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Short Communication

Existence of the solutions to convolution equations with distributional kernels

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Abstract

It is proved that a class of convolution integral equations of the Volterra type has a global solution, that is, solutions defined for all $t \geq 0$. Smoothness of the solution is studied.

Keywords: *Volterra equations; distributional kernels*

1. Introduction

Consider the equation:

$$u(t) = \int_0^t \frac{(t-s)^a}{\Gamma(a+1)} u(s)ds + f(t) := Vu + f, \tag{1}$$

where

$$t \geq 0; \quad a = const \neq -1, -2, ..., \tag{2}$$

and

$$Vu := V_a * u, \quad V_a := \frac{t_+^a}{\Gamma(a+1)}. \tag{3}$$

Here $\Gamma(z)$ is the Gamma function, and (3) is a convolution with the distribution V_a, see [1]. Thus, equation (1) is a Volterra equation with kernel that is not absolutely integrable for $a < -1$. There is a large literature on integral equations, [2], but the usual methods to study such equations are based on the assumption that the kernel of the operator V belongs to L^p with $p \geq 1$.
The goal of this paper is to develop a method to study (1) with a distributional kernel $\frac{t_+^a}{\Gamma(a+1)}$. The basic known result (see [1]) is the property of convolution

$$V_a * V_b = V_b * V_a = V_{a+b}. \tag{4}$$

Our result is formulated in Theorem 1.
Theorem 1. *Equation* (1) *with $a < -1$ is uniquely solvable. Its solution u exists for all $t \geq 0$. It belongs to the space of functions which is of the same smoothness as $V_{-a}f$.*
In the next section a proof is given.

2. Proof

Proof of Theorem 1. The idea of the proof is to apply V_{-a} to equation (1) and use the formula

$$V_a * V_{-a} = I, \tag{5}$$

where I is the identity operator whose kernel is the delta-function. Applying V_{-a} to (1) one gets

$$V_{-a}u = u + V_{-a}f, \tag{6}$$

or

$$u = -V_{-a}u + V_{-a}f. \tag{7}$$

Suppose that $a < -1$. Then $-a > 1$ and the operator V_{-a} is a convolution with a continuous kernel. Therefore equation (7) is a Volterra integral equation with a continuous kernel. Consequently, this equation has a unique solution u for all $t \geq 0$. This solution can be calculated by iterations. Equations (1) and (7) are equivalent because $V_a * V_{-a} = I$. Therefore, Theorem 1 is proved. □

References

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