Global Journal of Mathematical Analysis, 4 (1) (2016) 2-5

Global Journal of Mathematical Analysis

Website: www.sciencepubco.com/index.php/GJMA

doi: 10.14419/gjma.v4i1.5733

Research paper

Common fixed point theorems for weakly compatible non-self mappings in metric spaces of hyperbolic type

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Abstract

In this paper, we establish common fixed point theorems for a pair of weakly compatible nonself mappings satisfying generalized contractive conditions in metric space of hyperbolic type. The results generalize and extend some results in literature.

Keywords: *common fixed points, generalized contractive mapping, metric space of hyperbolic type, nonself mappings, weakly compatble mappings.*

1. Introduction

In literature, fixed point theory has diverse results on fixed point theorems for self-mappings in metric and Banach spaces. However, an area that seems not broadly investigated is the fixed point theorems for non-self mappings. Kirk [1] extended the metric space to metric space of hyperbolic type by replacing Krasnoselskii's result with the framework of convex metric space. The study of fixed point theorems for multivalued non-self mappings in a metric space (X,d) was initiated by Assad [2] and Assad and Kirk [3]. Many authors have studied the existence and uniqueness of fixed and common fixed points result for nonself contraction mappings in cone metric spaces [see; 4, 5, 6, 7]. Some authors studied common fixed point theorems for non-self mappings in metric spaces of hyperbolic type [See: 8, 9]. Motivated by Jankovic et al. [7], we prove some common fixed point theorems for a pair of weakly compatible non-self mappings satisfying a generalized contraction condition in the setting of metric space of hyperbolic type.

Throughout our consideration, we suppose that (X,d) is a metric space which contains a family L of metric segments (isometric images of real line segment) such that

a) each two points $x,y \in X$ are endpoints of exactly one number seg[x, y] of L, and

b) If $u,x,y \in X$ and if $z \in seg[x,y]$ satisfies $d(x,z) = \lambda d(x,y)$ for $\lambda \in [0,1]$ then

$$d(u,z) \leq (1-\lambda)d(u,x) + \lambda d(u,y) \tag{1.1}$$

A space of this type is called metric space of hyperbolic type.

The following definition was introduced by Jungck et al. [4] in the setting of cone metric spaces.

Definition 1.1 Let (X,d) be a complete cone metric space, let C be a non empty closed subset of X, and let $f,g : C \to X$ be non-self mappings. Denote for $x,y \in C$

$$M_1^{f,g} = \{d(gx,gy), d(fx,gx), d(fy,gy), \tfrac{d(fx,gy)+d(fy,gx)}{2}\} \tag{1.2}$$

Then f is called a generalized g_{M_1}-contractive mapping in C into X if, for some $\lambda \in (0, \sqrt{2}-1)$, there exists $U(x,y) \in M_1^{f,g}$ such that for all $x,y \in C$,

$d(fx, fy) \leq \lambda U(x,y)$

2. Main results

Jankovic et al. [7] proved the following fixed point theorem for a pair of non-self mappings defined on a nonempty closed subset of complete metrically convex cone metric spaces with new contractive conditions.

Theorem 2.1: Let (X,d) be a complete cone metric space, let K be a non empty closed subset of X such that for each $x \in C$ and $y \notin C$ there exists a point $z \in \delta K$ (the boundary of K) such that $d(x,z) + d(z,y) = d(x,y)$.

Suppose that $f,g : C \to X$ are such that f is a generalized g_{M_1}-contractive mapping of C into X and

(i) $\delta C \subseteq gC$, $fC \cap C \subseteq gC$,

(ii) $gx \in \delta C \Longrightarrow fx \in C$,

(iii) gC is closed in X.

Then the pair (f,g) has a coincidence point. Moreover, if (f, g) are coincidentally commuting, then f and g have a unique common fixed point.

In this paper, we extend the above theorem to fixed point theorem of weakly compatible non- self mappings in metric space of hyperbolic type.

We state and prove our main result as follows.

Theorem 2.2: Let X be a metric space of hyperbolic type, K a non-empty closed subset of X and δK the boundary of K. Let δK be nonempty and let $T : K \to X$ and $f : K \cap T(K) \to X$ be two non-self-mappings satisfying the following conditions:
$d(fx, fy) \leq \lambda.\mu$
where
$$\mu \in \{d(Tx,Ty), d(Tx, fx), d(Ty, fy), \frac{d(Tx,fy)+d(Ty,fx)}{2}\}) \qquad (2.1)$$

for all $x, y \in C$, $0 < \lambda < 1$. If
(i) $\delta K \subset TK$, $fK \cap K \subset TK$,
(ii) $Tx \in \delta K \Longrightarrow fx \in K$,
(iii) $fK \cap K$ is complete.
Then f and T have a coincidence point z in X. Moreover, if f and T are weakly compatible, then z is the unique common fixed point of f and T.

Proof: Let $x \in \delta K$ be arbitrary. We construct three sequences , $\{x_n\}$ and $\{z_n\}$ in K and a sequence $\{y_n\}$ in $fK \subset X$ as follows. Choose $z_0 = x$. Since $z_0 \in \delta K$ then there exists $x_0 \in K$ such that $z_0 = Tx_0 \in \delta K$. By (iii) $fx_0 \in K$. Now choose $y_1 = fx_0$ with $y_1 \in fK \subset X$. This implies that $fx_0 \in fK \cap K \subset TK$. Set $y_1 = fx_0$, we choose $x_1 \in K$ such that $Tx_1 = fx_0$. Hence $z_1 = Tx_1 = fx_0 = y_1$. This gives $y_2 = fx_1$.
Since $y_2 \in fK \cap K$ then $y_2 \in TK$ by (ii). Let $x_1 \in K$ with $z_1 = Tx_1 \in \delta K$ such that $z_2 = Tx_2 = fx_1 = y_2$. If $fx_1 = y_2 \notin K$, then there exists $z_2 \in \delta K (z_2 \notin y_2)$ such that $z_2 \in seg[y_1, y_2]$. Since $x_2 \in K$, then by (i) we have $Tx_2 = z_2$. Hence
$z_2 \in \delta K \cap seg[y_1, y_2]$.
We can choose $y_3 \in fK \cap K$, and by (ii), $y_3 \in TK$ and let $x_2 \in K$ such that $Tx_3 = y_3 = fx_2$. Continuing in the process, we construct three sequence $\{x_n\} \subseteq K$, $\{z_n\} \subseteq K$ and $\{y_n\} \subseteq fK \subset X$ such that
(a)$y_n = fx_{n-1}$
(b) $z_n = Tx_n$,
(c) $z_n = y_n$ if and only if $y_n \in K$
(d) $z_n \notin y_n$ whenever $y_n \notin K$ and $z_n \in \delta K$ such that
$z_n \in \delta K \cap seg[fx_{n-2}, fx_{n-1}]$.
This proves that f and T are non-self mappings.

Remark 2.3: By (d) if $z_n \neq y_n$, then $z_n \in \delta K$ and combining (b), (ii) and (a) we have $z_{n+1} = y_{n+1}$. Likewise $z_{n-1} = y_{n-1} \in K$. If $z_{n-1} \in \delta K$, then it implies $z_n = y_n \in K$.
Next, we show that $x_n \neq x_{n+1}$ for all n. From (a), (b), (c) and (d) we can establish three possibilities.
(1) $z_n = y_n \in K$ and $z_{n+1} = y_{n+1}$;
(2) $z_n = y_n \in K$ but $z_{n+1} \neq y_{n+1}$;
(3) $z_n \neq y_n \in K$ in which case $z_n \in \delta K \cap seg[fx_{n-2}, fx_{n-1}]$.
Now
Case (1)
Let $z_n = y_n \in K$ and $z_{n+1} = y_{n+1}$. Using (2.1) we obtain
$d(z_n, z_{n+1}) = d(y_n, y_{n+1}) = d(fx_{n-1}, fx_n) \leq \lambda.\mu_n$

where $\mu_n \in \{d(Tx_{n-1}, Tx_n), d(Tx_{n-1}, fx_{n-1}), d(Tx_n, fx_n),$
$$\frac{d(Tx_{n-1},fx_n)+d(Tx_n,fx_{n-1})}{2}\})$$
$$= \{d(z_{n-1}, z_n), d(z_{n-1}, y_n), d(z_n, y_{n+1}), \frac{d(z_{n-1},y_{n+1})+d(z_n,y_n)}{2}\}$$
$$= \{d(z_{n-1}, z_n), d(z_{n-1}, z_n), d(z_n, z_{n+1}), \frac{d(z_{n-1},z_{n+1})+0}{2}\}$$
$$= \{d(z_{n-1}, z_n), d(z_{n-1}, z_n), d(z_n, z_{n+1}), \frac{d(z_{n-1},z_n)+d(z_n,z_{n+1})}{2}\}$$

Obviously, there are infinite many n such that at least one of the following cases holds:
I: $d(z_n, z_{n+1}) \leq \lambda d(z_{n-1}, z_n)$

II: $d(z_n, z_{n+1}) \leq \lambda.d(z_{n-1}, z_n)$

III: $d(z_n, z_{n+1}) \leq \lambda.d(z_n, z_{n+1})$. A contradiction.

IV: $d(z_n, z_{n+1}) \leq \lambda.\frac{d(z_{n-1},z_n)+d(z_n,z_{n+1})}{2}\}$

$\leq \frac{\lambda}{2}(d(z_{n-1}, z_n) + \frac{\lambda}{2}d(z_n, z_{n+1}))$ implies
$d(z_n, z_{n+1}) \leq \lambda d(z_{n-1}, z_n)$

From I, II, III, IV it follows that
$$d(z_n, z_{n+1}) \leq \lambda.d(z_{n-1}, z_n) \qquad (2.2)$$

Case 2
Let $z_n = y_n \in K$ but $z_{n+1} \neq y_{n+1}$. Then $z_{n+1} \in \delta K \cap seg[y_n, y_{n+1}]$. From (1.1) with $u = y$, we obtain
$d(y, z) \leq (1-\lambda)d(x, y)$

Therefore
$d(x,y) \leq d(x,z) + d(z,y) \leq \lambda d(x,y) + (1-\lambda)d(x,y) = d(x,y)$
Hence
$z \in seg[x, y] \Longrightarrow d(x, z) + d(z, y) = d(x, y)$.
Since $z_{n+1} \in seg[y_n, y_{n+1}] = seg[z_n, y_{n+1}]$, we have

$d(z_n, z_{n+1}) = d(y_n, z_{n+1}) = d(y_n, y_{n+1}) - d(z_{n+1}, y_{n+1}) < d(y_n, y_{n+1})$

In view of case (1), we obtain
$d(y_n, y_{n+1}) \leq \lambda.d(z_{n-1}, z_n)$.

This implies that $d(z_n, z_{n+1}) \leq \lambda.d(z_{n-1}, z_n)$.

Case (3)
$z_n \neq y_n$. Then $z_n \in \delta K \cap seg[fx_{n-2}, fx_{n-1}]$. i.e. $z_n \in \delta K \cap seg[y_{n-1}, y_n]$
By remark (2.3) we have $z_{n+1} = y_{n+1}$ and $z_{n-1} = y_{n-1}$. This implies that
$d(z_n, z_{n+1}) = d(z_n, y_{n+1})$
$$\leq d(z_n, y_n) + d(y_n, y_{n+1})$$
$$= d(z_{n-1}, y_n) - d(z_n, z_{n-1}) + d(y_n, y_{n+1})$$
$$= d(y_{n-1}, y_n) - d(z_n, z_{n-1}) + d(y_n, y_{n+1}) \qquad (2.3)$$

We shall find $d(y_{n-1}, y_n)$ and $d(y_n, y_{n+1})$. Since $z_{n-1} = y_{n-1}$ then we can conclude that
$$d(y_{n-1}, y_n) \leq \lambda.d(z_{n-2}, z_{n-1}), \qquad (2.4)$$
with respect to case (2).
Now
$d(y_n, y_{n+1}) = d(fx_{n-1}, fx_n) \leq \lambda.\mu_n$

where $\mu_n \in \{d(Tx_{n-1}, Tx_n), d(Tx_{n-1}, fx_{n-1}), d(Tx_n, fx_n),$
$$\frac{d(Tx_{n-1},fx_n)+d(Tx_n,fx_{n-1})}{2}\})$$
$$= \{d(z_{n-1}, z_n), d(z_{n-1}, y_n), d(z_n, y_{n+1}), \frac{d(z_{n-1},y_{n+1})+d(z_n,y_n)}{2}\}$$
$$= \{d(z_{n-1}, z_n), d(z_{n-1}, y_n), d(z_n, z_{n+1}), \frac{d(z_{n-1},z_{n+1})+d(z_n,y_n)}{2}\}$$
$$\leq \{d(z_{n-1}, z_n), d(z_{n-1}, y_n), d(z_n, z_{n+1}),$$
$$\frac{d(z_{n-1},z_n)+d(z_n,z_{n+1})+d(z_n,z_{n-1})-d(z_{n-1},y_n)}{2}\}$$
$$\leq \{d(z_{n-1}, z_n), d(y_{n-1}, y_n), d(z_n, z_{n+1}), \frac{2d(z_{n-1},z_n)+d(z_n,z_{n+1})}{2}\}$$

Clearly, there are infinite many n such that at least one of the following cases holds:

I: $d(y_n,y_{n+1}) \leq \lambda.d(z_{n-1},z_n)$

II: $d(y_n,y_{n+1}) \leq \lambda d(y_{n-1},y_n) \leq \lambda^2.d(z_{n-2},z_{n-1})$

III: $d(y_n,y_{n+1}) \leq \lambda.d(z_n,z_{n+1})$

IV: $d(y_n,y_{n+1}) \leq \lambda.d(z_n,z_{n-1}) + \frac{\lambda}{2}d(z_n,z_{n+1})$
Substituting I, II, III, IV in (2.4) yields
$d(z_n,z_{n+1}) \leq \lambda.d(z_{n-2},z_{n-1}) - d(z_n,z_{n-1}) + \lambda.\mu_n$

from which we have four cases:
V: $d(z_n,z_{n+1}) \leq \lambda.d(z_{n-2},z_{n-1}) - d(z_n,z_{n-1}) + \lambda.d(z_{n-1},z_n)$

$$\leq \lambda.d(z_{n-2},z_{n-1}) - (1-\lambda)d(z_n,z_{n-1})$$

$$\leq \lambda.d(z_{n-2},z_{n-1})$$

VI: $d(z_n,z_{n+1}) \leq \lambda.d(z_{n-2},z_{n-1}) - d(z_n,z_{n-1}) + \lambda^2.d(z_{n-2},z_{n-1})$

$$\leq (\lambda+\lambda^2)d(z_{n-2},z_{n-1}) - d(z_n,z_{n-1})$$

$$\leq (\lambda+\lambda^2)d(z_{n-2},z_{n-1})$$

VII: $d(z_n,z_{n+1}) \leq \lambda.d(z_{n-2},z_{n-1}) - d(z_n,z_{n-1}) + \lambda.d(z_n,z_{n+1})$

$$\leq \frac{\lambda}{1-\lambda}d(z_{n-2},z_{n-1}) - \frac{1}{1-\lambda}d(z_n,z_{n-1})$$

$$\leq \frac{\lambda}{1-\lambda}d(z_{n-2},z_{n-1})$$

VIII: $d(z_n,z_{n+1}) \leq \lambda.d(z_{n-2},z_{n-1}) - d(z_n,z_{n-1}) + \lambda.d(z_n,z_{n-1}) + \frac{\lambda}{2}d(z_n,z_{n+1})$

$$\leq \lambda.d(z_{n-2},z_{n-1}) - (1-\lambda)d(z_n,z_{n-1}) + \frac{\lambda}{2}d(z_n,z_{n+1})$$

$$\leq \frac{2\lambda}{2-\lambda}.d(z_{n-2},z_{n-1}) - \frac{2(1-\lambda)}{2-\lambda}d(z_n,z_{n-1})$$

$$\leq \frac{2\lambda}{2-\lambda}.d(z_{n-2},z_{n-1})$$

From V, VI, VII, VIII we obtain
$d(z_n,z_{n+1}) \leq k.d(z_{n-2},z_{n-1})$ where

$k = max\{\lambda, \lambda+\lambda^2, \frac{\lambda}{1-\lambda}, \frac{2\lambda}{2-\lambda}\}$

Combining Cases 1, 2, 3 we get
$d(z_n,z_{n+1}) \leq k.\omega_n$

where $\omega_n \in \{d(z_{n-2},z_{n-1}), d(z_{n-1},z_n)\}$ and

$k = max\{\lambda, \lambda+\lambda^2, \frac{\lambda}{1-\lambda}, \frac{2\lambda}{2-\lambda}\}$

Following the procedure of Assad and Kirk [3], it can be easily verify by induction that for $n > 1$

$$d(z_n,z_{n+1}) \leq k^{\frac{n-1}{2}}.\omega_2 \tag{2.5}$$

where $\omega_2 \in \{d(z_0,z_1), d(z_1,z_2)\}$.

For $n > m$ and using (2.5) and the triangle inequality we have
$d(z_n,z_m) \leq d(z_n,z_{n-1}) + d(z_{n-1},z_{n-2}) + \cdots + d(z_{m+1},z_m)$

$$\leq (k^{\frac{n-1}{2}} + k^{\frac{n-2}{2}} + \cdots + k^{\frac{m-1}{2}}).\omega_2$$

$$\leq \frac{\sqrt{k}^{m-1}}{1-\sqrt{k}}.\omega_2 \to 0, \quad \text{as } m \to \infty.$$

The sequence is Cauchy. Since $z_n = fx_{n-1} \in fK \cap K$ is complete, there is some $z \in fK \cap K$ such that $z_n \to z$. Let w in K be such that $Tw = z$. By the construction of $\{z_n\}$, there is a subsequence $\{z_{n_k}\}$ such that $z_{n_k} = y_{n_k} = fx_{n_{k-1}}$ and $fx_{n_{k-1}} \to z$ We show that $fw = z$.
$d(fw,z) \leq d(fw, fx_{n_{k-1}}) + d(fx_{n_{k-1}}, z) \leq \lambda.\mu_{n_k} + d(fx_{n_{k-1}}, z)$
where
$\mu_{n_k} \in \{(d(Tw,Tx_{n_{k-1}}), d(Tx_{n_{k-1}}, fx_{n_{k-1}}), d(Tw,fw),$

$\frac{d(Tw,fx_{n_{k-1}})+d(Tx_{n_{k-1}},fw)}{2}\}$

Taking $z_{n_k} = y_{n_k} = fx_{n_{k-1}} \to z$ as $n \to \infty$ yields
$\mu_n \in \{0, d(z,fw), 0, \frac{d(z,fw)}{2}\}$

$\mu_n \in \{d(z,fw), \frac{d(z,fw)}{2}\}$

Thus, we have
i) $d(fw,z) \leq \lambda d(z,fw) + d(fx_{n_{k-1}}, z) \leq \lambda d(z,fw)$

Since $\lambda < 1$ then $d(fw,z) = 0$. This implies $z = fw$

ii) $d(fw,z) \leq \frac{\lambda}{2}d(fw,z)$

Since $\lambda < 1$ then $d(fw,z) = 0$. Hence $z = fw$. In all cases we have $z = fw$.
Suppose that T and f are weakly compatible, then we have $z = fw = Tw \Longrightarrow fz = fTw = Tfw = Tz$.
Next we prove that $z = fz = Tz$. Suppose $z \neq fz$ then using 2.1 we obtain
$d(fz,z) = d(fz,fw) \leq \lambda.\mu$
where
$\mu \in \{d(Tz,Tw), d(Tz,fz), d(Tw,fw),$
$\frac{d(Tz,fw)+d(Tw,fz)}{2}\}$

$\leq \{d(z,z), d(z,fz), d(z,z), \frac{d(z,z)+d(z,fz)}{2}\}$

$\leq \{d(z,fz), \frac{d(z,fz)}{2}\}$

Case (i)
$d(fz,z) \leq \lambda d(fz,z)$ It is a contradiction. Hence $z = fz$
Case(ii)
$d(fz,z) \leq \frac{\lambda}{2}d(fz,z)$

It is also a contradiction. This imples that $z = fz$. Therefore we obtain $z = fz = Tz$. Thus T and f have a common fixed point. The uniqueness of the common fixed point follows easily from (2, 1).

Remark 2.4 : Theorem 2.2 is an extension of the result of jankovic [7] .
Setting $T = I_x$, the identity mapping of X in Theorem 2.2 , we obtain the following result.

Corollary 2.5: Let (X,d) be metric space of hyperbolic type, K a non-empty closed subset of X and δK the boundary of K. Let δK be nonempty such that $f : K \to K$ satisfies the condition
$d(fx,fy) \leq \lambda.\mu$
where

$$\mu \in \{d(x,y), d(x,fx), d(y,fy), \frac{d(x,fy)+d(y,fx)}{2}\}) \tag{2.6}$$

for all $x,y \in k$, $0 < \lambda < 1$ and f has the additional property that for each $x \in \delta K$ and $fx \in K$. Then f has a unique fixed point.

Corollary 2.6: Let X be a metric space of hyperbolic type, K a non-empty closed subset of X and δK the boundary of K. Let δK be nonempty and let $T : K \to X$ and $f : K \cap T(K) \to X$ be two non-self- mappings satisfying the following conditions:

$$d(fx,fy) \leq \lambda(d(Tx,fx) + d(Ty,fy)) \tag{2.7}$$

for all $x,y \in C$, $0 < \lambda < \frac{1}{2}$. If

(i) $\delta K \subset TK$, $fK \cap K \subset TK$,
(ii) $Tx \in \delta K \Longrightarrow fx \in K$,
(iii) $fK \cap K$ is complete.
Then f and T have a coincidence point z in X. Moreover, if f and T are weakly compatible, then z is the unique common fixed point of f and T.

Example 2.7 : Let X be the set of real numbers with the usual metric, $K = [0, +\infty)$ and let $T : K \to X$ and $f : K \cap T(K) \to X$ be two non-self mappings defined by $Tx = 4x$ and $fx = \frac{4x}{1+4x}$ for all $x \in K$.
Taking $x = \frac{1}{2}$ and $y = \frac{1}{4}$ we obtain $\lambda = \frac{1}{6}$. Thus T and f satisfied (2. 1) and all the hypotheses in Theorem 2.2 are satisfied. T and f have a unique common fixed point $z = 0$.

3. Conclusion

In this section, we proved that in a metric space of hyperbolic type, two non-self mappings f and T satisfying certain contractive conditions have a coincidence point. Moreover, if the maps are weakly compatible then f and T have a unique common fixed point. We gave an example to validate our results.

References

[1] W. A. Kirk, Krasnoselskii's iteration process in hyperbolic space, Numer. Funct. Anal. and Optimiz., 4(1982), 371-381.
[2] N. A. Assad, On a fixed point theorem in Banach space, Tamkang J. Math., 7(1976), 91-94.
[3] N. A. Assad and W. A. Kirk, Fixed point theorems for set-valued mappings of contractive type, Pacific J. Math. 43(1972), 553-562.
[4] G. Jungck, S. Radenovic, V. Radejovic, V. Rakocevic, Common fixed point theorems for weakly compatible pairs on cone metric spaces, Fixed Point Theory Appl. 2009, Article ID 643840(2009), doi:10.1155/2009/643840.
[5] R. Sumitra, V. R. Uthariaraj, R. Hemavathy, P . Vijayaraju, Common fixed point theorem for non-self mappings satisfying generalized Ciric type contraction condition in cone metric space, Fixed Point Theory and Applications, (2010), Article ID 408086, 17 pages.
[6] S. Radenovic and B. E. Rhoades, Fixed point theorem for two non-self mappings in cone metric spaces, Computers and Mathematics with Applications, 57(2009), 1701-1707.
[7] S. Jankovic, Z. Kadelburg, S. Radenovic, BE Rhoades, Assad-Kirk type fixed point theorems for a pair of non -self mappings on cone metric spaces, Fixed Point Theory Appl. 2009, Article ID 761086(2009), doi:10.1155/2009/761086.
[8] L. B. Ciric, Contractive-type non-self mappings on metric spaces of hyperbolic type, J. Math. Anal. Appl., 317(2006), 28-42.
[9] L. Ciric, V. Rakocevic, S. Radenovic, M. Rajovic, R. Lazovic, Common fixed point theorems for non-self mappings in metric spaces of hyperbolic type, Journal of Computational and Applied Mathematics, 233(2010), 2966-2974.
[10] W. Takahashi, A convexity in metric spaces and nonexpansive mappings, I, Kodai Math. Sem. Rep. 22(1970), 142-149.