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Research paper

A New Hilbert-type integral inequality with a non-homogeneous kernel and its extension

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Abstract

By introducing some parameters , using the weight function and the technique of real analysis, a new Hilbert-type integral inequality with a non-homogeneous kernel as $\frac{1}{|1-axy|^{\lambda_2}}(a \geq 1)$ and its equivalent form are established. As application, the constant factor on the plane is the best value and its extension form with some parameters is also considered.

Keywords: *Some parameters, Hilbert-Type Integral Inequality, Best value, Extension.*

1. Introduction

If $0 < \int_0^\infty f^2(x)dx < \infty, 0 < \int_0^\infty g^2(x)dx < \infty$, ,Hilbert established the famous integral inequality[1]

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{x+y}dxdy < \pi(\int_0^\infty f^2(x)dx \int_0^\infty g^q(y)dy)^{\frac{1}{2}}, \quad (1)$$

where the constant factor π is the best possible.(1) is important in analysis and its application[2−3]. Since 1998, by introduced the conjugate index,a number of extensions (1) were given by Yang et al.[4−7].

In this paper, by introducing some parameters and using the way of weight function, a new inequality with a non-homogeneous kernel as follows is established:

$\int_0^\infty \int_0^\infty \frac{1}{|1-axy|^{1/2}}(a \geq 1) f(x)g(y)dxdy <$

$$2B(\frac{1}{2},\frac{1}{4})(\int_0^\infty x^{1/2} f^2(x)dx)^{\frac{1}{2}}(\int_0^\infty y^{1/2} g^2(y)dy)^{\frac{1}{2}}. \quad (2)$$

where the constant factor$2B(\frac{1}{2},\frac{1}{4})$ is the best possible. As application, the equivalent form and its extension form are obtained.

2. Some Lemmas

First,β function is given[9]:

$$B(u,v) = \int_0^1 x^{u-1}(1-x)^{v-1} = B(v,u)(u,v>0) \quad (3)$$

Lemma 2.1. $0 < \lambda_1 < \lambda_2 < 1, a \geq 1, \eta(a,\lambda_1,\lambda_2) := \frac{1}{a^{\lambda_1}}[B(\lambda_1, 1-\lambda_2) + B(\lambda_2-\lambda_1, 1-\lambda_2)], 0 < \lambda_1 < \lambda_2 < 1, a \geq 1, \eta(a,\lambda_1,\lambda_2) := \frac{1}{a^{\lambda_1}}[B(\lambda_1,1-\lambda_2)+B(\lambda_2-\lambda_1,1-\lambda_2)]$,*define the following weight function:*

$\omega(y) := \int_0^\infty \frac{1}{|1-axy|^{\lambda_2}} \frac{y^{\lambda_1}}{x^{1-\lambda_1}}dy \quad (y>0),$

$\omega(x) := \int_0^\infty \frac{1}{|1-axy|^{\lambda_2}} \frac{x^{\lambda_1}}{y^{1-\lambda_1}}dy \quad (x>0),$

then we have

$$\omega(x) = \omega(y) = \eta(a,\lambda_1,\lambda_2). \quad (4)$$

Proof *Setting $u = axy$ and by (3),we obtain*

$$\begin{aligned}\omega(y) &= \frac{1}{|1-u|^{\lambda_2}} \frac{y^{\lambda_1}}{(uy^{-1}a^{-1})^{1-\lambda_1}} y^{-1}a^{-1}du \\ &= \frac{1}{a^{\lambda_1}}[\int_0^1 \frac{1}{(1-u)^{\lambda_2}}u^{\lambda_1-1}du + \int_1^\infty \frac{1}{(u-1)^{\lambda_2}}u^{\lambda_1-1}du] \\ &= \frac{1}{a^{\lambda_1}}[B(\lambda_1,1-\lambda_2)+B(\lambda_2-\lambda_1,1-\lambda_2)] = \eta(a,v_1,v_2).\end{aligned}$$

Similarly, we can calculate that $\omega(x) = \eta(a,v_1,v_2)$.

Lemma 2.2. *As the assumption of Lemma 2.1, if $p > 1, \frac{1}{p}+\frac{1}{q} = 1, f(x) \geq 0$,we have*

$J := \int_0^\infty y^{p\lambda_1 - 1}\left[\int_0^\infty \frac{1}{|1-axy|^{\lambda_2}} f(x)dx\right]^p dy \leq$

$$[\eta(a,\lambda_1,\lambda_2)]^p \int_0^\infty x^{p(1-\lambda_1)-1} f^p(x)dx. \quad (5)$$

Proof *By Hölder's inequality with weight[8],we obtain*

$\int_0^\infty f(x)\frac{1}{|1-axy|^{\lambda_2}}dx = \int_0^\infty \frac{1}{|1-axy|^{\lambda_2}}[\frac{x^{(1-\lambda_1)/q}}{y^{(1-\lambda_1)/p}}f(x)][\frac{y^{(1-\lambda_1)/p}}{x^{(1-\lambda_1)/q}}]dx$

$\leq \{\int_0^\infty \frac{1}{|1-axy|^{\lambda_2}} \frac{x^{(1-\lambda_1)(p-1)}}{y^{(1-\lambda_1)}} f^p(x)dx\}^{1/p}\{\int_0^\infty \frac{1}{|1-axy|^{\lambda_2}} \frac{y^{(1-\lambda_1)(q-1)}}{x^{(1-\lambda_1)}}dx\}^{1/q}$

$$= y^{\frac{1}{p}-\lambda_1}[\omega(y)]^{1/q}\{\int_0^\infty \frac{1}{|1-axy|^{\lambda_2}} \frac{x^{(1-\lambda_1)(p-1)}}{y^{(1-\lambda_1)}} f^p(x)dx\}^{1/p}. \quad (6)$$

By (4)and Fubini's theorem[9], we obtain

$$\begin{aligned}J &\leq [\eta(a,\lambda_1,\lambda_2]^{p-1}\int_0^\infty \int_0^\infty \frac{1}{|1-axy|^{\lambda_2}} \frac{x^{(1-\lambda_1)(p-1)}}{y^{(1-\lambda_1)}} f^p(x)dxdy \\ &= [\eta(a,\lambda_1,\lambda_2]^{p-1}\int_0^\infty \omega(x)x^{p(1-\lambda_1)-1}f^p(x)dx.\end{aligned}$$

Hence by the above results, we have (5). The lemma is proved.

3. Main results

Theorem 3.1. *As the assumption of Lemma 2.1, if* $p > 1, \frac{1}{p} + \frac{1}{q} = 1, f(x), g(y) \geq 0, 0 < \int_0^\infty x^{p(1-\lambda_1)-1} f^p(x)dx < \infty, 0 < \int_0^\infty y^{q(1-\lambda_1)-1} g^q(y)dy < \infty$, *then we have two equivalent inequalities as*

$$I := \int_0^\infty \int_0^\infty \frac{1}{|1-axy|^{\lambda_2}} f(x)g(y)dxdy < \eta(a,\lambda_1,\lambda_2)\times$$

$$\left\{\int_0^\infty x^{p(1-\lambda_1)-1} f^p(x)dx\right\}^{\frac{1}{p}} \left\{\int_0^\infty y^{q(1-\lambda_1)-1} g^q(y)dy\right\}^{\frac{1}{q}}, \tag{7}$$

$$J = \int_0^\infty y^{p\lambda_1-1} \left[\int_0^\infty \frac{1}{|1-axy|^{\lambda_2}} f(x)dx\right]^p dy <$$

$$[\eta(a,\lambda_1,\lambda_2)]^p \int_0^\infty x^{p(1-\lambda_1)-1} f^p(x)dx. \tag{8}$$

Proof *By the conditions of theorem and Hölder's inequality,we conform that the middle of (6) keeps the form of strict inequality,Hence(6) takes the strict sign-inequality,so does (5). In addition, we have (7). By Hölder's inequality*[8] *,we find*

$$I = \int_0^\infty [y^{\lambda_1-1/p} \int_0^\infty \frac{1}{|1-axy|^{\lambda_2}} f(x)dx][y^{-\lambda_1+1/p} g(y)]dy \leq$$

$$J^{1/p}[\int_0^\infty y^{q(1-\lambda_1)-1} g^q(y)dy]^{1/q}. \tag{9}$$

Then by (8), we have (7).On the other hand,assuming that (7) is valid ,setting $g(y) := y^{p\lambda_1-1}[\int_0^\infty \frac{1}{|1-axy|^{\lambda_2}} f(x)]dx]^{p-1}$ *,then we have* $J = \int_0^\infty y^{q(1-\lambda_1)-1} g^q(y)dy$*,Through (5),it follows* $J < \infty$*.If* $J = 0$*,then (8)is naturally valid.If* $0 < J < \infty$*, then by(7),we find*

$$0 < \int_0^\infty y^{q(1-\lambda_1)-1} g^q(y)dy = J = I < \eta(a,\lambda_1,\lambda_2)\times$$

$$\left\{\int_0^\infty x^{p(1-\lambda_1)-1} f^p(x)dx\right\}^{\frac{1}{p}} \left\{\int_0^\infty y^{q(1-\lambda_1)-1} g^q(y)dy\right\}^{\frac{1}{q}}, \tag{10}$$

$$J^{1/p} = \left\{\int_0^\infty y^{q(1-\lambda_1)-1} g^q(y)dy\right\}^{\frac{1}{p}} <$$

$$\eta(a,\lambda_1,\lambda_2)\left\{\int_0^\infty x^{p(1-\lambda_1)-1} f^p(x)dx\right\}^{\frac{1}{p}}. \tag{11}$$

and then we have (8),which is equivalent to (7). The Theorem is proved.

Theorem 3.2. *Under the conditions of Theorem 3.1,the constants* $\eta(a,\lambda_1,\lambda_2)$*and* $[\eta(a,\lambda_1,\lambda_2)]^p$*in (7)and (8) are the best value.*

Proof *For* $0 < \varepsilon < pv_1$*,if*

$$\tilde{f}(x) := \begin{cases} x^{\lambda_1+\varepsilon/p-1}, & x \in (0,1] \\ 0, & x \in (1,\infty) \end{cases},$$

$$\tilde{g}(y) := \begin{cases} 0, & y \in (0,1) \\ y^{\lambda_1-\varepsilon/q-1}, & y \in [1,\infty) \end{cases}.$$

we can calculate

$$\tilde{J} := \left\{\int_0^\infty x^{p(1-\lambda_1)-1} \tilde{f}^p(x)dx\right\}^{\frac{1}{p}} \left\{\int_0^\infty y^{q(1-\lambda_1)-1} \tilde{g}^q(y)dy\right\}^{\frac{1}{q}} = \frac{1}{\varepsilon}$$

.by Fubini's theorem[9]*,we obtain*

$$I := \int_0^\infty \int_0^\infty \frac{1}{|1-axy|^{\lambda_2}} \tilde{f}(x)\tilde{g}(x)dxdy$$

$$= \int_1^\infty y^{\lambda_1-\varepsilon/q-1} \left[\int_1^\infty \frac{1}{|1-axy|^{\lambda_2}} x^{\lambda_1+\varepsilon/p-1} dx\right] dy$$

$$\underset{=}{(u = axy)} \frac{1}{a^{\lambda_1+\varepsilon/p}} \int_1^\infty y^{-\varepsilon-1} \left[\int_0^{ay} \frac{1}{|1-u|^{\lambda_2}} u^{\lambda_1+\varepsilon/p-1} du\right] dy$$

$$= \frac{1}{a^{\lambda_1+\varepsilon/p}} \left\{\int_1^\infty y^{-\varepsilon-1} \left[\int_0^1 \frac{u^{\lambda_1+\varepsilon/p-1}}{(1-u)^{\lambda_2}} du + \int_1^{ay} \frac{u^{\lambda_1+\varepsilon/p-1}}{(u-1)^{\lambda_2}} du\right] dy\right\}$$

$$= \frac{1}{a^{\lambda_1+\varepsilon/p}} \left\{\frac{1}{\varepsilon} \int_0^1 \frac{u^{\lambda_1+\varepsilon/p-1}}{(1-u)^{\lambda_2}} du + \int_1^\infty (\int_{a^{-1}u}^\infty y^{-\varepsilon-1}) \frac{u^{\lambda_1+\varepsilon/p-1}}{(u-1)^{\lambda_2}} du\right\}$$

$$= \frac{1}{a^{\lambda_1+\varepsilon/p}} \left\{\frac{1}{\varepsilon} \int_0^1 \frac{u^{\lambda_1+\varepsilon/p-1}}{(1-u)^{\lambda_2}} du + \frac{a^\varepsilon}{\varepsilon} \int_1^\infty \frac{u^{\lambda_1-\varepsilon/q-1}}{(u-1)^{\lambda_2}} du\right\}.$$

If there exists a positive number $k \leq \eta(a,\lambda_1,\lambda_2)$*,such that (7)is still valid when we replace* $\eta(a,\lambda_1,\lambda_2)$ *by k,then in particular* $\tilde{f}, \tilde{g}$*, by the above results ,we find*

$$\frac{1}{a^{\lambda_1+\varepsilon/p}} \left\{\int_0^1 \frac{u^{\lambda_1+\varepsilon/p-1}}{(1-u)^{\lambda_2}} du + a^\varepsilon \int_1^\infty \frac{u^{\lambda_1-\varepsilon/q-1}}{(u-1)^{\lambda_2}} du\right\} =$$

$$\varepsilon\tilde{I} < \varepsilon k\tilde{J} = k. \tag{12}$$

By Fatou lemma[9]*and(12),we obtain*

$$\eta(a,\lambda_1,\lambda_2) =$$

$$\frac{1}{a^{\lambda_1}} \left[\int_0^1 \underset{\varepsilon\to 0^+}{\underline{lim}} \frac{1}{(1-u)^{\lambda_2}} u^{\lambda_1+\varepsilon/p-1} du + \int_1^\infty \underset{\varepsilon\to 0^+}{\underline{lim}} \frac{1}{(1-u)^{\lambda_2}} u^{\lambda_1-\varepsilon/q-1} u\right] \leq$$

$$\frac{1}{a^{\lambda_1}} \underset{\varepsilon\to 0^+}{\underline{lim}} \left[\int_0^1 \frac{1}{(1-u)^{\lambda_2}} u^{\lambda_1+\varepsilon/p-1} du + \int_1^\infty \frac{1}{(u-1)^{\lambda_2}} u^{\lambda_1-\varepsilon/q-1} du\right] \leq k.$$

Hence $k = \eta(a,\lambda_1,\lambda_2)$*is the best value of (7).We conform that* $[\eta(a,\lambda_1,\lambda_2)]^p$ *in (8)is the best possible, otherwise we can get a contradiction by (9) that the constant in (7) is not the best possible.The theorem is proved.*

4. Conclusion

For $a = 1, \lambda_1 = \frac{1}{4}, \lambda_2 = \frac{1}{2}, p = q = 2$ in (7), it deduces to (2). Hence inequality (7) is the best extension of (2).

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References

[1] Hardy GH.Note on a Theorem of H ilbert Concern ing Series of Positive Terms [J]. Proc London Math Soc,1925,23 (2):XLV-XLVL.

[2] Hardy GH,Littewood JE,PolyaG.Inequalities[M].Cambridge:Cambridge University Press,1952.

[3] Mitrinovic DS, Pecaric J, Fink AM. Inequalities Involving Functions and The ir Integrals and Derivatives [M]. Boston:Kluwer Academic Publishers, 1991.

[4] Yongjing Li and Bing He. On inequalities of Hilbert's type [J].Bulletin of the Australian Mathematical Society,,2007,76(1):1-13.

[5] Qiong Liu. A Hilbert-type Integral Inequality with Several Parameters[J].J of Jinlin University:science edition,,2009,47(5):903-908.

[6] Bicheng Yang.On a Hilbert-type integral inequality with the homogeneous kernel of degree[J].J of Shanghai University(EnglEd):Natual Science,2010,14(6):391-395.

[7] Weiliang Wu.A new Hilbert-type integral inequality with multiple independent parameters[J].J of Lanzhou university of technology,2015,41 (1):160-163.

[8] Jichang Kuang, Applied Inequalities, Shangdong Science Press, Jinan,2004.

[9] Jichang Kuang, Introduction to real analysis, Hunan Education Press,Changsha,1996.