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Short communication

Alternative proofs for summation formulas of some trigonometric series

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Abstract

In the paper, the authors supply alternative proofs for some summation formulas of trigonometric series.

Keywords: *summation formula; trigonometric series; alternative proof.*

1. Introduction

In 2016, after establishing

$$\sum_{n=1}^{\infty} 4^n \sin^4 \frac{x}{2^n} = x^2 - \sin^2 x, \tag{1.1}$$

$$\sum_{n=1}^{\infty} \frac{\sin^4(2^n x)}{4^n} = \frac{\sin^2(2x)}{4}, \tag{1.2}$$

$$\sum_{n=1}^{\infty} (-1)^n \frac{\cos^4(2^n x)}{4^n} = -\frac{3}{40} - \frac{\cos(4x)}{8}, \tag{1.3}$$

$$\sum_{n=1}^{\infty} 3^n \sin^3 \frac{x}{3^n} = \frac{3}{4}(x - \sin x), \tag{1.4}$$

$$\sum_{n=1}^{\infty} \frac{\sin^3(3^n x)}{3^n} = \frac{\sin(3x)}{4}, \tag{1.5}$$

$$\sum_{n=1}^{\infty} (-1)^n \frac{\cos^3(3^n x)}{3^n} = -\frac{\cos(3x)}{4} \tag{1.6}$$

by means of telescoping method and after listing some formulas from [2, p. 39], Chu [1] posed that there may exist summation formulas for the trigonometric series

$$U(\alpha,\beta;x) = \sum_{n=1}^{\infty} \alpha^n \sin(\beta^n x)$$

and

$$V(\alpha,\beta;x) = \sum_{n=1}^{\infty} \alpha^n \cos(\beta^n x),$$

where $\alpha,\beta \notin \{0,\pm1\}$ and $x \ne 0$ are real constants.
It is obvious that

$$V(\alpha,\beta;x) \pm iU(\alpha,\beta;x) = \sum_{n=1}^{\infty} \alpha^n e^{\pm i\beta^n x},$$

where i denotes the imaginary unit in the complex numbers.
For $\alpha,\beta \notin \{0,\pm1\}$, $\gamma \in \mathbb{N}$, $x \ne 0$, and $m \in \mathbb{N} \cup \{\infty\}$, let

$$U(\alpha,\beta,\gamma;x;m) = \sum_{n=1}^{m} \alpha^n \sin^\gamma(\beta^n x)$$

and

$$V(\alpha,\beta,\gamma;x;m) = \sum_{n=1}^{m} \alpha^n \cos^\gamma(\beta^n x).$$

It is clear that

1. $U(\alpha,\beta,1;x;\infty) = U(\alpha,\beta;x)$ and $V(\alpha,\beta,1;x;\infty) = V(\alpha,\beta;x)$;
2. when $0 < |\alpha| < 1$, the infinite trigonometric series $U(\alpha,\beta,\gamma;x;\infty)$ and $V(\alpha,\beta,\gamma;x;\infty)$ are absolutely convergent;
3. when $0 < |\beta| < 1$ and $0 < |\alpha\beta^\gamma| < 1$, the infinite trigonometric series $U(\alpha,\beta,\gamma;x;\infty)$ is absolutely convergent.

In this paper, we supply alternative proofs for summation formulas (1.1) to (1.6) by means of discussing the trigonometric series $U(\alpha,\beta,\gamma;x;\infty)$ and $V(\alpha,\beta,\gamma;x;\infty)$.

2. Alternative proofs of the formulas (1.1) to (1.6)

We now start out to prove those formulas (1.1) to (1.6) alternatively. In [3, Corollary 2.1], Guo and Qi obtained

$$\cos^\ell x = \frac{1}{2^\ell} \sum_{q=0}^{\ell} \binom{\ell}{q} \cos[(2q-\ell)x] \tag{2.1}$$

and

$$\sin^\ell x = \frac{(-1)^\ell}{2^\ell} \sum_{q=0}^{\ell} (-1)^q \binom{\ell}{q} \cos\left[(2q-\ell)x - \frac{\ell}{2}\pi\right] \tag{2.2}$$

for $\ell \in \mathbb{N}$. Accordingly, when $\gamma \in \mathbb{N}$, we have

$$
\begin{aligned}
&U(\alpha,\beta,\gamma;x;\infty)\\
&= \frac{(-1)^\gamma}{2^\gamma}\sum_{q=0}^{\gamma}(-1)^q\binom{\gamma}{q}\sum_{n=1}^{\infty}\alpha^n\cos\left[(2q-\gamma)\beta^n x-\frac{\gamma}{2}\pi\right]\\
&= \begin{cases}
\frac{(-1)^\gamma}{2^\gamma}\sum_{q=0}^{\gamma}(-1)^q\binom{\gamma}{q}\sum_{n=1}^{\infty}\alpha^n\sin\left[(2q-\gamma)\beta^n x\right], \\
\gamma = 4k+1;\\
\frac{(-1)^{\gamma+1}}{2^\gamma}\sum_{q=0}^{\gamma}(-1)^q\binom{\gamma}{q}\sum_{n=1}^{\infty}\alpha^n\cos\left[(2q-\gamma)\beta^n x\right], \\
\gamma = 4k+2;\\
\frac{(-1)^{\gamma+1}}{2^\gamma}\sum_{q=0}^{\gamma}(-1)^q\binom{\gamma}{q}\sum_{n=1}^{\infty}\alpha^n\sin\left[(2q-\gamma)\beta^n x\right], \\
\gamma = 4k+3;\\
\frac{(-1)^\gamma}{2^\gamma}\sum_{q=0}^{\gamma}(-1)^q\binom{\gamma}{q}\sum_{n=1}^{\infty}\alpha^n\cos\left[(2q-\gamma)\beta^n x\right], \\
\gamma = 4k+4
\end{cases} \qquad (2.3)
\end{aligned}
$$

and

$$
V(\alpha,\beta,\gamma;x;\infty) = \frac{1}{2^\gamma}\sum_{q=0}^{\gamma}\binom{\gamma}{q}\sum_{n=1}^{\infty}\alpha^n\cos\left[(2q-\gamma)\beta^n x\right] \qquad (2.4)
$$

for $k \in \{0\}\cup\mathbb{N}$. Consequently, taking $\alpha = \frac{1}{4}$, $\beta = 2$, and $\gamma = 4$ in the above expression for $U(\alpha,\beta,\gamma;x;\infty)$ leads to

$$
\begin{aligned}
&U\left(\frac{1}{4},2,4;x;\infty\right) = \frac{1}{16}\sum_{q=0}^{4}(-1)^q\binom{4}{q}\sum_{n=1}^{\infty}\frac{1}{4^n}\cos[(2q-4)2^n x]\\
&= \frac{1}{16}\left[2\sum_{n=1}^{\infty}\frac{1}{4^n}\cos\left(2^{n+2}x\right)-8\sum_{n=1}^{\infty}\frac{1}{4^n}\cos\left(2^{n+1}x\right)+\binom{4}{2}\sum_{n=1}^{\infty}\frac{1}{4^n}\right]\\
&= \frac{1}{16}\left[2\sum_{n=1}^{\infty}\frac{1}{4^n}\cos\left(2^{n+2}x\right)-8\sum_{n=0}^{\infty}\frac{1}{4^{n+1}}\cos\left(2^{n+2}x\right)+2\right]\\
&= \frac{1}{16}[-2\cos(4x)+2] = \frac{1}{8}[1-\cos(4x)] = \frac{\sin^2(2x)}{4}.
\end{aligned}
$$

The formula (1.2) is thus recovered.
Similarly, we have

$$
\begin{aligned}
&V\left(-\frac{1}{4},2,4;x;\infty\right) = \frac{1}{16}\sum_{q=0}^{4}\binom{4}{q}\sum_{n=1}^{\infty}(-1)^n\frac{1}{4^n}\cos[(2q-4)2^n x]\\
&= \frac{1}{16}\left[2\sum_{n=1}^{\infty}\frac{(-1)^n}{4^n}\cos\left(2^{n+2}x\right)+8\sum_{n=1}^{\infty}\frac{(-1)^n}{4^n}\cos\left(2^{n+1}x\right)\right.\\
&\quad\left.+\binom{4}{2}\sum_{n=1}^{\infty}\frac{(-1)^n}{4^n}\right]\\
&= \frac{1}{16}\left[2\sum_{n=1}^{\infty}\frac{(-1)^n}{4^n}\cos\left(2^{n+2}x\right)+8\sum_{n=0}^{\infty}\frac{(-1)^{n+1}}{4^{n+1}}\cos\left(2^{n+2}x\right)-\frac{6}{5}\right]\\
&= \frac{1}{16}\left[-2\cos(4x)-\frac{6}{5}\right] = -\frac{3}{40}-\frac{\cos(4x)}{8}.
\end{aligned}
$$

The formula (1.3) is thus proved again.
Furthermore, we have

$$
\begin{aligned}
&U\left(\frac{1}{3},3,3;x;\infty\right) = \frac{1}{2^3}\sum_{q=0}^{3}(-1)^q\binom{3}{q}\sum_{n=1}^{\infty}\frac{1}{3^n}\sin\left[(2q-3)3^n x\right]\\
&= \frac{1}{4}\left[-\sum_{n=1}^{\infty}\frac{1}{3^n}\sin\left(3^{n+1}x\right)+3\sum_{n=1}^{\infty}\frac{1}{3^n}\sin\left(3^n x\right)\right]\\
&= \frac{1}{4}\left[-\sum_{n=1}^{\infty}\frac{1}{3^n}\sin\left(3^{n+1}x\right)+3\sum_{n=0}^{\infty}\frac{1}{3^{n+1}}\sin\left(3^{n+1}x\right)\right]\\
&= \frac{\sin(3x)}{4}
\end{aligned}
$$

and

$$
\begin{aligned}
&V\left(-\frac{1}{3},3,3;x;\infty\right) = \frac{1}{2^3}\sum_{q=0}^{3}\binom{3}{q}\sum_{n=1}^{\infty}\frac{(-1)^n}{3^n}\cos\left[(2q-3)3^n x\right]\\
&= \frac{1}{4}\left[\sum_{n=1}^{\infty}\frac{(-1)^n}{3^n}\cos\left(3^{n+1}x\right)+3\sum_{n=1}^{\infty}\frac{(-1)^n}{3^n}\cos\left(3^n x\right)\right]\\
&= \frac{1}{4}\left[\sum_{n=1}^{\infty}\frac{(-1)^n}{3^n}\cos\left(3^{n+1}x\right)+3\sum_{n=0}^{\infty}\frac{(-1)^{n+1}}{3^{n+1}}\cos\left(3^{n+1}x\right)\right]\\
&= -\frac{\cos(3x)}{4}.
\end{aligned}
$$

The formulas (1.5) and (1.6) are verified once again.
By (2.2), it is straightforward that

$$
\begin{aligned}
&U\left(3,\frac{1}{3},3;x;\infty\right)\\
&= \sum_{n=1}^{\infty}3^n\frac{(-1)^3}{2^3}\sum_{q=0}^{3}(-1)^{q+1}\binom{3}{q}\sin\left[(2q-3)\frac{1}{3^n}x\right]\\
&= \frac{1}{8}\sum_{n=1}^{\infty}3^n\left(-2\sin\frac{x}{3^{n-1}}+6\sin\frac{x}{3^n}\right)\\
&= \frac{1}{4}\sum_{n=1}^{\infty}\left(3^{n+1}\sin\frac{x}{3^n}-3^n\sin\frac{x}{3^{n-1}}\right)\\
&= \frac{1}{4}\left[\lim_{n\to\infty}\left(3^{n+1}\sin\frac{x}{3^n}\right)-3\sin x\right]\\
&= \frac{3}{4}(x-\sin x)
\end{aligned}
$$

and that

$$
\begin{aligned}
&U\left(4,\frac{1}{2},4;x;\infty\right) = \sum_{n=1}^{\infty}4^n\frac{1}{2^4}\sum_{q=0}^{4}(-1)^q\binom{\ell}{q}\cos\left[(2q-4)\frac{1}{2^n}x\right]\\
&= \frac{1}{16}\sum_{n=1}^{\infty}4^n\left[2\cos\frac{x}{2^{n-2}}-8\cos\frac{x}{2^{n-1}}+\binom{4}{2}\right]\\
&= \frac{1}{8}\sum_{n=1}^{\infty}4^n\left(\cos\frac{x}{2^{n-2}}-4\cos\frac{x}{2^{n-1}}+3\right)\\
&= \frac{1}{8}\sum_{n=1}^{\infty}4^n\left[4\left(1-\cos\frac{x}{2^{n-1}}\right)-\left(1-\cos\frac{x}{2^{n-2}}\right)\right]\\
&= \frac{1}{8}\sum_{n=1}^{\infty}\left[4^{n+1}\left(1-\cos\frac{x}{2^{n-1}}\right)-4^n\left(1-\cos\frac{x}{2^{n-2}}\right)\right]\\
&= \frac{1}{8}\left[\lim_{n\to\infty}\left(4^{n+1}\left(1-\cos\frac{x}{2^{n-1}}\right)\right)-4(1-\cos(2x))\right]\\
&= \frac{1}{8}\left[8x^2-4(1-\cos(2x))\right]\\
&= x^2-\sin^2 x.
\end{aligned}
$$

The formulas (1.4) and (1.1) are recovered alternatively. The proofs of (1.1) to (1.6) are complete.

3. Remarks

Remark 3.1. By (2.3) and (2.4), it is not difficult to see that, in order to find summation formulas for $U(\alpha,\beta,\gamma;x;\infty)$ and $V(\alpha,\beta,\gamma;x;\infty)$, it is sufficient to find summation formulas for the trigonometric series

$$
u(\alpha,\beta,\gamma,q;x;\infty) = \sum_{n=1}^{\infty}\alpha^n\sin\left[(2q-\gamma)\beta^n x\right]
$$

and

$$v(\alpha,\beta,\gamma,q;x;\infty)=\sum_{n=1}^{\infty}\alpha^n\cos\left[(2q-\gamma)\beta^n x\right]$$

for given $\gamma\in\mathbb{N}$ and for all $0\le q\le\gamma$. Since

$$u(\alpha,\beta,\gamma,q;x;\infty)=U(\alpha,\beta;(2q-\gamma)x)$$

and

$$v(\alpha,\beta,\gamma,q;x;\infty)=V(\alpha,\beta;(2q-\gamma)x),$$

it is sufficient to find summation formulas for the infinite trigonometric series $U(\alpha,\beta;x)$ and $V(\alpha,\beta;x)$.

Remark 3.2. In [3, Theorem 3.1], Guo and Qi obtained

$$\sin(kx)=\sum_{\ell=0}^{k}\binom{k}{\ell}\sin\frac{\ell\pi}{2}\sin^{\ell}x\cos^{k-\ell}x$$

and

$$\cos(kx)=\sum_{\ell=0}^{k}\binom{k}{\ell}\cos\frac{\ell\pi}{2}\sin^{\ell}x\cos^{k-\ell}x$$

for $k\ge 2$. These two identities and the identities (2.1) and (2.2) can be regarded as inversions each other.

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