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Short communication

A determinantal representation for derangement numbers

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Abstract

In the note, the author finds a representation for derangement numbers in terms of a tridiagonal determinant whose elements are the first few natural numbers.

Keywords: *derangement number; determinantal representation; tridiagonal determinant.*

In combinatorics, a derangement is a permutation of the elements of a set, such that no element appears in its original position. The number of derangements of a set of size n is called derangement number and sometimes denoted by $!n$. The first ten derangement numbers $!n$ for $0 \le n \le 9$ are

$$1, \quad 0, \quad 1, \quad 2, \quad 9, \quad 44, \quad 265, \quad 1854, \quad 14833, \quad 133496.$$

We now discover that derangement numbers $!n$ can be beautifully expressed as a certain explicitly written down tridiagonal determinant. To the best of our knowledge, we have not seen such a representation in the context earlier.

Theorem 1. *For $n \in \{0\} \cup \mathbb{N}$, derangement numbers $!n$ can be expressed by a tridiagonal $(n+1)\times(n+1)$ determinant*

$$!n = -\begin{vmatrix} -1 & 1 & 0 & 0 & \dots & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & \dots & 0 & 0 & 0 \\ 0 & -1 & 1 & 1 & \dots & 0 & 0 & 0 \\ 0 & 0 & -2 & 2 & \dots & 0 & 0 & 0 \\ 0 & 0 & 0 & -3 & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & n-3 & 1 & 0 \\ 0 & 0 & 0 & 0 & \dots & -(n-2) & n-2 & 1 \\ 0 & 0 & 0 & 0 & \dots & 0 & -(n-1) & n-1 \end{vmatrix}$$
$$= -|e_{ij}|_{(n+1)\times(n+1)},$$

where

$$e_{ij} = \begin{cases} 1, & i-j=-1, \\ i-2, & i-j=0, \\ 2-i, & i-j=1, \\ 0, & i-j \ne 0, \pm 1. \end{cases}$$

Proof. Once we write down the determinant, the proof of Theorem 1 can be made into a single line! Indeed, if the determinant written down in Theorem 1 is denoted by a_n, then an induction immediately gives $a_{n+1} = n(a_n + a_{n-1})$. This clearly produces derangement numbers $!n$ which are determined by this recursion. Once discovered, the proof is just a single line. □

Remark 1. Recently, an alternative, although slightly complicated, proof of Theorem 1 was supplied in [1].

References

[1] F. Qi, J.-L. Wang, and B.-N. Guo, *A recovery of two determinantal representations for derangement numbers*, Cogent Math. (2016), in press; Available online at http://dx.doi.org/10.1080/23311835.2016.1232878.