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Research Paper

On inequalities of Hermite-Hadamard type for co-ordinated (α_1, m_1)-(α_2, m_2)-convex functions

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Abstract

In the paper, the authors establish some integral inequalities of Hermite-Hadamard type for co-ordinated (α_1, m_1)-(α_2, m_2)-convex functions on a rectangle of the first quadrant in a plane.

Keywords: (α_1, m_1)-(α_2, m_2)-*convex function; co-ordinates; rectangle of the plane; Hermite-Hadamard type inequality*

1. Introduction

The following definitions are known in the literature.

Definition 1.1. A function $f : I \subseteq \mathbb{R} = (-\infty, \infty) \to \mathbb{R}$ is said to be convex if $f(\lambda x + (1-\lambda)y) \le \lambda f(x) + (1-\lambda) f(y)$ holds for all $x, y \in I$ and $\lambda \in [0, 1]$.

Definition 1.2 ([7])**.** For $f : [0, b] \to \mathbb{R}$ and $m \in (0, 1]$, if $f(\lambda x + m(1-\lambda)y) \le \lambda f(x) + m(1-\lambda) f(y)$ is valid for all $x, y \in [0, b]$ and $\lambda \in [0, 1]$, then we say that $f(x)$ is an m-convex function on $[0, b]$.

Definition 1.3 ([6])**.** For $f : [0, b] \to \mathbb{R}$ and $(\alpha, m) \in (0, 1] \times (0, 1]$, if $f(\lambda x + m(1-\lambda)y) \le \lambda^\alpha f(x) + m(1-\lambda^\alpha) f(y)$ is valid for all $x, y \in [0, b]$ and $\lambda \in [0, 1]$, then we say that $f(x)$ is an (α, m)-convex function on $[0, b]$.

Definition 1.4 ([3, 4])**.** A function $f : \Delta = [a, b] \times [c, d] \subseteq \mathbb{R}^2 \to \mathbb{R}$ is said to be convex on the co-ordinates on Δ if the partial mappings $f_y : u \in [a, b] \to f(u, y) \in \mathbb{R}$ and $f_x : v \in [c, d] \to f(x, v) \in \mathbb{R}$ are convex for all $x \in (a, b)$ and $y \in (c, d)$.

Definition 1.5. [3, 4] A function $f : \Delta = [a, b] \times [c, d] \subseteq \mathbb{R}^2 \to \mathbb{R}$ is said to be convex on the co-ordinates on Δ if

$$f(tx + (1-t)z, \lambda y + (1-\lambda)w) \le t\lambda f(x, y) + t(1-\lambda) f(x, w) + (1-t)\lambda f(z, y) + (1-t)(1-\lambda) f(z, w)$$

holds for all $t, \lambda \in [0, 1]$ and $(x, y), (z, w) \in \Delta$.

We now recall some inequalities of Hermite-Hadamard type.

Theorem 1.1 ([5]). *Let $f:\mathbb{R}_0=[0,\infty)\to\mathbb{R}$ be m-convex and $m\in(0,1]$. If $f\in L_1([a,b])$ for $0\le a<b<\infty$, then*

$$\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}\,x\le\min\left\{\frac{f(a)+mf(b/m)}{2},\frac{mf(a/m)+f(b)}{2}\right\}.$$

Theorem 1.2 ([3, Theorem 2.2]). *Let $f:\Delta=[a,b]\times[c,d]\subseteq\mathbb{R}^2$ be convex on the co-ordinates on Δ. Then we have*

$$\begin{aligned}f\left(\frac{a+b}{2},\frac{c+d}{2}\right)&\le\frac{1}{2}\left[\frac{1}{b-a}\int_a^b f\left(x,\frac{c+d}{2}\right)\mathrm{d}\,x+\frac{1}{d-c}\int_c^d f\left(\frac{a+b}{2},y\right)\mathrm{d}\,y\right]\\&\le\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)\,\mathrm{d}\,y\,\mathrm{d}\,x\\&\le\frac{1}{4}\left[\frac{1}{b-a}\int_a^b[f(x,c)+f(x,d)]\,\mathrm{d}\,x+\frac{1}{d-c}\int_c^d[f(a,y)+f(b,y)]\,\mathrm{d}\,y\right]\\&\le\frac{1}{4}[f(a,c)+f(b,c)+f(a,d)+f(b,d)].\end{aligned}$$

In [1, 2], the authors introduced the following co-ordinated (α_1,m_1)-(α_2,m_2)-convex functions.

Definition 1.6 ([1, 2]). For $m_1,m_2,\alpha_1,\alpha_2\in(0,1]$, a function $f:[0,b]\times[0,d]\to\mathbb{R}$ is said to be co-ordinated (α_1,m_1)-(α_2,m_2)-convex if

$$\begin{aligned}&f(tx+m_1(1-t)z,\lambda y+m_2(1-\lambda)w)\\&\quad\le t^{\alpha_1}\lambda^{\alpha_2}f(x,y)+m_1(1-t^{\alpha_1})\lambda^{\alpha_2}f(z,y)+m_2t^{\alpha_1}(1-\lambda^{\alpha_2})f(x,w)+m_1m_2(1-t^{\alpha_1})(1-\lambda^{\alpha_2})f(z,w)\end{aligned}\tag{1.1}$$

holds for all $(t,\lambda)\in[0,1]\times(0,1)$ and $(x,y),(z,w)\in[0,b]\times[0,d]$.

For more information on Hermite-Hadamard type inequalities for various kinds of convex functions, please refer to the monograph [4], recently published papers [1, 2, 8, 9], and closely related references therein.

In this paper, we will establish some integral inequalities of Hermite-Hadamard type for co-ordinated (α_1,m_1)-(α_2,m_2)-convex functions.

2. Integral inequalities of Hermite-Hadamard type

Theorem 2.1. *Let $f:\left[0,\frac{b}{m_1^2}\right]\times\left[0,\frac{d}{m_2^2}\right]\to\mathbb{R}$ be an integrable function with $0\le a<b$ and $0\le c<d$ for some fixed $m_1,m_2\in(0,1]$. If f is co-ordinated (α_1,m_1)-(α_2,m_2)-convex on $\left[0,\frac{b}{m_1^2}\right]\times\left[0,\frac{d}{m_2^2}\right]$ for $\alpha_1,\alpha_2\in(0,1]$, then*

$$\begin{aligned}f\left(\frac{a+b}{2},\frac{c+d}{2}\right)&\le\frac{1}{2^{\alpha_1+\alpha_2+1}}\left[\frac{1}{b-a}\int_a^b G\left(x,\frac{c+d}{2}\right)\mathrm{d}\,x+\frac{1}{d-c}\int_c^d G\left(\frac{a+b}{2},y\right)\mathrm{d}\,y\right]\\&\le\frac{1}{2^{2(\alpha_1+\alpha_2)}(b-a)(d-c)}\int_c^d\int_a^b\left[G(x,y)+m_1(2^{\alpha_1}-1)G\left(\frac{x}{m_1},y\right)\right.\\&\quad\left.+m_2(2^{\alpha_2}-1)G\left(x,\frac{y}{m_2}\right)+m_1m_2(2^{\alpha_1}-1)(2^{\alpha_2}-1)G\left(\frac{x}{m_1},\frac{y}{m_2}\right)\right]\mathrm{d}\,x\,\mathrm{d}\,y,\end{aligned}\tag{2.1}$$

where

$$G(u,v)=f(u,v)+m_1(2^{\alpha_1}-1)f\left(\frac{u}{m_1},v\right)+m_2(2^{\alpha_2}-1)f\left(u,\frac{v}{m_2}\right)+m_1m_2(2^{\alpha_1}-1)(2^{\alpha_2}-1)f\left(\frac{u}{m_1},\frac{v}{m_2}\right)$$

for $(u,v)\in\left[a,\frac{b}{m_1}\right]\times\left[c,\frac{d}{m_2}\right]$.

Proof. Using the (α_1,m_1)-(α_2,m_2)-convexity of f with $t=\lambda=\frac{1}{2}$ in (1.1), we have

$$\begin{aligned}f\left(\frac{a+b}{2},\frac{c+d}{2}\right)&=\int_0^1 f\left(\frac{ta+(1-t)b+(1-t)a+tb}{2},\frac{c+d}{2}\right)\mathrm{d}\,t\\&\le\int_0^1\left[\frac{1}{2^{\alpha_1+\alpha_2}}f\left(ta+(1-t)b,\frac{c+d}{2}\right)+m_1\left(1-\frac{1}{2^{\alpha_1}}\right)\frac{1}{2^{\alpha_2}}f\left(\frac{(1-t)a+tb}{m_1},\frac{c+d}{2}\right)+m_2\frac{1}{2^{\alpha_1}}\left(1-\frac{1}{2^{\alpha_2}}\right)\right.\\&\quad\left.\times f\left(ta+(1-t)b,\frac{c+d}{2m_2}\right)+m_1m_2\left(1-\frac{1}{2^{\alpha_1}}\right)\left(1-\frac{1}{2^{\alpha_2}}\right)f\left(\frac{(1-t)a+tb}{m_1},\frac{c+d}{2m_2}\right)\right]\mathrm{d}\,t\\&=\frac{1}{2^{\alpha_1+\alpha_2}(b-a)}\int_a^b\left\{f\left(x,\frac{c+d}{2}\right)+m_1(2^{\alpha_1}-1)f\left(\frac{x}{m_1},\frac{c+d}{2}\right)\right.\end{aligned}$$

$$+ m_2(2^{\alpha_2}-1)f\left(x, \frac{c+d}{2m_2}\right) + m_1m_2(2^{\alpha_1}-1)(2^{\alpha_2}-1)f\left(\frac{x}{m_1}, \frac{c+d}{2m_2}\right)\bigg\} \,\mathrm{d}\, x$$
$$= \frac{1}{2^{\alpha_1+\alpha_2}(b-a)} \int_a^b G\left(x, \frac{c+d}{2}\right) \mathrm{d}\, x.$$

Taking $y = \lambda c + (1-\lambda)d$ for $0 \le \lambda \le 1$ and using the (α_1, m_1)-(α_1, m_2)-convexity of f with $t = \lambda = \frac{1}{2}$ in (1.1) give

$$\begin{aligned} f\left(x, \frac{c+d}{2}\right) &= \int_0^1 f\left(x, \frac{\lambda c + (1-\lambda)d + (1-\lambda)c + \lambda d}{2}\right) \mathrm{d}\,\lambda \\ &\le \frac{1}{2^{\alpha_1+\alpha_2}} \int_0^1 \bigg\{ f(x, \lambda c + (1-\lambda)d) + m_1(2^{\alpha_1}-1)f\left(\frac{x}{m_1}, (1-\lambda)c + \lambda d\right) \\ &\quad + m_2(2^{\alpha_2}-1)f\left(x, \frac{\lambda c + (1-\lambda)d}{m_2}\right) + m_1m_2(2^{\alpha_1}-1)(2^{\alpha_2}-1)f\left(\frac{x}{m_1}, \frac{(1-\lambda)c+\lambda d}{m_2}\right)\bigg\} \,\mathrm{d}\,\lambda \\ &= \frac{1}{2^{\alpha_1+\alpha_2}(d-c)} \int_c^d G(x,y) \,\mathrm{d}\, y \end{aligned}$$

for $x \in [a, b]$. A direct calculation gives

$$f\left(\frac{x}{m_1}, \frac{c+d}{2}\right) \le \frac{1}{2^{\alpha_1+\alpha_2}(d-c)} \int_c^d G\left(\frac{x}{m_1}, y\right) \mathrm{d}\, y, \quad f\left(x, \frac{c+d}{2m_2}\right) \le \frac{1}{2^{\alpha_1+\alpha_2}(d-c)} \int_c^d G\left(x, \frac{y}{m_2}\right) \mathrm{d}\, y,$$

and

$$f\left(\frac{x}{m_1}, \frac{c+d}{2m_2}\right) \le \frac{1}{2^{\alpha_1+\alpha_2}(d-c)} \int_c^d G\left(\frac{x}{m_1}, \frac{y}{m_2}\right) \mathrm{d}\, y.$$

Combining the above inequalities arrives at

$$\begin{aligned} f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) &\le \frac{1}{2^{\alpha_1+\alpha_2}(b-a)} \int_a^b G\left(x, \frac{c+d}{2}\right) \mathrm{d}\, x \\ &\le \frac{1}{2^{2(\alpha_1+\alpha_2)}(b-a)(d-c)} \int_c^d \int_a^b \bigg[G(x,y) + m_1(2^{\alpha_1}-1)G\left(\frac{x}{m_1}, y\right) \\ &\quad + m_2(2^{\alpha_2}-1)G\left(x, \frac{y}{m_2}\right) + m_1m_2(2^{\alpha_1}-1)(2^{\alpha_2}-1)G\left(\frac{x}{m_1}, \frac{y}{m_2}\right)\bigg] \,\mathrm{d}\, x \,\mathrm{d}\, y. \end{aligned}$$

Similarly, we have

$$\begin{aligned} f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) &\le \frac{1}{2^{\alpha_1+\alpha_2}(d-c)} \int_c^d G\left(\frac{a+b}{2}, y\right) \mathrm{d}\, y \\ &\le \frac{1}{2^{2(\alpha_1+\alpha_2)}(b-a)(d-c)} \int_a^b \int_c^d \bigg[G(x,y) + m_1(2^{\alpha_1}-1)G\left(\frac{x}{m_1}, y\right) \\ &\quad + m_2(2^{\alpha_2}-1)G\left(x, \frac{y}{m_2}\right) + m_1m_2(2^{\alpha_1}-1)(2^{\alpha_2}-1)G\left(\frac{x}{m_1}, \frac{y}{m_2}\right)\bigg] \,\mathrm{d}\, x \,\mathrm{d}\, y. \end{aligned}$$

Utilizing the last two inequalities leads to the inequality (2.1). The proof of Theorem 2.1 is complete. □

Corollary 2.1.1. *Under the conditions of Theorem 2.1,*

1. *if $\alpha_1 = \alpha_2 = \alpha$ and $m_1 = m_2 = m$, then*

$$\begin{aligned} &f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) \le \frac{1}{2^{2\alpha+1}}\left[\frac{1}{b-a}\int_a^b G\left(x, \frac{c+d}{2}\right) \mathrm{d}\, x + \frac{1}{d-c}\int_c^d G\left(\frac{a+b}{2}, y\right) \mathrm{d}\, y\right] \\ &\le \frac{1}{16^\alpha(b-a)(d-c)} \int_c^d \int_a^b \left[G(x,y) + m(2^\alpha-1)G\left(\frac{x}{m}, y\right) + m(2^\alpha-1)G\left(x, \frac{y}{m}\right) + m^2(2^\alpha-1)^2 G\left(\frac{x}{m}, \frac{y}{m}\right)\right] \mathrm{d}\, x \,\mathrm{d}\, y; \end{aligned}$$

2. *if $\alpha_1 = \alpha_2 = 1$ and $m_1 = m_2 = m$, then*

$$\begin{aligned} f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) &\le \frac{1}{8}\left[\frac{1}{b-a}\int_a^b G\left(x, \frac{c+d}{2}\right) \mathrm{d}\, x + \frac{1}{d-c}\int_c^d G\left(\frac{a+b}{2}, y\right) \mathrm{d}\, y\right] \\ &\le \frac{1}{16(b-a)(d-c)} \int_c^d \int_a^b \left[G(x,y) + mG\left(\frac{x}{m}, y\right) + mG\left(x, \frac{y}{m}\right) + m^2 G\left(\frac{x}{m}, \frac{y}{m}\right)\right] \mathrm{d}\, x \,\mathrm{d}\, y. \end{aligned}$$

Theorem 2.2. *Let $f : \left[0, \frac{b}{m_1^2}\right] \times \left[0, \frac{d}{m_2^2}\right] \to \mathbb{R}$ be an integrable function with $0 \le a < b$ and $0 \le c < d$ for some fixed $m_1, m_2 \in (0,1]$. If f is co-ordinated (α_1, m_1)-(α_2, m_2)-convex on $\left[0, \frac{b}{m_1^2}\right] \times \left[0, \frac{d}{m_2^2}\right]$ for some fixed $\alpha_1, \alpha_2 \in (0,1]$, then*

$$\frac{1}{(b-a)(d-c)}\int_c^d\int_a^b f(x,y)\,\mathrm{d}\,x\,\mathrm{d}\,y \le \frac{1}{2^{\alpha_1+1}(\alpha_2+1)(b-a)}\int_a^b M(x,c,d)\,\mathrm{d}\,x + \frac{1}{2^{\alpha_2+1}(\alpha_1+1)(d-c)}\int_c^d N(a,b,y)\,\mathrm{d}\,y$$
$$\le \frac{1}{2^{\alpha_1+\alpha_2+1}(\alpha_1+1)(\alpha_2+1)}\bigg[2N(a,b,c) + m_1(2^{\alpha_1}+\alpha_1-1)N\left(\frac{a}{m_1},\frac{b}{m_1},c\right)$$
$$+ m_2(2^{\alpha_2}+\alpha_2-1)N\left(a,b,\frac{d}{m_2}\right) + m_1m_2[\alpha_2(2^{\alpha_1}-1)+\alpha_1(2^{\alpha_2}-1)]N\left(\frac{a}{m_1},\frac{b}{m_1},\frac{d}{m_2}\right)\bigg],$$

where

$$M(x,w,z) = f(x,w) + m_1(2^{\alpha_1}-1)f\left(\frac{x}{m_1},z\right) + m_2\alpha_2 f\left(x,\frac{w}{m_2}\right) + m_1m_2\alpha_2(2^{\alpha_1}-1)f\left(\frac{x}{m_1},\frac{z}{m_2}\right)$$

and

$$N(u,v,y) = f(u,y) + m_1\alpha_1 f\left(\frac{v}{m_1},y\right) + m_2(2^{\alpha_2}-1)f\left(u,\frac{y}{m_2}\right) + m_1m_2\alpha_1(2^{\alpha_2}-1)f\left(\frac{v}{m_1},\frac{y}{m_2}\right)$$

for $x,u,v \in \left[a, \frac{b}{m_1}\right]$ and $y,w,z \in \left[c, \frac{d}{m_2}\right]$.

Proof. Setting $y = \lambda c + (1-\lambda)d$ for $0 < \lambda < 1$. From the (α_1, m_1)-(α_2, m_2)-convexity of f with $t = \frac{1}{2}$ and $0 < \lambda < 1$ in (1.1), we obtain

$$\frac{1}{(b-a)(d-c)}\int_c^d\int_a^b f(x,y)\,\mathrm{d}\,x\,\mathrm{d}\,y = \frac{1}{b-a}\int_0^1\int_a^b f(x,\lambda c+(1-\lambda)d)\,\mathrm{d}\,x\,\mathrm{d}\,\lambda$$
$$\le \frac{1}{2^{\alpha_1}(b-a)}\int_0^1\int_a^b\bigg\{\lambda^{\alpha_2}f(x,c) + m_1(2^{\alpha_1}-1)\lambda^{\alpha_2}f\left(\frac{x}{m_1},c\right)$$
$$+ m_2(1-\lambda^{\alpha_2})f\left(x,\frac{d}{m_2}\right) + m_1m_2(2^{\alpha_1}-1)(1-\lambda^{\alpha_2})f\left(\frac{x}{m_1},\frac{d}{m_2}\right)\bigg\}\,\mathrm{d}\,x\,\mathrm{d}\,\lambda$$
$$= \frac{1}{2^{\alpha_1}(\alpha_2+1)(b-a)}\int_a^b M(x,c,d)\,\mathrm{d}\,x.$$

Obviously, if letting $x = ta + (1-t)b$ for $0 \le t \le 1$ and using the (α_1, m_1)-(α_2, m_2)-convexity of f with $0 \le t \le 1$ and $\lambda = \frac{1}{2}$ in (1.1), it follows that

$$\frac{1}{b-a}\int_a^b f(x,c)\,\mathrm{d}\,x \le \frac{1}{2^{\alpha_2}}\int_0^1\bigg\{t^{\alpha_1}f(a,c) + m_1(1-t^{\alpha_1})f\left(\frac{b}{m_1},c\right)$$
$$+ m_2t^{\alpha_1}(2^{\alpha_2}-1)f\left(a,\frac{c}{m_2}\right) + m_1m_2(1-t^{\alpha_1})(2^{\alpha_2}-1)f\left(\frac{b}{m_1},\frac{c}{m_2}\right)\bigg\}\,\mathrm{d}\,t$$
$$= \frac{1}{2^{\alpha_2}(\alpha_1+1)}N(a,b,c)$$

and

$$\frac{1}{b-a}\int_a^b f\left(\frac{x}{m_1},c\right)\mathrm{d}\,x \le \frac{1}{2^{\alpha_2}(\alpha_1+1)}N\left(\frac{a}{m_1},\frac{b}{m_1},c\right), \quad \frac{1}{b-a}\int_a^b f\left(x,\frac{d}{m_2}\right)\mathrm{d}\,x \le \frac{1}{2^{\alpha_2}(\alpha_1+1)}N\left(a,b,\frac{d}{m_2}\right),$$
$$\frac{1}{b-a}\int_a^b f\left(\frac{x}{m_1},\frac{d}{m_2}\right)\mathrm{d}\,x \le \frac{1}{2^{\alpha_2}(\alpha_1+1)}N\left(\frac{a}{m_1},\frac{b}{m_1},\frac{d}{m_2}\right).$$

When combining the above inequalities, we find

$$\frac{1}{(b-a)(d-c)}\int_c^d\int_a^b f(x,y)\,\mathrm{d}\,x\,\mathrm{d}\,y \le \frac{1}{2^{\alpha_1}(\alpha_2+1)(b-a)}\int_a^b M(x,c,d)\,\mathrm{d}\,x$$
$$\le \frac{1}{2^{\alpha_1+\alpha_2}(\alpha_1+1)(\alpha_2+1)}\bigg[N(a,b,c) + m_1(2^{\alpha_1}-1)N\left(\frac{a}{m_1},\frac{b}{m_1},c\right)$$
$$+ m_2\alpha_2N\left(a,b,\frac{d}{m_2}\right) + m_1m_2\alpha_2(2^{\alpha_1}-1)N\left(\frac{a}{m_1},\frac{b}{m_1},\frac{d}{m_2}\right)\bigg].$$

Similarly, we have

$$\frac{1}{(b-a)(d-c)}\int_c^d\int_a^b f(x,y)\,\mathrm{d}\,x\,\mathrm{d}\,y \le \frac{1}{2^{\alpha_2}(\alpha_1+1)(d-c)}\int_c^d N(a,b,y)\,\mathrm{d}\,y$$

$$\le \frac{1}{2^{\alpha_1+\alpha_2}(\alpha_1+1)(\alpha_2+1)}\bigg[N(a,b,c)+m_1\alpha_1 N\bigg(\frac{a}{m_1},\frac{b}{m_1},c\bigg) + m_2(2^{\alpha_2}-1)N\bigg(a,b,\frac{d}{m_2}\bigg)+m_1m_2\alpha_1(2^{\alpha_2}-1)N\bigg(\frac{a}{m_1},\frac{b}{m_1},\frac{d}{m_2}\bigg)\bigg].$$

Combining the last two inequalities leads to the inequality in Theorem 2.2. The proof of Theorem 2.2 is thus complete. □

Corollary 2.2.1. *Under the conditions of Theorem 2.2, if $\alpha_1=\alpha_2=\alpha$ and $m_1=m_2=m$, then*

$$\frac{1}{(b-a)(d-c)}\int_c^d\int_a^b f(x,y)\,\mathrm{d}\,x\,\mathrm{d}\,y \le \frac{1}{2^{\alpha+1}(\alpha+1)}\bigg[\frac{1}{b-a}\int_a^b M(x,c,d)\,\mathrm{d}\,x+\frac{1}{(d-c)}\int_c^d N(a,b,y)\,\mathrm{d}\,y\bigg]$$
$$\le \frac{1}{2^{2\alpha+1}(\alpha+1)^2}\bigg[2N(a,b,c)+m(2^\alpha+\alpha-1)N\bigg(\frac{a}{m},\frac{b}{m},c\bigg) + m(2^\alpha+\alpha-1)N\bigg(a,b,\frac{d}{m}\bigg)+2m^2\alpha(2^\alpha-1)N\bigg(\frac{a}{m},\frac{b}{m},\frac{d}{m}\bigg)\bigg].$$

Corollary 2.2.2. *Under the assumptions of Theorems 2.1 and 2.2, if $\alpha_1=\alpha_2=\alpha$ and $m_1=m_2=m$, then*

$$f\bigg(\frac{a+b}{2},\frac{c+d}{2}\bigg)\le\frac{1}{2}\bigg[\frac{1}{b-a}\int_a^b f\bigg(x,\frac{c+d}{2}\bigg)\,\mathrm{d}\,x+\frac{1}{d-c}\int_c^d f\bigg(\frac{a+b}{2},y\bigg)\,\mathrm{d}\,y\bigg]$$
$$\le\frac{1}{(b-a)(d-c)}\int_c^d\int_a^b f(x,y)\,\mathrm{d}\,x\,\mathrm{d}\,y$$
$$\le\frac{1}{2^{\alpha+1}}\bigg[\frac{1}{b-a}\int_a^b[f(x,c)+(2^\alpha-1)f(x,d)]\,\mathrm{d}\,x+\frac{1}{d-c}\int_c^d[f(a,y)+(2^\alpha-1)f(b,y)]\,\mathrm{d}\,y\bigg]$$
$$\le \frac{1}{2^{2\alpha+1}(\alpha+1)^2}\bigg[2N(a,b,c)+m(2^\alpha+\alpha-1)N\bigg(\frac{a}{m},\frac{b}{m},c\bigg) + m(2^\alpha+\alpha-1)N\bigg(a,b,\frac{d}{m}\bigg)+2m^2\alpha(2^\alpha-1)N\bigg(\frac{a}{m},\frac{b}{m},\frac{d}{m}\bigg)\bigg].$$

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