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***Research paper***

# A simple proof of the closed graph theorem

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**Abstract**

Assume that $A$ is a closed linear operator defined on all of a Hilbert space $H$. Then $A$ is bounded. A new short proof of this classical theorem is given on the basis of the uniform boundedness principle. The proof can be easily extended to Banach spaces.

***Keywords:*** *closed graph theorem; closed linear operator; uniform boundedness principle; new short proof of the closed graph theorem.*

## 1. Introduction

We denote by $D(A)$ the domain of definition of $A$, by $A^*$ the adjoint operator, by $||A||$ the norm of $A$, by $(u,v)$ the inner product in $H$, by $c>0$ various estimation constants are denoted. Let $A$ be a linear operator in $H$ and $u_n \in D(A)$. Suppose that $u_n \to u$ and $Au_n \to v$. If the above implies that $u \in D(A)$ and $Au = v$ then the operator $A$ is called closed (see [5]). It is well known (see, [1], [5]) that any bounded sequence in $H$ contains a convergent subsequence.

The following result is classical:

**Theorem 1.** *Let $A$ be a closed linear operator in a Hilbert space $H$, and $D(A) = H$. Then $A$ is bounded.*

Theorem 1 is known as the closed graph theorem. Its proof can be found in [1], [5], [7], and in many other texts in functional analysis. These proofs are based on the Baire cathegory theorem. The aim of this note is to give a simple new proof of Theorem 1 using the well-known uniform boundedness principle, which we state as Theorem 2, and a new result, stated as Theorem 3, which is proved in Section 2.

In [2] a proof of Theorem 1 is given, which is different from ours. Our proof of Theorem 1 is not only new but also very short.

Proofs of Theorem 2 which are not based on Baire's theorem can be found in [2], problem 27, [3], [4], [6].

**Theorem 2.** *If $\sup_n |(Au_n,v)| \le \infty$ for every $v \in H$, then* $\sup_n ||Au_n|| \le \infty$

We *assume* Theorem 2 known.

The new result we use in the proof of Theorem 1 is the following:

**Theorem 3.** *If $A$ is a linear closed operator with $D(A) = H$, then $D(A^*) = H$.*

In Section 2 proofs are given.

## 2. Proofs

**Proof of Theorem 3**.

If $A$ is a linear closed operator and $D(A) = H$, then $A^*$ exists, is closed and densely defined. To prove that $D(A^*) = H$, let $v \in H$ be arbitrary, and $v_n \to v$, $v_n \in D(A^*)$. Let $u \in H$ be arbitrary. Then $(Au, v_n) = (u, A^* v_n)$ and

$$\sup_n |(u, A^* v_n)| \le \sup_n ||v_n|| ||Au|| \le c(u). \quad (1)$$

By Theorem 2 one has $\sup_n ||A^* v_n|| \le c$. Therefore, a subsequence, denoted again $A^* v_n$, converges weakly in $H$: $A^* v_n \rightharpoonup v^*$, and $(Au, v) = (u, v^*)$. Thus, $v \in D(A^*)$, and $D(A^*) = H$ since $v \in H$ was arbitrary.

Theorem 3 is proved. □

**Proof of Theorem 1**. Consider the relation $(Au, v) = (u, v^*)$. Since $D(A) = H$ and $A$ is closed, Theorem 3 says that $D(A^*) = H$, the above relation holds for every $v \in H$, and $v^* = A^* v$. Suppose that $A$ is unbounded. Then there exists a sequence $u_n$, $||u_n|| = 1$, such that

$$||Au_n|| \to \infty. \quad (2)$$

On the other hand, one has:

$$\sup_n |(Au_n, v)| = \sup_n |(u_n, A^* v)| \le \sup_n ||u_n|| \cdot ||A^* v|| = ||A^* v|| := c(v). \quad (3)$$

By Theorem 2 one concludes that $\sup_n ||Au_n|| < c$. This contradicts (2). Thus, one concludes that $||A|| < c$. Theorem 1 is proved. □

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