1+\lambda_2 v_2)}g^q(y)}{Ax}a.e.in x\in(0,\infty)$.*This contradicts the fact that* $0<\int_0^\infty x^{p(1+\lambda_1 v_1)-1}f^p(x)dx<\infty$.*Hence(5) takes the strict sign-inequality,so does (4). In addition, we have (7). By Hölder's inequality*[8] *,we find*

$$I=\int_0^\infty [y^{-\lambda_2 v_2-1/p}\int_0^\infty k(x,y)f(x)dx][y^{\lambda_2 v_2+1/p}g(y)]dy\leq J^{1/p}[\int_0^\infty y^{q(1+\lambda_2 v_2)-1}g^q(y)dy]^{1/q}. \tag{8}$$

Then by (7), we have (6).On the other hand,assuming that (7) is valid ,setting $g(y):=y^{-p\lambda_2 v_2-1}[\int_0^\infty k(x,y)f(x)]dx]^{p-1}$ *,then we have* $J=\int_0^\infty y^{q(1+\lambda_2 v_2)-1}g^q(y)dy$,*Through (4),it follows* $J<\infty$.*If*$J=0$,*then 7is naturally valid.If* $0<J<\infty$, *then by(6),we find*

$$0<\int_0^\infty y^{q(1+\lambda_2 v_2)-1}g^q(y)dy=J=I<\frac{\eta(v_1,v_2)}{\lambda_1^{1/q}\lambda_2^{1/p}}\left\{\int_0^\infty x^{p(1+\lambda_1 v_1)-1}f^p(x)dx\right\}^{\frac{1}{p}}\left\{\int_0^\infty y^{q(1+\lambda_2 v_2)-1}g^q(y)dy\right\}^{\frac{1}{q}}, \tag{9}$$

$$J^{1/p}=\left\{\int_0^\infty y^{q(1+\lambda_2 v_2)-1}g^q(y)dy\right\}^{\frac{1}{p}}<\frac{\eta(v_1,v_2}{\lambda_1^{1/q}\lambda_2^{1/p}}\left\{\int_0^\infty x^{p(1+\lambda_1 v_1)-1}f^p(x)dx\right\}^{\frac{1}{p}}. \tag{10}$$

and then we have (7),which is equivalent to (6). The Theorem is proved.

Theorem 3.2 *Under the conditions of Theorem 3.1the constants* $\frac{\eta(v_1,v_2)}{\lambda_1^{1/q}\lambda_2^{1/p}}$ *and* $\left[\frac{\eta(v_1,v_2)}{\lambda_1^{1/q}\lambda_2^{1/p}}\right]^p$ *in (6)and (7) are the best possible.*

Proof *For* $0<\varepsilon<pv_1$,*if*

$\tilde{f}(x):=\begin{cases}0, & x\in(0,1)\\ x^{-\lambda_1 v_1-\lambda_1\varepsilon/p-1}, & x\in[1,\infty)\end{cases}$, $\tilde{g}(y):=\begin{cases}0, & y\in(0,1)\\ y^{-\lambda_2 v_2-\lambda_2\varepsilon/q-1}, & y\in[1,\infty)\end{cases}$.

we can calculate

$\tilde{J}:=\left\{\int_0^\infty x^{p(1+\lambda_1 v_1)-1}\tilde{f}^p(x)dx\right\}^{\frac{1}{p}}\left\{\int_0^\infty y^{q(1+\lambda_2 v_2)-1}\tilde{g}^q(y)dy\right\}^{\frac{1}{q}}=\frac{1}{\lambda_1^{1/p}\lambda_2^{1/q}\varepsilon}.$

by Fubini's theorem[9]*,we obtain*

$$\begin{aligned}
I: &= \int_0^\infty\int_0^\infty k(x,y)\tilde{f}(x)\tilde{g}(x)dxdy\\
&= \int_1^\infty y^{-\lambda_2 v_2-\lambda_2\varepsilon/q-1}\left[\int_1^\infty k(x,y)x^{-\lambda_1 v_1-\lambda_1\varepsilon/p-1}dx\right]dy\\
&\underset{u=(x^{\lambda_1}/y^{\lambda_2})}{=} \frac{1}{\lambda_1}\int_1^\infty y^{-\lambda_2\varepsilon-1}\left[\int_{y^{-\lambda_2}}^\infty min\{1,u\}u^{-v_1-\varepsilon/p-1}du\right]dy\\
&= \frac{1}{\lambda_1}\left\{\int_1^\infty y^{-\lambda_2\varepsilon-1}\left[\int_{y^{-\lambda_2}}^1 u^{v_2-\varepsilon/p-1}du+\int_1^\infty u^{-v_1-\varepsilon/p-1}du\right]dy\right\}\\
&= \frac{1}{\lambda_1}\left\{\int_0^1(\int_{u^{-1/\lambda_2}}^\infty y^{-\lambda_2\varepsilon-1}dy)u^{v_2-\varepsilon/p-1}du+\frac{1}{\lambda_2\varepsilon}\int_1^\infty u^{-v_1-\varepsilon/p-1}du\right\}\\
&= \frac{1}{\lambda_1\lambda_2\varepsilon}\left[\int_0^1 u^{v_2+\varepsilon/q-1}du+\int_1^\infty u^{-v_1-\varepsilon/p-1}du\right].
\end{aligned}$$

If there exists a positive number $k\leq\frac{\eta(v_1,v_2,)}{\lambda_1^{1/q}\lambda_2^{1/p}}$,*such that (6)is still valid when we replace* $\frac{\eta(v_1,v_2)}{\lambda_1^{1/q}\lambda_2^{1/p}}$ *by k then in particular* $\tilde{f},\tilde{g}$, *by the above results ,we find*

$$\frac{1}{\lambda_1\lambda_2}\left[\int_0^1 u^{v_2+\varepsilon/q-1}du+\int_1^\infty u^{-v_1-\varepsilon/p-1}du\right]=\varepsilon\tilde{I}<\varepsilon k\tilde{J}=k\frac{1}{\lambda_1^{1/p}\lambda_2^{1/q}}. \tag{11}$$

By Fatou lemma[9] *and(11)we obtain*

$\frac{\eta(v_1,v_2)}{\lambda_1^{1/q}\lambda_2^{1/p}}=\frac{1}{\lambda_1^{1/q}\lambda_2^{1/p}}\left[\int_0^1\underset{\varepsilon\to0^+}{\underline{lim}}u^{v_2+\varepsilon/q-1}du+\int_1^\infty\underset{\varepsilon\to0^+}{\underline{lim}}u^{-v_1-\varepsilon/p-1}du\right]\leq$

$\frac{1}{\lambda_1^{1/q}\lambda_2^{1/p}} \underset{\varepsilon\to 0^+}{lim} \left[\int_0^1 u^{v_2+\varepsilon/q-1}du + \int_1^\infty u^{-v_1-\varepsilon/p-1}du\right] \leq k.$

Hence $k = \frac{\eta(v_1,v_2)}{\lambda_1^{1/q}\lambda_2^{1/p}}$ *is the best value of (6).We conform that* $\left[\frac{\eta(v_1,v_2)}{\lambda_1^{1/q}\lambda_2^{1/p}}\right]^p$ *in (7)is the best possible, otherwise we can get a contradiction by 8that the constant in (6) is not the best possible.The theorem is proved.*

Theorem 3.3 *As the assumption of Theorem 3.1, if* $0 < p < 1, I, J$*are defined by (6)and (4),then we have equivalent inequalities as*

$$I > \frac{\eta(v_1, v_2,)}{\lambda_1^{1/q}\lambda_2^{1/p}} \left\{\int_0^\infty x^{p(1+\lambda_1 v_1)-1} f^p(x)dx\right\}^{\frac{1}{p}} \left\{\int_0^\infty y^{q(1+\lambda_2 v_2)-1} g^q(y)dy\right\}^{\frac{1}{q}}. \tag{12}$$

$$J > \left[\frac{\eta(v_1, v_2)}{\lambda_1^{1/q}\lambda_2^{1/p}}\right]^p \int_0^\infty x^{p(1+\lambda_1 v_1)-1} f^p(x)dx. \tag{13}$$

where the constants in (12)and (13) are the best possible.

Proof *By the reverse Hölder's inequality*[9]*, we have the reverse of (5),(4),(8). By the same way of Theorem 3.1,we show that the reverse of(5) keeps the strict sign-inequality and then we have (13).By the reverse of (8) ,we have (12).Assuming that (12)is valid,setting g(y) as Theorem 1.In view of (4), we have* $J > 0$*.If* $J = \infty$*, then (13) is valid,if*$0 < J < \infty$*,then by12we easily find the reverses of (9) and (10),hence the reverse of 7is valid which is equivalent to (12).*

If there exists a positive number $k \leq \frac{\eta(v_1,v_2,)}{\lambda_1^{1/q}\lambda_2^{1/p}}$*,such that (12)is still valid when we replace* $\frac{\eta(v_1,v_2)}{\lambda_1^{1/q}\lambda_2^{1/p}}$ *by kthen by the reverse of (11),we find*

$$\frac{1}{\lambda_1^{1/q}\lambda_2^{1/p}} \left[\int_0^1 u^{v_2+\varepsilon/q-1}du + \int_1^\infty u^{-v_1-\varepsilon/p-1}du\right] > k. \tag{14}$$

If $0 < \varepsilon_0 < |q|\, v_2$*, when* $0 < \varepsilon \leq \varepsilon_0 u \in (0, 1]$*,*$u^{\varepsilon/q} \leq u^{\varepsilon_0/q}$ *is valid ,and we obtain*
$\int_0^1 u^{v_2+\varepsilon_0/q-1}du \leq \frac{1}{\varepsilon_0/q+v_2}$
by Lebesgue's control convergence theorem [10]*, we obtain*
$\int_0^1 u^{v_2+\varepsilon/q-1}du = \int_0^1 u^{v_2-1}du + o(1) \quad (\varepsilon \to 0^+).$
We find $\frac{\eta(v_1,v_2,\alpha)}{\lambda_1^{1/q}\lambda_2^{1/p}} \geq k(\varepsilon \to 0^+) of(14)$*, hence* $\frac{\eta(v_1,v_2,\alpha)}{\lambda_1^{1/q}\lambda_2^{1/p}} = k$ *is the best value of (12). We conform that the constant factor of (13) is the best possible,otherwise we can get a contradiction that the constant in reverse of (8) is not the best possible.The theorem is proved.*

4. Conclusion

For $\lambda_1 = \lambda_2 = \lambda, v_1 = \frac{1}{r}, v_2 = \frac{1}{s}$ in (6), it deduces to (2). Hence inequality (6) is the best extension of (2).

Acknowledgements

The author would like to thank the editor and the referees. This work is supported by the key projects of Jiaying University(2014KJZ02) and College students' innovative training program of Guangdong(201510582021).

References

[1] Hardy G H.Note on a Theorem of H ilbert Concern ing Series of Positive Terms [J]. Proc London Math Soc,1925,23 2XLV-XLVL.

[2] Hardy GH,Littewood J E,Polya G.Inequalities[M].Cambridge:Cambridge University Press,1952.

[3] Mitrinovic D S, Pecaric J, Fink A M. Inequalities Involving Functions and The ir Integrals and Derivatives [M]. Boston:Kluwer Academic Publishers, 1991.

[4] Bicheng Yang. A new Hilbert's type integral inequality[J].Journal Mathematics,2007,33(4):849-859.

[5] Bicheng Yang.On the norm of an integral operator and applications[J].J Math And Appl,2006,321:182-192.

[6] Weiliang Wu.A new Hilbert-type integral inequality with some parameters and its applications[J].J of Xibei Normal University:Natual Science2012,48 626-30.

[7] Bicheng Yang.On a Hilbert-type integral inequality with the homogeneous kernel of degree[J].J of Shanghai University(EnglEd):Natual Science2010,146391-395.

[8] Jichang Kuang,Applied Inequalities,Shangdong Science Press,Jinan,2004.

[9] Jichang Kuang,Introduction to real analysis,Hunan Education Press,Changsha,1996.