Global Journal of Mathematical Analysis, *3 (x) (2015) xxx-xxx*
©Science Publishing Corporation
www.sciencepubco.com/index.php/GJMA
doi:
Research Paper

A logarithmically completely monotonic function involving the gamma function and originating from the Catalan numbers

Fang-Fang Liu*, Xiao-Ting Shi, Feng Qi

Department of Mathematics, College of Science, Tianjin Polytechnic University, Tianjin City, 300387, China

**Corresponding author e-mail: fang-liu@qq.com*

Abstract

In the paper, the author finds necessary and sufficient conditions for a function involving the gamma function and originating from investigation of properties of the Catalan numbers in combinatorics to be logarithmically completely monotonic.

Keywords: *necessary and sufficient condition; logarithmically completely monotonic function; gamma function; Catalan number*

1. Introduction

It is known [24] that, in combinatorics, the Catalan numbers C_n for $n \ge 0$ form a sequence of natural numbers that occur in tree enumeration problems such as "In how many ways can a regular n-gon be divided into $n-2$ triangles if different orientations are counted separately?" The solution is the Catalan number C_{n-2}. The first few Catalan numbers C_n for $0 \le n \le 11$ are

1, 1, 2, 5, 14, 42, 132, 429, 1430, 4862, 16796, 58786.

Explicit formulas of C_n for $n \ge 0$ include

$$C_n = \frac{1}{n+1}\binom{2n}{n} = \frac{(2n)!}{n!(n+1)!} = \frac{2^n(2n-1)!!}{(n+1)!} = (-1)^n 2^{2n+1}\binom{\frac12}{n+1} = \frac{1}{n}\binom{2n}{n-1} = {}_2F_1(1-n,-n;2;1)$$

and

$$C_n = \frac{4^n\Gamma(n+1/2)}{\sqrt{\pi}\,\Gamma(n+2)}, \tag{1}$$

where

$$\Gamma(z) = \int_0^\infty t^{z-1}e^{-t}\,\mathrm{d}\,t, \quad \Re(z) > 0$$

is the classical Euler gamma function and

$$ {}_pF_q(a_1,\dots,a_p;b_1,\dots,b_q;z)=\sum_{n=0}^{\infty}\frac{(a_1)_n\cdots(a_p)_n}{(b_1)_n\cdots(b_q)_n}\frac{z^n}{n!} $$

is the generalized hypergeometric series defined for complex numbers $a_i\in\mathbb{C}$ and $b_i\in\mathbb{C}\setminus\{0,-1,-2,\dots\}$, for positive integers $p,q\in\mathbb{N}$, and in terms of the rising factorials

$$ (x)_n=\begin{cases}x(x+1)(x+2)\cdots(x+n-1), & n\ge1,\\ 1, & n=0.\end{cases} $$

The asymptotic form for the Catalan numbers is

$$ C_x\sim\frac{4^x}{\sqrt{\pi}}\left(x^{-3/2}-\frac{9}{8}x^{-5/2}+\frac{145}{128}x^{-7/2}+\cdots\right). $$

For more detailed information on the Catalan numbers C_n, please refer to the monographs [1, 2] and references therein.

In the paper [23], motivated by the explicit expression (1) and by virtue of an integral representation of the gamma function $\Gamma(x)$, the authors established an integral representation of the Catalan numbers C_x for $x\ge0$.

Theorem 1 ([23, Theorem 1])**.** *For $x\ge0$, we have*

$$ C_x=\frac{e^{3/2}4^x(x+1/2)^x}{\sqrt{\pi}\,(x+2)^{x+3/2}}\exp\left[\int_0^\infty\beta(t)\left(e^{-t/2}-e^{-2t}\right)e^{-xt}\,\mathrm{d}\,t\right], \tag{2} $$

where

$$ \beta(t)=\frac{1}{t}\left(\frac{1}{e^t-1}-\frac{1}{t}+\frac{1}{2}\right). $$

Recall from [4, Chapter XIII], [22, Chapter 1], and [25, Chapter IV] that an infinitely differentiable function f is said to be completely monotonic on an interval I if it satisfies

$$ 0\le(-1)^kf^{(k)}(x)<\infty $$

on I for all $k\ge0$. Recall from [9, 10] that an infinitely differentiable and positive function f is said to be logarithmically completely monotonic on an interval I if

$$ 0\le(-1)^k[\ln f(x)]^{(k)}<\infty $$

hold on I for all $k\in\mathbb{N}$. For more information on logarithmically completely monotonic functions, please refer to [11, 12, 15, 21].

The formula (2) can be rearranged as

$$ \ln\left[\frac{\sqrt{\pi}\,(x+2)^{x+3/2}}{e^{3/2}4^x(x+1/2)^x}C_x\right]=\int_0^\infty\beta(t)\left(e^{-t/2}-e^{-2t}\right)e^{-xt}\,\mathrm{d}\,t. \tag{3} $$

Since the function $\beta(t)$ is positive on $(0,\infty)$, see [3, 16, 26] and references therein, the right hand side of (3) is a completely monotonic function on $(0,\infty)$. This means that the function

$$ \frac{(x+2)^{x+3/2}}{4^x(x+1/2)^x}C_x \tag{4} $$

is logarithmically completely monotonic on $(0,\infty)$. Because any logarithmically completely monotonic function must be completely monotonic, see [12, Eq. (1.4)] and references therein, the function (4) is also completely monotonic on $(0,\infty)$.

By virtue of (1), the function (4) can be rewritten as

$$ \frac{(x+2)^{x+3/2}\Gamma(x+1/2)}{(x+1/2)^x\Gamma(x+2)},\quad x>0. \tag{5} $$

Hence, the logarithmically complete monotonicity of (4) implies the logarithmically complete monotonicity of (5). The function (5) is a special case $F_{1/2,2}(x)$ of the general function

$$ F_{a,b}(x)=\frac{\Gamma(x+a)}{(x+a)^x}\frac{(x+b)^{x+b-a}}{\Gamma(x+b)},\quad a,b\in\mathbb{R},\quad a\ne b\quad x>-\min\{a,b\}. \tag{6} $$

It is clear that

$$ F_{a,b}(x)F_{b,a}(x)=\left(\frac{x+b}{x+a}\right)^{b-a}=\left(1+\frac{b-a}{x+a}\right)^{b-a}. $$

It is noted that the function $F_{a,b}(x)$ does not appear in the expository and survey articles [6, 7, 12, 13, 14] and plenty of references therein. Therefore, it is significant to naturally pose an open problem below.

Open Problem 1 ([23, Open Problem 1]). *What are the necessary and sufficient conditions on $a, b \in \mathbb{R}$ such that the function $F_{a,b}(x)$ defined by (6) is (logarithmically) completely monotonic in $x \in (-\min\{a,b\}, \infty)$?*

The aim of this paper is to give solutions to the above open problem.

Theorem 2. *The sufficient conditions on a, b for the function $[F_{a,b}(x)]^{\pm 1}$ defined by (6) to be logarithmically completely monotonic in $x \in (-\min\{a,b\}, \infty)$ are $(a,b) \in D_{\pm}(a,b)$, where*

$$D_+(a,b) = \{a \ge 1, a > b\} \cup \left\{a \le \frac{1}{2}, a < b\right\}$$

and

$$D_-(a,b) = \{a \ge 1, b > a\} \cup \left\{a \le \frac{1}{2}, b < a\right\}.$$

The necessary conditions on a, b for the function $[F_{a,b}(x)]^{\pm 1}$ to be logarithmically completely monotonic in $x \in (-\min\{a,b\}, \infty)$ are

$$a(a-b) \gtreqless \frac{a-b}{2}.$$

2. Proof of Theorem 2

Taking the logarithm of $F_{a,b}(x)$ gives

$$\begin{aligned}\ln F_{a,b}(x) &= \ln\Gamma(x+a) - x\ln(x+a) - [\ln\Gamma(x+b) - (x+b-a)\ln(x+b)] \\ &\triangleq f_a(x) - f_a(x+b-a).\end{aligned}$$

Differentiating with respect to the variable x of $f_a(x)$ yields

$$f_a'(x) = \psi(x+a) - \ln(x+a) + \frac{a}{x+a} - 1$$

and

$$f_a''(x) = \psi'(x+a) - \frac{1}{x+a} - \frac{a}{(x+a)^2}. \tag{7}$$

In [8, Theorem 1.3], it was found that

1. the function $\psi(x) - \ln x + \frac{\alpha}{x}$ is completely monotonic on $(0, \infty)$ if and only if $\alpha \ge 1$,
2. the function $\ln x - \frac{\alpha}{x} - \psi(x)$ is completely monotonic on $(0, \infty)$ if and only if $\alpha \le \frac{1}{2}$.

This means that

1. if $\alpha \ge 1$, the function $\frac{1}{x} + \frac{\alpha}{x^2} - \psi'(x)$ is completely monotonic on $(0, \infty)$;
2. if $\alpha \le \frac{1}{2}$, the function $\psi'(x) - \frac{1}{x} - \frac{\alpha}{x^2}$ is completely monotonic on $(0, \infty)$.

Equivalently,

1. if $a \ge 1$, the function $-f_a''(x-a)$ is completely monotonic on $(0, \infty)$;
2. if $a \le \frac{1}{2}$, the function $f_a''(x-a)$ is completely monotonic on $(0, \infty)$.

Consequently,

1. if $a \ge 1$, the function $-f_a''(x)$ is completely monotonic on $(-a, \infty)$;
2. if $a \le \frac{1}{2}$, the function $f_a''(x)$ is completely monotonic on $(-a, \infty)$.

As a result,

1. when $a \ge 1$ and $b - a > 0$ the negativity of the function

   $$[\ln F_{a,b}(x)]'' = f_a''(x) - f_a''(x+b-a)$$

   is completely monotonic on $(-a, \infty)$;
2. when $a \ge 1$ and $b - a < 0$, the function $[\ln F_{a,b}(x)]''$ is completely monotonic on $(-b, \infty)$;
3. when $a \le \frac{1}{2}$ and $b - a > 0$, the function $[\ln F_{a,b}(x)]''$ is completely monotonic on $(-a, \infty)$;
4. when $a \le \frac{1}{2}$ and $b - a < 0$, the function $-[\ln F_{a,b}(x)]''$ is completely monotonic on $(-b, \infty)$.

In conclusion,

1. when $(a,b)\in D_+(a,b)$, the function $[\ln F_{a,b}(x)]''$ is completely monotonic on $(-\min\{a,b\},\infty)$;
2. when $(a,b)\in D_-(a,b)$, the function $-[\ln F_{a,b}(x)]''$ is completely monotonic on $(-\min\{a,b\},\infty)$.

By straightforward computation, we see that

$$\begin{aligned}\lim_{x\to\infty}[\ln F_{a,b}(x)]' &= \lim_{x\to\infty}\left[\psi(x+a)-\ln(x+a)+\frac{a}{x+a}-1-\psi(x+b)+\ln(x+b)-\frac{a}{x+b}+1\right]\\ &= \lim_{x\to\infty}\left[\psi(x+a)-\psi(x+b)+\ln\frac{x+b}{x+a}+\frac{a(b-a)}{(x+a)(x+b)}\right]\\ &= 0.\end{aligned}$$

This implies that, when $(a,b)\in D_\pm(a,b)$, the first logarithmic derivative satisfies $[\ln F_{a,b}(x)]'\lesseqgtr 0$. By the definition of logarithmically completely monotonic functions, we conclude that, when $(a,b)\in D_\pm(a,b)$, the function $[F_{a,b}(x)]^{\pm1}$ is logarithmically completely monotonic on $(-\min\{a,b\},\infty)$.

Conversely, if the function $[F_{a,b}(x)]^{\pm1}$ is logarithmically completely monotonic on $(-\min\{a,b\},\infty)$, then $\pm[\ln F_{a,b}(x)]'\le 0$ which is equivalent to

$$\psi(x+a)-\psi(x+b)+\ln\frac{x+b}{x+a}+\frac{a(b-a)}{(x+a)(x+b)}\lesseqgtr 0.$$

This can be rearranged as

$$a(a-b)\gtreqless (x+a)(x+b)\left[\psi(x+a)-\psi(x+b)+\ln\frac{x+b}{x+a}\right]\to\frac{a-b}{2}$$

as $x\to\infty$. Therefore, the necessary conditions are $a(a-b)\gtreqless\frac{a-b}{2}$. The proof of Theorem 2 is complete.

Remark 1. About recent development on investigation of the function (7), please refer to the manuscript [17] and plenty of references therein.

Remark 2. This paper is a companion of the articles [18, 19, 20, 23] and a slightly revised version of the preprint [5].

References

[1] L. Comtet, *Advanced Combinatorics: The Art of Finite and Infinite Expansions*, Revised and Enlarged Edition, D. Reidel Publishing Co., Dordrecht and Boston, 1974.

[2] R. L. Graham, D. E. Knuth, and O. Patashnik, *Concrete Mathematics—A Foundation for Computer Science*, 2nd ed., Addison-Wesley Publishing Company, Reading, MA, 1994.

[3] A.-Q. Liu, G.-F. Li, B.-N. Guo, and F. Qi, *Monotonicity and logarithmic concavity of two functions involving exponential function*, Internat. J. Math. Ed. Sci. Tech. **39** (2008), no. 5, 686–691; Available online at http://dx.doi.org/10.1080/00207390801986841.

[4] D. S. Mitrinović, J. E. Pečarić, and A. M. Fink, *Classical and New Inequalities in Analysis*, Kluwer Academic Publishers, Dordrecht-Boston-London, 1993; Available online at http://dx.doi.org/10.1007/978-94-017-1043-5.

[5] F. Qi, *A logarithmically completely monotonic function involving the gamma function and originating from the Catalan numbers*, ResearchGate Research, available online at http://dx.doi.org/10.13140/RG.2.1.1401.2009.

[6] F. Qi, *Bounds for the ratio of two gamma functions*, J. Inequal. Appl. **2010** (2010), Article ID 493058, 84 pages; Available online at http://dx.doi.org/10.1155/2010/493058.

[7] F. Qi, *Bounds for the ratio of two gamma functions: from Gautschi's and Kershaw's inequalities to complete monotonicity*, Turkish J. Anal. Number Theory **2** (2014), no. 5, 152–164; Available online at http://dx.doi.org/10.12691/tjant-2-5-1.

[8] F. Qi, *Three classes of logarithmically completely monotonic functions involving gamma and psi functions*, Integral Transforms Spec. Funct. **18** (2007), no. 7, 503–509; Available online at http://dx.doi.org/10.1080/10652460701358976.

[9] F. Qi and C.-P. Chen, *A complete monotonicity property of the gamma function*, J. Math. Anal. Appl. **296** (2004), 603–607; Available online at http://dx.doi.org/10.1016/j.jmaa.2004.04.026.

[10] F. Qi and B.-N. Guo, *Complete monotonicities of functions involving the gamma and digamma functions*, RGMIA Res. Rep. Coll. **7** (2004), no. 1, Art. 8, 63–72; Available online at http://rgmia.org/v7n1.php.

[11] F. Qi, S. Guo, and B.-N. Guo, *Complete monotonicity of some functions involving polygamma functions*, J. Comput. Appl. Math. **233** (2010), no. 9, 2149–2160; Available online at http://dx.doi.org/10.1016/j.cam.2009.09.044.

[12] F. Qi and W.-H. Li, *A logarithmically completely monotonic function involving the ratio of gamma functions*, J. Appl. Anal. Comput. **5** (2015), no. 4, 626–634; Available online at http://dx.doi.org/10.11948/2015049.

[13] F. Qi and Q.-M. Luo, *Bounds for the ratio of two gamma functions—From Wendel's and related inequalities to logarithmically completely monotonic functions*, Banach J. Math. Anal. **6** (2012), no. 2, 132–158.

[14] F. Qi and Q.-M. Luo, *Bounds for the ratio of two gamma functions: from Wendel's asymptotic relation to Elezović-Giordano-Pečarić's theorem*, J. Inequal. Appl. 2013, **2013**:542, 20 pages; Available online at http://dx.doi.org/10.1186/1029-242X-2013-542.

[15] F. Qi, Q.-M. Luo, and B.-N. Guo, *Complete monotonicity of a function involving the divided difference of digamma functions*, Sci. China Math. **56** (2013), no. 11, 2315–2325; Available online at http://dx.doi.org/10.1007/s11425-012-4562-0.

[16] F. Qi, Q.-M. Luo, and B.-N. Guo, *The function* $(b^x - a^x)/x$*: Ratio's properties*, In: *Analytic Number Theory, Approximation Theory, and Special Functions*, G. V. Milovanović and M. Th. Rassias (Eds), Springer, 2014, pp. 485–494; Available online at http://dx.doi.org/10.1007/978-1-4939-0258-3_16.

[17] F. Qi, F.-F. Liu, and X.-T. Shi, *Comments on two completely monotonic functions involving the q-trigamma function*, ResearchGate Dataset, available online at http://dx.doi.org/10.13140/RG.2.1.3267.4401.

[18] F. Qi, X.-T. Shi, and F.-F. Liu, *An exponential representation for a function involving the gamma function and originating from the Catalan numbers*, ResearchGate Research, available online at http://dx.doi.org/10.13140/RG.2.1.1086.4486.

[19] F. Qi, X.-T. Shi, and F.-F. Liu, *An integral representation, complete monotonicity, and inequalities of the Catalan numbers*, ResearchGate Technical Report, available online at http://dx.doi.org/10.13140/RG.2.1.3230.1927.

[20] F. Qi, X.-T. Shi, and F.-F. Liu, *Several formulas for special values of the Bell polynomials of the second kind and applications*, ResearchGate Technical Report, available online at http://dx.doi.org/10.13140/RG.2.1.3230.1927.

[21] F. Qi, C.-F. Wei, and B.-N. Guo, *Complete monotonicity of a function involving the ratio of gamma functions and applications*, Banach J. Math. Anal. **6** (2012), no. 1, 35–44; Available online at http://dx.doi.org/10.15352/bjma/1337014663.

[22] R. L. Schilling, R. Song, and Z. Vondraček, *Bernstein Functions—Theory and Applications*, 2nd ed., de Gruyter Studies in Mathematics **37**, Walter de Gruyter, Berlin, Germany, 2012; Available online at http://dx.doi.org/10.1515/9783110269338.

[23] X.-T. Shi, F.-F. Liu, and F. Qi, *An integral representation of the Catalan numbers*, ResearchGate Research, available online at http://dx.doi.org/10.13140/RG.2.1.2273.6485.

[24] R. Stanley and E. W. Weisstein, *Catalan Number*, From MathWorld–A Wolfram Web Resource; Available online at http://mathworld.wolfram.com/CatalanNumber.html.

[25] D. V. Widder, *The Laplace Transform*, Princeton Mathematical Series **6**, Princeton University Press, Princeton, N. J., 1941.

[26] S.-Q. Zhang, B.-N. Guo, and F. Qi, *A concise proof for properties of three functions involving the exponential function*, Appl. Math. E-Notes **9** (2009), 177–183.