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Short communication

Completeness of the set $\{e^{ik\beta\cdot s}\}|_{\forall\beta\in S^2}$

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Abstract

Let S^2 be the unit sphere in $\mathbb{R}^3$, $k>0$ be a fixed constant, $s\in S$, and S is a smooth, closed, connected surface, the boundary of a bounded domain D in $\mathbb{R}^3$. It is proved that the set $\{e^{ik\beta\cdot s}\}|_{\forall\beta\in S^2}$ is total in $L^2(S)$ if and only if k^2 is not a Dirichlet eigenvalue of the Laplacian in D.

Keywords: *completeness; scattering theory.*

1. Introduction

Let $D\subset\mathbb{R}^3$ be a bounded domain with a connected closed C^2-smooth boundary S, $D':=\mathbb{R}^3\setminus D$ be the unbounded exterior domain and S^2 be the unit sphere in $\mathbb{R}^3$, $\beta\in S^2$, $s\in S$.
We are interested in the following problem:
Is the set $\{e^{ik\beta\cdot s}\}|_{\forall\beta\in S^2}$ total in $L^2(S)$?
A set $\{\phi(s,\beta)\}$ is total (complete) in $L^2(S)$ if the relation $\int_S f(s)\phi(s,\beta)ds=0$ for all $\beta\in S^2$ implies $f=0$, where $f\in L^2(S)$ is an arbitrary fixed function.
The above question is of interest by itself, but also it is of interest in scattering problems and in inverse problems, see [1]–[5].
Our result is:
Theorem 1. *The set $\{e^{ik\beta\cdot s}\}|_{\forall\beta\in S^2}$ is total in $L^2(S)$ if and only if k^2 is not a Dirichlet eigenvalue of the Laplacian in D.*

2. Proof of Theorem 1

Necessity. Let $f\in L^2(S)$ and

$$\int_S f(s)e^{ik\beta\cdot s}ds=0 \quad \forall\beta\in S^2, \tag{1}$$

and there is a $u\not\equiv 0$ such that

$$(\nabla^2+k^2)u=0 \quad in \quad D, \qquad u|_S=0. \tag{2}$$

Choose $f=u_N$, where N is the unit normal to S pointing out of D. Then, by Green's formula, equation (1) holds and $f\not\equiv 0$ by the uniqueness of the solution to the Cauchy problem for elliptic equation (2). Necessity is proved.
Sufficiency. Assume that $f\in L^2(S)$ is and arbitrary fixed function, $f\not\equiv 0$, and (1) holds. Let $h\in L^2(S^2)$ be arbitrary and

$$w(x):=\int_{S^2}h(\beta)e^{ik\beta\cdot x}d\beta. \tag{3}$$

Then

$$(\nabla^2+k^2)w=0 \quad in \quad \mathbb{R}^3. \tag{4}$$

If (1) holds, then

$$\int_S f(s)w(s)ds=0 \tag{5}$$

for all w of the form (3). Let us now apply the following Lemma:
Lemma 1. *The set $\{w|_S\}$ for all $h\in L^2(S^2)$ is the orthogonal complement in $L^2(S)$ to the linear span of the set $\{v_N\}$, where v solve equation (4) and $v|_S=0$.*
If k^2 is not a Dirichlet eigenvalue of the Laplacian in D, then Lemma 1 implies that the set $\{w|_S\}$ is total in $L^2(S)$, so (1) implies $f=0$. Sufficiency and Theorem 1 are proved. □
Lemma 1 is similar to Theorem 6 in [3].
Proof of Lemma 1. Let $w|_S:=\psi$. Choose an arbitrary $F\in C^2(D)$ such that $F|_S=\psi$. Define $G:=F-w$ in D. Then

$$(\nabla^2+k^2)G=(\nabla^2+k^2)F \quad in \quad D; \quad G|_S=0. \tag{6}$$

For (6) to hold it is necessary and sufficient that

$$0=\int_D(\nabla^2+k^2)Fvdx, \tag{7}$$

where v is an arbitrary function in the set of solutions of equation (2). Using Green's formula one reduces condition (7) to the following condition:

$$\int_S\psi v_N ds=0. \tag{8}$$

Therefore the set $\{\psi\}$ is the orthogonal complement in $L^2(S)$ of the linear span of the functions $\{v_N\}$. Lemma 1 is proved. □

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