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Some inequalities for (s_1,m_1)-(s_2,m_2)-convex functions on the co-ordinates

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Abstract

In this paper, the authors introduce a new concept "(s_1,m_1)-(s_2,m_2)-convex function on co-ordinates" and establish some inequalities for (s_1,m_1)-(s_2,m_2)-convex functions of 2-variables on the co-ordinates.

Keywords: *co-ordinated convex function; (s_1,m_1)-(s_2,m_2)-convex function; inequality.*

1 Introduction

The following inequality, named after Simpson, is one of the best known results in the literature.

Theorem A. *Let $f: I \subset \mathbb{R}_0 = [0,\infty) \to \mathbb{R} = (-\infty,\infty)$ be a four times continuously differentiable mapping on $[a,b]$ and $\|f^{(4)}\|_\infty = \sup_{x\in[a,b]} |f^{(4)}(x)| < \infty$. Then*

$$\left|\frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]-\frac{1}{b-a}\int_a^b f(x)\mathrm{d}x\right| \le \frac{(b-a)^4}{2880}\left\|f^{(4)}\right\|_\infty. \tag{1.1}$$

For more information and recent development on Simpson's type inequalities, please refer to [1, 2, 3] and closely related references therein.

In [4], J. Park obtained the following generalized identity for some partial differentiable mappings on a bi-dimensional interval.

Lemma 1. *Let $f:\Delta\to\mathbb{R}$ be a partial differentiable mapping on $\Delta=[a,b]\times[c,d]\subset\mathbb{R}^2$. If $\frac{\partial^2 f}{\partial t\,\partial\lambda}\in L(\Delta)$, then for $r_1,r_2\ge 2$ and $h_1,h_2\in(0,1)$ with $\frac{1}{r_1}\le h_1\le\frac{r_1-1}{r_1}$, $\frac{1}{r_2}\le h_2\le\frac{r_2-1}{r_2}$, we have*

$$\begin{aligned}
I(f)(h_1,h_2,r_1,r_2) &= \left[\frac{(r_1-2)(r_2-2)}{r_1r_2}\right] f(h_1a+(1-h_1)b,h_2c+(1-h_2)d) \\
&\quad+\left[\frac{r_1-2}{r_1r_2}\right]\left[f(h_1a+(1-h_1)b,c)+f(h_1a+(1-h_1)b,d)\right]+\left[\frac{r_2-2}{r_1r_2}\right]\left[f(a,h_2c+(1-h_2)d)+f(b,h_2c+(1-h_2)d)\right] \\
&\quad+\frac{1}{r_1r_2}\left[f(a,c)+f(a,d)+f(b,c)+f(b,d)\right]-\frac{1}{r_2(b-a)}\int_a^b\left[f(x,c)+(r_2-2)f(x,h_2c+(1-h_2)d)+f(x,d)\right]dx \\
&\quad-\frac{1}{r_1(d-c)}\int_c^d\left[f(a,y)+(r_1-2)f(h_1a+(1-h_1)b,y)+f(b,y)\right]dy+\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)\,dy\,dx \\
&=(b-a)(d-c)\int_0^1\int_0^1 p(h_1,r_1,t)\,q(h_2,r_2,\lambda)\frac{\partial^2 f}{\partial t\,\partial\lambda}(ta+(1-t)b,\lambda c+(1-\lambda)d)\,dt\,d\lambda,
\end{aligned}$$

where

$$p(h_1,r_1,t)==\begin{cases} t-\dfrac{1}{r_1}, & t\in[0,h_1] \\ t-\dfrac{r_1-1}{r_1}, & t\in[h_1,1] \end{cases} \quad and \quad q(h_2,r_2,t)=\begin{cases} \lambda-\dfrac{1}{r_2}, & \lambda\in[0,h_2], \\ \lambda-\dfrac{r_2-1}{r_2}, & \lambda\in[h_2,1]. \end{cases}$$

Moreover, J. Park established some Simpson-like type inequalities for co-ordinated s -convex mappings in the second sense.

The main purpose of this paper is to introduce the new concept "co-ordinated (s_1,m_1) - (s_2,m_2) -convex function" and establish some new inequalities for such kind of functions on the co-ordinates.

2 Co-ordinated (s_1,m_1)-(s_2,m_2)-convex functions

In [2, 4-10], Dragomir et al. and Park considered among others the class of mappings which are convex and s -convex on the co-ordinates.

Let us now consider a bi-dimensional interval $\Delta=[a,b]\times[c,d]$ in $\mathbb{R}^2$ with $b>a$ and $d>c$.

Definition 1. ([8]) *A mapping $f:\Delta\to\mathbb{R}$ is said to be convex on Δ if the inequality*

$$f(\alpha x+(1-\alpha)z,\alpha y+(1-\alpha)w)\le\alpha f(x,y)+(1-\alpha)f(z,w)$$

holds for all $(x,y),(z,w)\in\Delta$ and $\alpha\in[0,1]$. If the inequality is reversed, then f is said to be concave on Δ.

A modification for convex functions, which is also known as co-ordinated convex functions, was introduced by Dragomir in [6] as follows.

Definition 2. *A function $f:\Delta\to\mathbb{R}$ is said to be convex on the co-ordinates on Δ if the partial mappings $f_y:[a,b]\to R, f_y(u)=f(u,y)$ and $f_x:[c,d]\to R, f_x(v)=f(x,v)$ are convex for all $x\in[a,b]$ and $y\in[c,d]$.*

A formal definition for co-ordinated convex functions may be stated as follows.

Definition 3. *A function $f:\Delta\to\mathbb{R}$ is said to be convex on the co-ordinates on Δ if the inequality*

$$f(tx+(1-t)z,\lambda y+(1-\lambda)w)\le t\lambda f(x,y)+t(1-\lambda)f(x,w)+\lambda(1-t)f(z,y)+(1-t)(1-\lambda)f(z,w)$$

holds for all $(x,y),(z,w),(x,w),(z,y)\in\Delta$ and $t,\lambda\in[0,1]$.

Clearly, every convex mapping $f:\Delta\to\mathbb{R}$ is convex on the co-ordinates. Furthermore, there exists co-ordinated convex functions which are not convex. See [9].

Definition 4 ([4, 8, 10]) *For the bi-dimensional interval $\Delta=[a,b]\times[c,d]$ in $\mathbb{R}_0^2=[0,\infty)^2$ with $a<b$ and $c<d$, a mapping $f:\Delta\to\mathbb{R}$ is said to be s -convex on Δ if the inequality*

$$f\left(\alpha x+(1-\alpha)z,\alpha y+(1-\alpha)w\right)\le\alpha^s f\left(x,y\right)+\left(1-\alpha\right)^s f\left(z,w\right)$$

holds for all $(x,y),(z,w)\in\Delta$ and $\alpha\in[0,1]$ and for some fixed $s\in(0,1]$.

Definition 5. ([4, 8]) *A function $f:\Delta=[a,b]\times[c,d]\subseteq\mathbb{R}_0^2\to\mathbb{R}$ is called s-convex on co-ordinates on Δ if the partial mappings $f_y:[a,b]\to\mathbb{R}, f_y(u)=f(u,y)$ and $f_x:[c,d]\to\mathbb{R}$, $f_x(v)=f(x,v)$ are s -convex for all $x\in[a,b]$, $y\in[c,d]$ and $s\in(0,1]$, i.e., f_y and f_x are s -convex for some fixed $s\in(0,1]$.*

Theorem B. ([8]) *Every s -convex mapping $f:\Delta=[a,b]\times[c,d]\subset\mathbb{R}_0^2\to\mathbb{R}$ is s -convex on the co-ordinates. But, the converse is not true in general.*

A formal definition for co-ordinated s -convex mappings may be stated as follow.

Definition 6. ([4,8]) A *mapping $f:\Delta\to\mathbb{R}$ is called s-convex on the co-ordinates on Δ if the inequality*

$$f\left(tx+(1-t)z,\lambda y+(1-\lambda)w\right)\le t^s\lambda^s f(x,y)+t^s(1-\lambda)^s f(x,w)+\lambda^s(1-t)^s f(z,y)+(1-t)^s(1-\lambda)^s f(z,w)$$

holds for all $(x,y),(z,w),(x,w),(z,y)\in\Delta$ *and* $t,\lambda\in[0,1]$ *and* $s\in(0,1]$.

J. Park gave the notions of (s,m)-convexity and (s,m)–convexity on the co-ordinates for the function f on a rectangle from the plane $\mathbb{R}^2$ as follows.

Definition 7 ([11]). *A mapping* $f:\Delta_0=[0,b]\times[0,d]\subset\mathbb{R}_0^2\to\mathbb{R}$ *is* (s,m)*-convex on* Δ *if the inequality*

$$f\left(\alpha x+(1-\alpha)z,\alpha y+m(1-\alpha)w\right)\le\alpha^s f(x,y)+m\left(1-\alpha^s\right)f(z,w)$$

holds for all $(x,y),(z,w)\in\Delta$ *and* $\alpha\in[0,1]$ *and for some fixed* $s,m\in(0,1]$.

Definition 8 ([11]). *A function* $f:\Delta_0=[0,b]\times[0,d]\subset\mathbb{R}_0^2\to\mathbb{R}$ *is called* (s,m)*- convex on co-ordinates on* Δ *if the partial mappings* $f_y:[0,b]\to\mathbb{R}, f_y(u)=f(u,y)$ *and* $f_x:[0,d]\to\mathbb{R},\ f_x(v)=f(x,v)$ *are* (s,m)*-convex for all* $x\in[0,b], y\in[0,d]$ *and* $s,m\in(0,1]$*, i.e.,* f_y *and* f_x *are* (s,m)*-convex with some fixed* $s,m\in(0,1]$.

Motivated by the above definitions, we now introduce the (s_1,m_1) - (s_2,m_2) -convexity on the co-ordinates for a function $f(x)$ on a rectangle from the plane $\mathbb{R}^2$ as follows:

Definition 9. *A function* $f:\Delta_0=[0,b]\times[0,d]\subset\mathbb{R}_0^2\to\mathbb{R}$ *is called* (s_1,m_1) - (s_2,m_2) *convex on co-ordinates on* Δ *if the inequality*

$$f\left(tx+m_1(1-t)z,\lambda y+m_2(1-\lambda)w\right)$$
$$\le t^{s_1}\lambda^{s_2}f(x,y)+m_1t^{s_1}\left(1-\lambda^{s_2}\right)f(x,w)+m_2\lambda^{s_2}\left(1-t^{s_1}\right)f(z,y)+m_1m_2\left(1-t^{s_1}\right)\left(1-\lambda^{s_2}\right)f(z,w)$$

holds for all $(x,y),(z,w),(x,w),(z,y)\in\Delta$ with $t,\lambda\in[0,1]$ and $s_1,s_2,m_1,m_2\in(0,1]$.

3 Some inequalities for (s_1,m_1)-(s_2,m_2)-convex functions

Theorem 1 *Let* $f:\mathbb{R}_0^2\to\mathbb{R}$ *be a partial differentiable mapping on* $\Delta_m=[0,\frac{b}{m_1}]\times[0,\frac{d}{m_2}]$ *for* $b>a>0$ *and* $d>c>0$. *If* $\frac{\partial^2 f}{\partial t\,\partial\lambda}\in L(\Delta_m)$ *is a co-ordinated* (s_1,m_1) - (s_2,m_2) *-convex mapping on* Δ_m *for* $s_1,s_2,m_1,m_2\in(0,1]$, *then for* $r_1,r_2\ge 2$ *and* $h_1,h_2\in(0,1)$ *with* $\frac{1}{r_1}\le h_1\le\frac{r_1-1}{r_1},\frac{1}{r_2}\le h_2\le\frac{r_2-1}{r_2}$, *the inequality*

$$\begin{aligned}\frac{1}{(b-a)(d-c)}\left|I(f)(h_1,h_2,r_1,r_2)\right| &\le \mu(h_1,r_1,s_1)\left\{\mu(h_2,r_2,s_2)\left|\frac{\partial^2 f}{\partial t\,\partial\lambda}(a,c)\right|+m_2\nu(h_2,r_2,s_2)\left|\frac{\partial^2 f}{\partial t\,\partial\lambda}\left(a,\frac{d}{m_2}\right)\right|\right\}\\ &+\nu(h_1,r_1,s_1)\left\{m_1\mu(h_2,r_2,s_2)\left|\frac{\partial^2 f}{\partial t\,\partial\lambda}\left(\frac{b}{m_1},c\right)\right|+m_1m_2\nu(h_2,r_2,s_2)\left|\frac{\partial^2 f}{\partial t\,\partial\lambda}\left(\frac{b}{m_1},\frac{d}{m_2}\right)\right|\right\},\end{aligned}\quad (3.1)$$

where

$$\mu(h,r,s)=M(r,s)+N(h,s),\ \nu(h,r,s)=W(h,r)-M(r,s)-N(1-h,s),$$

and

$$M(r,s)=\frac{2+2(r-1)^{s+2}+r^{s+1}(s-r+2)}{(s+1)(s+2)r^{s+2}},\ N(h,s)=\frac{h^{s+1}((2h-1)s+2(h-1))}{(s+1)(s+2)},$$

$$W(h,r)=\frac{1}{2}-h+h^2+\frac{(2-r)}{r^2}.$$

Proof. From Lemma 1 and the co-ordinated (s_1,m_1) - (s_2,m_2) -convexity of $\frac{\partial^2 f}{\partial t\,\partial\lambda}$, we have

$$
\begin{aligned}
&\frac{1}{(b-a)(d-c)}\left|I(f)(h_1,h_2,r_1,r_2)\right| \\
&\le \int_0^1\int_0^1 |p(h_1,r_1,t)q(h_2,r_2,\lambda)| \left|\frac{\partial^2 f}{\partial t\,\partial\lambda}\left(ta+m_1(1-t)\frac{b}{m_1},\lambda c+m_2(1-\lambda)\frac{d}{m_2}\right)\right| \mathrm{d}t\mathrm{d}\lambda \\
&\le \int_0^1\int_0^1 |p(h_1,r_1,t)q(h_2,r_2,\lambda)| \Bigg\{ t^{s_1}\lambda^{s_2}\left|\frac{\partial^2 f}{\partial t\,\partial\lambda}(a,c)\right| + m_2 t^{s_1}(1-\lambda^{s_2})\left|\frac{\partial^2 f}{\partial t\,\partial\lambda}\left(a,\frac{d}{m_2}\right)\right| \\
&\quad + m_1(1-t^{s_1})\lambda^{s_2}\left|\frac{\partial^2 f}{\partial t\,\partial\lambda}\left(\frac{b}{m_1},c\right)\right| + m_1m_2(1-t^{s_1})(1-\lambda^{s_2})\left|\frac{\partial^2 f}{\partial t\,\partial\lambda}\left(\frac{b}{m_1},\frac{d}{m_2}\right)\right|\Bigg\}\mathrm{d}t\mathrm{d}\lambda \\
&= \left(\int_0^1 |p(h_1,r_1,t)| t^{s_1}\mathrm{d}t\right)\Bigg\{\left(\int_0^1 |q(h_2,r_2,\lambda)|\lambda^{s_2}\mathrm{d}\lambda\right)\left|\frac{\partial^2 f}{\partial t\,\partial\lambda}(a,c)\right| \\
&\quad + m_2\left(\int_0^1 |q(h_2,r_2,\lambda)|(1-\lambda^{s_2})\mathrm{d}\lambda\right)\left|\frac{\partial^2 f}{\partial t\,\partial\lambda}\left(a,\frac{d}{m_2}\right)\right|\Bigg\} \\
&\quad + \left(\int_0^1 |p(h_1,r_1,t)|(1-t^{s_1})\mathrm{d}t\right)\Bigg\{m_1\left(\int_0^1 |q(h_2,r_2,\lambda)|\lambda^{s_2}\mathrm{d}\lambda\right)\left|\frac{\partial^2 f}{\partial t\,\partial\lambda}\left(\frac{b}{m_1},c\right)\right| \\
&\quad + m_1m_2\left(\int_0^1 |q(h_2,r_2,\lambda)|(1-\lambda^{s_2})\mathrm{d}\lambda\right)\left|\frac{\partial^2 f}{\partial t\,\partial\lambda}\left(\frac{b}{m_1},\frac{d}{m_2}\right)\right|\Bigg\}. \qquad (3.2)
\end{aligned}
$$

Note that

$$
\int_0^1 |p(h_1,r_1,t)| t^{s_1}\mathrm{d}t = \mu(h_1,r_1,s_1);\ \int_0^1 |q(h_2,r_2,\lambda)|\lambda^{s_2}\mathrm{d}t = \mu(h_2,r_2,s_2);
$$

$$
\int_0^1 |p(h_1,r_1,t)|\mathrm{d}t = \frac{1}{2}-h_1+h_1^2+\frac{(2-r_1)}{r_1^2} = W(h_1,r_1);
$$

$$
\int_0^1 |q(h_2,r_2,t)|\mathrm{d}t = \frac{1}{2}-h_2+h_2^2+\frac{(2-r_2)}{r_2^2} = W(h_2,r_2);
$$

$$
\int_0^1 |p(h_1,r_1,t)|(1-t^{s_1})\mathrm{d}t = \int_0^1 |p(h_1,r_1,t)|\mathrm{d}t - \int_0^1 |p(h_1,r_1,t)| t^{s_1}\mathrm{d}t = W(h_1,r_1)-\mu(h_1,r_1,s_1);
$$

$$
\int_0^1 |q(h_2,r_2,\lambda)|(1-\lambda^{s_2})\mathrm{d}t = W(h_2,r_2)-\mu(h_2,r_2,s_2).
$$

Substituting the above equations into (3.2) and rearranging leads to (3.1). This completes the proof.

Corollary 1. *Under the conditions of Theorem 1,*

(i) *if we choose* $h_1=h_2=\frac{1}{2}$, $r_1=r_2=6$, *and* $s_1=s_2=m_1=m_2=1$ *in* (3.1), *then*

$$
\left|I(f)\left(\frac{1}{2},\frac{1}{2},6,6\right)\right| \le \left(\frac{5}{72}\right)^2 M(b-a)(d-c).
$$

(ii) *if we choose* $h_1=h_2=\frac{1}{2}$, $r_1=r_2=2$, *and* $s_1=s_2=m_1=m_2=1$ *in* (3.1), *then*

$$
\left|I(f)\left(\frac{1}{2},\frac{1}{2},2,2\right)\right| \le \left(\frac{1}{8}\right)^2 M(b-a)(d-c),
$$

where

$$
M = \left|\frac{\partial^2 f}{\partial t\,\partial\lambda}(a,c)\right| + \left|\frac{\partial^2 f}{\partial t\,\partial\lambda}(a,d)\right| + \left|\frac{\partial^2 f}{\partial t\,\partial\lambda}(b,c)\right| + \left|\frac{\partial^2 f}{\partial t\,\partial\lambda}(b,d)\right|.
$$

Theorem 2 *Let* $f:\mathbb{R}_0^2\to\mathbb{R}$ *be a partial differentiable mapping on* $\Delta_m=[0,\frac{b}{m_1}]\times[0,\frac{d}{m_2}]$ *for* $b>a>0$ *and* $d>c>0$. *If* $\left|\frac{\partial^2 f}{\partial t\,\partial\lambda}\right|^q\in L(\Delta_m)$ *is a co-ordinated* (s_1,m_1) - (s_2,m_2) *-convex mapping, where* $s_1,s_2,m_1,m_2\in(0,1]$ *and* $\frac{1}{p}+\frac{1}{q}=1$ *for* $q>1$, *then for* $r_1,r_2\ge 2$ *and* $h_1,h_2\in(0,1)$ *with* $\frac{1}{r_1}\le h_1\le\frac{r_1-1}{r_1},\frac{1}{r_2}\le h_2\le\frac{r_2-1}{r_2}$, *the inequality*

$$\frac{1}{(b-a)(d-c)}\left|I(f)(h_1,h_2,r_1,r_2)\right|\le\mu_3^{\frac{1}{p}}\nu_3^{\frac{1}{p}}\left\{\frac{1}{(s_1+1)(s_2+1)}\left[\left|\frac{\partial^2 f}{\partial t\,\partial\lambda}(a,c)\right|^q+m_2s_2\left|\frac{\partial^2 f}{\partial t\,\partial\lambda}\left(a,\frac{d}{m_2}\right)\right|^q\right.\right.$$

$$\left.\left.+m_1s_1\left|\frac{\partial^2 f}{\partial t\,\partial\lambda}\left(\frac{b}{m_1},c\right)\right|^q+m_1s_1m_2s_2\left|\frac{\partial^2 f}{\partial t\,\partial\lambda}\left(\frac{b}{m_1},\frac{d}{m_2}\right)\right|^q\right]\right\}^{\frac{1}{q}}\tag{3.3}$$

is true, where

$$\mu_3=\frac{2+(r_1-r_1h_1-1)^{p+1}+(r_1h_1-1)^{p+1}}{r_1^{p+1}(p+1)}\ \text{and}\ \nu_3=\frac{2+(r_2-r_2h_2-1)^{p+1}+(r_2h_2-1)^{p+1}}{r_2^{p+1}(p+1)}.$$

Proof. From Lemma 1, we have

$$\frac{1}{(b-a)(d-c)}\left|I(f)(h_1,h_2,r_1,r_2)\right|$$

$$\le\int_0^1\int_0^1\left|p(h_1,r_1,t)q(h_2,r_2,\lambda)\right|\left|\frac{\partial^2 f}{\partial t\,\partial\lambda}\left(ta+m_1(1-t)\frac{b}{m_1},\lambda c+m_2(1-\lambda)\frac{d}{m_2}\right)\right|\mathrm{d}t\mathrm{d}\lambda$$

$$\le\left\{\int_0^1\int_0^1\left|p(h_1,r_1,t)q(h_2,r_2,\lambda)\right|^p\mathrm{d}t\mathrm{d}\lambda\right\}^{\frac{1}{p}}\left\{\int_0^1\int_0^1\left|\frac{\partial^2 f}{\partial t\partial\lambda}\left(ta+m_1(1-t)\frac{b}{m_1},\lambda c+m_2(1-\lambda)\frac{d}{m_2}\right)\right|^q\mathrm{d}t\mathrm{d}\lambda\right\}^{\frac{1}{q}}.\tag{3.4}$$

Hence, by the inequality (3.4) and the co-ordinated (s_1,m_1) - (s_2,m_2) -convexity of $\left|\frac{\partial^2 f}{\partial t\,\partial\lambda}\right|^q$, it follows that

$$\frac{1}{(b-a)(d-c)}\left|I(f)(h_1,h_2,r_1,r_2)\right|\le\left\{\int_0^1\int_0^1\left|p(h_1,r_1,t)q(h_2,r_2,\lambda)\right|^p\mathrm{d}t\mathrm{d}\lambda\right\}^{\frac{1}{p}}\left\{\frac{1}{(s_1+1)(s_2+1)}\left[\left|\frac{\partial^2 f}{\partial t\partial\lambda}(a,c)\right|^q\right.\right.$$

$$\left.\left.+m_2s_2\left|\frac{\partial^2 f}{\partial t\,\partial\lambda}\left(a,\frac{d}{m_2}\right)\right|^q+m_1s_1\left|\frac{\partial^2 f}{\partial t\,\partial\lambda}\left(\frac{b}{m_1},c\right)\right|^q+m_1m_2s_1s_2\left|\frac{\partial^2 f}{\partial t\,\partial\lambda}\left(\frac{b}{m_1},\frac{d}{m_2}\right)\right|^q\right]\right\}^{\frac{1}{q}}.$$

Note that

$$\int_0^1\left|p(h_1,r_1,t)\right|^p\mathrm{d}t=\frac{2+(r_1-r_1h_1-1)^{p+1}+(r_1h_1-1)^{p+1}}{r_1^{p+1}(p+1)}$$

and

$$\int_0^1\left|q(h_2,r_2,\lambda)\right|^p\mathrm{d}t=\frac{2+(r_2-r_2h_2-1)^{p+1}+(r_2h_2-1)^{p+1}}{r_2^{p+1}(p+1)}.$$

Substituting the above equations into (3.4) and simplifying results in (3.3). The proof is complete.

Corollary 2 *Under the conditions of Theorem 2,*

(i) *if we choose* $h_1=h_2=\frac{1}{2}$, $r_1=r_2=6$, *and* $s_1=s_2=m_1=m_2=1$, *then*

$$\left|I(f)\left(\frac{1}{2},\frac{1}{2},6,6\right)\right|\le\left[\frac{2(1+2^{p+1})}{6^{p+1}(p+1)}\right]^{\frac{2}{p}}(b-a)(d-c)M_q^{\frac{1}{q}}.$$

(ii) *if we choose* $h_1 = h_2 = \frac{1}{2}, r_1 = r_2 = 2$, *and* $s_1 = s_2 = m_1 = m_2 = 1$, *then*

$$\left|I(f)\left(\frac{1}{2},\frac{1}{2},2,2\right)\right| \le \left[\frac{1}{2^p(p+1)}\right]^{\frac{2}{p}}(b-a)(d-c)M_q^{\frac{1}{q}}.$$

(iii) *if we choose* $h_1 = h_2 = \frac{1}{2}, r_1 = r_2 = 6$, $s_1 = s_2 = m_1 = m_2 = 1$, *and* $p = 1$, *then*

$$\left|I(f)\left(\frac{1}{2},\frac{1}{2},6,6\right)\right| \le \left(\frac{5}{36}\right)^2(b-a)(d-c)M_q^{\frac{1}{q}},$$

where

$$M_q = \frac{1}{4}\left(\left|\frac{\partial^2 f}{\partial t\,\partial\lambda}(a,c)\right|^q + \left|\frac{\partial^2 f}{\partial t\,\partial\lambda}(a,d)\right|^q + \left|\frac{\partial^2 f}{\partial t\,\partial\lambda}(b,c)\right|^q + \left|\frac{\partial^2 f}{\partial t\,\partial\lambda}(b,d)\right|^q\right).$$

Theorem 3 *Let* $f:\mathbb{R}_0^2 \to \mathbb{R}$ *be a partial differentiable mapping on* $\Delta_m = [0,\frac{b}{m_1}]\times[0,\frac{d}{m_2}]$ *for* $b > a > 0$ *and* $d > c > 0$. *If* $\left|\frac{\partial^2 f}{\partial t\,\partial\lambda}\right|^q \in L(\Delta_m)$ *is a co-ordinated* (s_1,m_1) - (s_2,m_2) - *convex mapping for* $s_1,s_2,m_1,m_2 \in (0,1]$, $q>1$, *and* $\frac{1}{p}+\frac{1}{q}=1$, *then, for* $r_1,r_2 \ge 2$ *and* $h_1,h_2 \in (0,1)$ *with* $\frac{1}{r_1} \le h_1 \le \frac{r_1-1}{r_1}, \frac{1}{r_2} \le h_2 \le \frac{r_2-1}{r_2}$, *the inequality*

$$\frac{1}{(b-a)(d-c)}\left|I(f)(h_1,h_2,r_1,r_2)\right| \le \left\{\left(\frac{1}{2}-h_1+h_1^2+\frac{(2-r_1)}{r_1^2}\right)\left(\frac{1}{2}-h_2+h_2^2+\frac{(2-r_2)}{r_2^2}\right)\right\}^{1-\frac{1}{q}}$$

$$\times\left\{\mu(h_1,r_1,s_1)\left[\mu(h_2,r_2,s_2)\left|\frac{\partial^2 f}{\partial t\,\partial\lambda}(a,c)\right|^q + m_2\nu(h_2,r_2,s_2)\left|\frac{\partial^2 f}{\partial t\,\partial\lambda}\left(a,\frac{d}{m_2}\right)\right|^q\right]\right.$$

$$\left.+\nu(h_1,r_1,s_1)\left[m_1\mu(h_2,r_2,s_2)\left|\frac{\partial^2 f}{\partial t\,\partial\lambda}\left(\frac{b}{m_1},c\right)\right|^q + m_1 m_2\nu(h_2,r_2,s_2)\left|\frac{\partial^2 f}{\partial t\,\partial\lambda}\left(\frac{b}{m_1},\frac{d}{m_2}\right)\right|^q\right]\right\}^{\frac{1}{q}}. \quad (3.5)$$

holds true, where μ *and* ν *are given in Theorem 2.1.*

Proof. From Lemma 1, it follows that

$$\frac{1}{(b-a)(d-c)}\left|I(f)(h_1,h_2,r_1,r_2)\right| \le \int_0^1\int_0^1\left|p(h_1,r_1,t)q(h_2,r_2,\lambda)\right|\left|\frac{\partial^2 f}{\partial t\partial\lambda}(ta+(1-t)b,\lambda c+(1-\lambda)d)\right|\mathrm{d}t\mathrm{d}\lambda$$

$$\le\left\{\int_0^1\int_0^1\left|p(h_1,r_1,t)q(h_2,r_2,\lambda)\right|\mathrm{d}t\mathrm{d}\lambda\right\}^{1-\frac{1}{q}}\left\{\int_0^1\int_0^1\left|p(h_1,r_1,t)q(h_2,r_2,\lambda)\right|\left|\frac{\partial^2 f}{\partial t\partial\lambda}(ta+(1-t)b,\lambda c+(1-\lambda)d)\right|^q\mathrm{d}t\mathrm{d}\lambda\right\}^{\frac{1}{q}}. \quad (3.6)$$

It is straightforward that

$$\int_0^1\left|p(h_1,r_1,t)\right|dt = \frac{1}{2}-h_1+h_1^2+\frac{(2-r_1)}{r_1^2},\ \int_0^1\left|q(h_2,r_2,\lambda)\right|d\lambda = \frac{1}{2}-h_2+h_2^2+\frac{(2-r_2)}{r_2^2}.$$

Since $\left|\frac{\partial^2 f}{\partial t\,\partial\lambda}\right|^q$ is a co-ordinated (s_1,m_1) - (s_2,m_2) -convex mapping on Δ, we have

$$\int_0^1\int_0^1\left|p(h_1,r_1,t)q(h_2,r_2,\lambda)\right|\left|\frac{\partial^2 f}{\partial t\,\partial\lambda}(ta+(1-t)b,\lambda c+(1-\lambda)d)\right|^q\mathrm{d}t\mathrm{d}\lambda$$

$$\le\int_0^1\int_0^1\left|p(h_1,r_1,t)q(h_2,r_2,\lambda)\right|\left\{t^{s_1}\lambda^{s_2}\left|\frac{\partial^2 f}{\partial t\,\partial\lambda}(a,c)\right|^q + m_2 t^{s_1}(1-\lambda^{s_2})\left|\frac{\partial^2 f}{\partial t\,\partial\lambda}\left(a,\frac{d}{m_2}\right)\right|^q\right.$$

$$+m_1(1-t^{s_1})\lambda^{s_2}\left|\frac{\partial^2 f}{\partial t\,\partial\lambda}\left(\frac{b}{m_1},c\right)\right|^q+m_1m_2(1-t^{s_1})(1-\lambda^{s_2})\left|\frac{\partial^2 f}{\partial t\,\partial\lambda}\left(\frac{b}{m_1},\frac{d}{m_2}\right)\right|^q\Bigg\}\mathrm{d}t\mathrm{d}\lambda$$

$$=\left(\int_0^1|p(h_1,r_1,t)|t^{s_1}\mathrm{d}t\right)\left\{\left(\int_0^1|q(h_2,r_2,\lambda)|\lambda^{s_2}\mathrm{d}\lambda\right)\left|\frac{\partial^2 f}{\partial t\,\partial\lambda}(a,c)\right|^q\right.$$

$$\left.+m_2\left(\int_0^1|q(h_2,r_2,\lambda)|(1-\lambda^{s_2})\mathrm{d}\lambda\right)\left|\frac{\partial^2 f}{\partial t\,\partial\lambda}\left(a,\frac{d}{m_2}\right)\right|^q\right\}$$

$$+\left(\int_0^1|p(h_1,r_1,t)|(1-t^{s_1})\mathrm{d}t\right)\left\{m_1\left(\int_0^1|q(h_2,r_2,\lambda)|\lambda^{s_2}\mathrm{d}\lambda\right)\left|\frac{\partial^2 f}{\partial t\,\partial\lambda}\left(\frac{b}{m_1},c\right)\right|^q\right.$$

$$\left.+m_1m_2\left(\int_0^1|q(h_2,r_2,\lambda)|(1-\lambda^{s_2})\mathrm{d}\lambda\right)\left|\frac{\partial^2 f}{\partial t\,\partial\lambda}\left(\frac{b}{m_1},\frac{d}{m_2}\right)\right|^q\right\}$$

$$=\mu(h_1,r_1,s_1)\left\{\mu(h_2,r_2,s_2)\left|\frac{\partial^2 f}{\partial t\,\partial\lambda}(a,c)\right|^q+m_2\nu(h_2,r_2,s_2)\left|\frac{\partial^2 f}{\partial t\,\partial\lambda}\left(a,\frac{d}{m_2}\right)\right|^q\right\}$$

$$+\nu(h_1,r_1,s_1)\left\{m_1\mu(h_2,r_2,s_2)\left|\frac{\partial^2 f}{\partial t\,\partial\lambda}\left(\frac{b}{m_1},c\right)\right|^q+m_1m_2\nu(h_2,r_2,s_2)\left|\frac{\partial^2 f}{\partial t\,\partial\lambda}\left(\frac{b}{m_1},\frac{d}{m_2}\right)\right|^q\right\}.$$

Combining the above equations and inequalities reveals the assertion (3.5). The proof is completed.

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