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Research Paper

Logarithmic gradient estimates to a Monge-Ampère Type equation on $\mathbb{S}^n$

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Abstract

Monge-Ampère type equations arise naturally from many geometric problems. In this paper, we are concerned with one of these Monge-Ampère type equations on n dimensional sphere $\mathbb{S}^n$ and obtain logarithmic gradient estimate by using Bernstein technique.

Keywords: *Monge-Ampère equation; logarithmic; gradient estimate.*

1. Introduction

It is well-known that Yau [1] gave logarithmic gradient estimate of positive solution to harmonic function on complete Riemannian manifold. As applications, he obtained the Liouville theorem which asserts that a positive harmonic function on a complete manifold with nonnegative Ricci curvature is constant. Logarithmic gradient estimates can also be used to derive Harnack inequality through path integral (see also [2]). Its generalization to some fully nonlinear elliptic equations are interesting problems and the Monge-Ampère type equations are the main concerns of this paper.

Many fully nonlinear elliptic equations arise from conformal geometry and optics geometry, and there have been some works concerning the existence or non-existence of global and local gradient estimates for these equations. Local estimates in general do not hold for fully nonlinear equation due to the counterexample of Pogorelov to Monge-Ampère equation, which asserted that no interior gradient estimates hold for dimensions $n \geq 3$. However, there are plenty of works concerning global or local gradient estimates for geometric fully nonlinear elliptic equations, the readers can see these in [3],[4], [5] and [6] etc. for example.

In this paper, we are mainly concerned with the global logarithmic gradient estimate to the following problem

$$det(u_{ij} + u\delta_{ij}) = (u^2 + |\nabla u|^2)^{\frac{n}{2}} f(x) \qquad \text{on} \qquad S^n, \tag{1}$$

where f is a positive function on $\mathbb{S}^n$. This kind of operator arises from geometry problems. It is related to the Minkowski problems and problems which prescribing Gauss curvature for a radial graph on $\mathbb{S}^n$. Our main results are stated in the following.

Theorem 1.1 *Let $f \in C^1(\mathbb{S}^n)$ be a positive function and $0 < u \in C^3(\mathbb{S}^n)$ be an admissible solution of equation (1), i.e., the matrix $\{u_{ij} + u\delta_{ij}\}$ be positive definite. We further assume that f satisfies $|\nabla f| < nf^{1-\frac{1}{n}}$, then we have $\sup_{x\in\mathbb{S}^n} |\nabla \log u| \leq C_0$, where $C_0 = C_0(n, f, |\nabla f|)$.*

The above logarithmic gradient estimate naturally implies the Harnack inequality for the solution, i.e., we obtain the following Corollary.

Corollary 1.2 *Assume f satisfy the same conditions as listed above in Theorem 1.1 and u be a positive admissible solution of equation (1), then we have the following Harnack inequality for u. Then $\max_{x\in\mathbb{S}^n} u \le C\min_{x\in\mathbb{S}^n} u$, where the constant C also depends on n, f and $|\nabla f|$.*

Since the logarithmic gradient estimate is obtained in Theorem 1.1, the proof of Harnack inequality is easy and standard. For convenience of the reader, we sketch the proof here. Let x_1 and x_2 be two points where $log(u)$ attains its maximum and minimum values on $\mathbb{S}^n$ respectively, and let γ be a minimal geodesic connecting x_1 and x_2, then

$$log\frac{max_{x\in\mathbb{S}^n}u}{min_{x\in\mathbb{S}^n}u} = logu(x_1) - logu(x_2) = |\int_\gamma \frac{dlogu}{ds}ds| \le \int_\gamma |\nabla logu|ds \int_\gamma \frac{|\nabla u|}{u}ds \le C,$$

where the logarithmic gradient estimate is used in the last inequality. Hence the Corollary 1.2 follows from the above inequality.

Remark 1.3 *Since the solution of equation (1) is invariant under multiplying by a constant, we can scale the solution such that $\min_{x\in\mathbb{S}^n} u = 1$. Therefore Harnack inequality in Corollary 1.2 implies global C^0 estimate and logarithmic gradient estimate further implies global C^1 estimate of the solution.*

In the next section, we will prove Theorem 1.1. We will use some facts concerning the properties of elementary symmetric function during the proving process. The definitions and the proof of these properties are standard, the readers can consult them on other reference books, such as [7] or [8], etc..

2. Proof of the logarithmic gradient estimates

To prove Theorem 1.1, we first rewrite (1) into another form with respect to $v = logu$ and then we use Bernstein technique to derive the gradient estimate of v. In this direction, we consider an auxiliary function $G = \frac{1}{2}|\nabla v|^2$ and calculate at the point where G attains its maximum. For calculation convenience, we choose appropriate orthonormal frame field at this maximum point.

Proof of Theorem 1.1.

Set $v = \log u$, i.e., $u = e^v$. Then we have $u_i = e^v v_i$, $u_{ij} = e^v(v_{ij} + v_i v_j)$ and $u^2 + |\nabla u|^2 = e^{2v}(1 + |\nabla v|^2)$. Hence the equation (1) can be written as

$$det(v_{ij} + v_i v_j + \delta_{ij}) = (1 + |\nabla v|^2)^{\frac{n}{2}} f(x), \tag{1}$$

To obtain gradient estimate of v, we consider the auxiliary function $G = \frac{1}{2}|\nabla v|^2$. Let G attain its maximum at $x_o \in S^n$. Then we can choose a local orthonormal frame field $\{e_1, ..., e_i, ..., e_n\}$ at x_o such that

$$\nabla v = v_1 e_1, \qquad v_1 > 0, \tag{2}$$

and therefore $v_i = 0$ for $2 \le i \le n$.

Since x_o is the maximum point of G, we see $G_i = 0$ and $G_{ij} \le 0$ which means non-positive definite at x_o. It follows from (2) that

$$G_i = \sum_{k=1}^n v_k v_{ki} = 0, \qquad \text{i.e.,} \qquad v_{1i} = 0 \qquad \forall\ 1 \le i \le n \tag{3}$$

and

$$G_{ij} = \sum_{k=1}^n v_{ki} v_{kj} + \sum_{k=1}^n v_k v_{kij} \le 0, \tag{4}$$

From (3), we can rotate the frame $e_2, ..., e_n$, such that the matrix $\{v_{ij}\}$ is diagonal at x_o. Therefore we may assume at x_o

$$\{a_{ij}\} := \{v_{ij} + v_i v_j + \delta_{ij}\} = diag\{1 + v_1^2, 1 + v_{22}, ..., 1 + v_{nn}\} =: diag\{\lambda_1, \lambda_2, ..., \lambda_n\}. \tag{5}$$

The following calculation will be fixed at the point x_o. Since the matrix $\{u_{ij} + u\delta_{ij}\}$ is positive definite by admissibility assumption of the solution u, we know the matrix $\{a_{ij}\}$ is also positive definite. Denote $F^{ij} = \frac{\partial det(a_{kl})}{\partial a_{ij}}$. From the admissibility of the solution we see the matrix $\{F^{ij}\}$ is also positive definite and therefore $F^{ij}G_{ij} \leq 0$. We also know that the matrix $\{F^{ij}\}$ is diagonal since $\{a_{ij}\}$ is diagonal.

Noting the following Ricci identity on $\mathbb{S}^n$

$$v_{jii} = v_{iji} = v_{iij} + v_s R_{siji} = v_{iij} + v_s(\delta_{sj}\delta_{ii} - \delta_{si}\delta_{ij}) = v_{iij} + v_j - v_i\delta_{ij},$$

we obtain from (4) that

$$F^{ii}G_{ii} = \sum_{i=1}^{n} F^{ii}v_{ii}^2 + \sum_{i,s=1}^{n} F^{ii}v_{iis}v_s + v_1^2\sum_{i=2}^{n} F^{ii} \leq 0. \tag{6}$$

To deal with the term involving three order derivative in (6), we first take covariant derivative of equation (1) in the direction e_s and contract with v_s to get

$$\sum_{i=1}^{n} F^{ii}(v_{iis} + 2v_iv_{is})v_s = (1 + |\nabla v|^2)^{\frac{n}{2}} f_s v_s + \sum_{i=1}^{n} n(1 + |\nabla v|^2)^{\frac{n}{2}-1} v_iv_{is}v_s f. \tag{7}$$

From (2) and (3) we obtain

$$\sum_{i,s=1}^{n} F^{ii}v_{iis}v_s = (1 + |\nabla v|^2)^{\frac{n}{2}} f_1 v_1.$$

Substituting the above equality into (6), we get

$$F^{ii}G_{ii} = \sum_{i=2}^{n} F^{ii}v_{ii}^2 + v_1^2\sum_{i=2}^{n} F^{ii} + (1 + |\nabla v|^2)^{\frac{n}{2}} f_1 v_1 \leq 0. \tag{8}$$

Recalling that $\lambda_1 = a_{11} = 1 + v_1^2$ and $\lambda_i = a_{ii} = 1 + v_{ii}$ for $2 \leq i \leq n$, therefore for $i \geq 2$, we have $v_{ii}^2 = \lambda_i^2 - 2\lambda_i + 1$ and (8) becomes

$$0 \geq \sum_{i=2}^{n} F^{ii}\lambda_i^2 - 2\sum_{i=2}^{n} F^{ii}\lambda_i + (1 + v_1^2)\sum_{i=2}^{n} F^{ii} + (1 + |\nabla v|^2)^{\frac{n}{2}} f_1 v_1 := I + II + III + IV. \tag{9}$$

We calculate $I - IV$ term by term. For convenience, we first introduce some notations. We denote by $S_k(\lambda)$ the $k-$th elementary symmetric function of $\lambda \in \mathbb{R}^n$ and $S_k(\lambda|i)$ the sum of the terms of $S_k(\lambda)$ not containing the factor λ_i. Therefore $F^{ii} = S_{n-1}(\lambda|i)$ by our notations.

For term I, we have

$$I = \sum_{i=2}^{n} S_{n-1}(\lambda|i)\lambda_i^2 = \sum_{i=2}^{n} S_n(\lambda)\lambda_i = S_n(\lambda)S_1(\lambda|1) > 0, \tag{10}$$

where we have used the fact for every i, $S_{n-1}(\lambda|i)\lambda_i = S_n(\lambda)$ and $S_1(\lambda|1) > 0$.

For term II, we obtain

$$II = -2\sum_{i=2}^{n} F^{ii}\lambda_i = -2\sum_{i=2}^{n} S_{n-1}(\lambda|i)\lambda_i = -2(n-1)S_n(\lambda). \tag{11}$$

For term III, we get

$$\begin{aligned} III =& (1 + v_1^2)\sum_{i=2}^{n} F^{ii} = (1 + v_1^2)(\sum_{i=1}^{n} S_{n-1}(\lambda|i) - S_{n-1}(\lambda|1)) \\ =& (1 + v_1^2)S_{n-1}(\lambda) - S_n(\lambda), \end{aligned} \tag{12}$$

where we have used the equality $\sum_{i=1}^{n} S_{n-1}(\lambda|i) = S_{n-1}(\lambda)$.

Using equation (1), we obtain

$$IV = S_n(\lambda)\frac{f_1}{f}v_1. \tag{13}$$

Substituting (10)-(13) into (9), we get

$$S_n(\lambda)S_1(\lambda|1) - 2(n-1)S_n(\lambda) + (1+v_1^2)S_{n-1}(\lambda) - S_n(\lambda) + S_n(\lambda)\frac{f_1}{f}v_1 \leq 0. \tag{14}$$

We still need to estimate $S_{n-1}(\lambda)$ in terms of $S_n(\lambda)$. Recalling the Newton-Maclaurin inequality

$$\left[S_n(\lambda)\right]^{\frac{1}{n}} \leq \left[\frac{S_{n-1}(\lambda)}{C_n^{n-1}}\right]^{\frac{1}{n-1}},$$

we get

$$S_{n-1}(\lambda) \geq nS_n(\lambda)^{\frac{n-1}{n}}. \tag{15}$$

Combining (14) and (15), we finally obtain

$$n(1+v_1^2)S_n(\lambda)^{\frac{n-1}{n}} + S_n(\lambda)\frac{f_1}{f}v_1 \leq (2n-1)S_n(\lambda).$$

Dividing by $S_n(\lambda)$ on both sides of the above inequality and using (1), we obtain

$$n(1+v_1^2)((1+v_1^2)^{\frac{n}{2}}f)^{-\frac{1}{n}} - \frac{|\nabla f|}{f}v_1 \leq 2n-1,$$

i.e.,

$$\left(nf^{-\frac{1}{n}} - \frac{|\nabla f|}{f}\right)\cdot v_1 \leq 2n-1. \tag{16}$$

If $nf^{-\frac{1}{n}} - \frac{|\nabla f|}{f} > 0$, then we have $v_1 \leq C(n, f, |\nabla f|)$ which is equivalent to $|\nabla logu| \leq C$ and therefore we complete the proof of Theorem 1.1.

3. Conclusion

We have obtained Harnack inequality of Equation (1) by using logarithmic gradient estimate. Local C^2 apriori estimate can also be obtained and the $C^{2,\alpha}$ bounds are standard by Evans-Krylov estimates. By virtue of continuity method, we can prove that if $f \in C^\infty(\mathbb{S}^n)$ is a positive function, then there exists a C^∞ solution to equation (1).

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