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Short communication

Simple forms for coefficients in two families of ordinary differential equations

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Abstract

In the paper, by virtue of the Faá di Bruno formula, properties of the Bell polynomials of the second kind, and the inversion formula for the Stirling numbers of the first and second kinds, the author finds simple, meaningful, and significant forms for coefficients in two families of ordinary differential equations.

Keywords: *simple form; coefficient; ordinary differential equation; Faá di Bruno formula; Bell polynomial of the second kind; inversion formula; Stirling number*

1. Motivation and main results

In [2, Theorems 2.1] and [3, Theorem 2.1], it was established inductively and recursively that the differential equations

$$(n-1)!e^{-nF(t)}=(-1)^{n-1}\sum_{k=1}^{n}\lambda^{n-k}(1+\lambda t)^k a_k(n)F^{(k)}(t) \tag{1}$$

and

$$F^{(n)}(t)=\frac{(-1)^{n-1}(n-1)!}{(1+\lambda t)^n}\sum_{k=1}^{n}(k-1)!\lambda^{n-k}H_{n-1,k-1}e^{-kF(t)} \tag{2}$$

for $n\in\mathbb{N}$ have the same solution

$$F(t)=\ln\left[1+\frac{\ln(1+\lambda t)}{\lambda}\right], \tag{3}$$

where

$$a_n(n)=a_1(n)=1,\quad H_{n,0}=1,\quad H_{n,1}=H_n=\sum_{k=1}^{n}\frac{1}{k},$$

$$a_k(n)=\sum_{i_{k-1}=0}^{n-k}\sum_{i_{k-2}=0}^{n-k-i_{k-1}}\cdots\sum_{i_1=0}^{n-k-i_{k-1}-\cdots-i_2}k^{i_{k-1}}\cdots 2^{i_1},\quad 2\le k<n, \tag{4}$$

$$H_{n,j}=\sum_{k=j}^{n}\frac{H_{k-1,j-1}}{k},\quad 2\le j\le n. \tag{5}$$

Hereafter, the expressions in (4) and (5) were employed in the whole papers [2, 3].
In this paper, since

1. the original proofs of [2, Theorems 2.1] and [3, Theorem 2.1] are long and tedious,
2. the expressions in (4) and (5) are too complicated to be remembered, understood, and computed easily,

we will provide a nice and standard proof for [2, Theorems 2.1] and [3, Theorem 2.1] and, more importantly, discover simple, meaningful, and significant forms for the quantities $a_k(n)$ and $H_{n,j}$.
Our main results can be stated as the following theorem.

Theorem 1. *For $k\in\mathbb{N}$, the function $F(t)$ defined by (3) satisfies*

$$F^{(n)}(t)=\left(\frac{\lambda}{1+\lambda t}\right)^n\sum_{k=1}^{n}\frac{(-1)^{k-1}(k-1)!}{[\lambda+\ln(1+\lambda t)]^k}s(n,k) \tag{6}$$

and

$$\sum_{k=1}^{n}\lambda^{n-k}(1+\lambda t)^k S(n,k)F^{(k)}(t)=(-1)^{n-1}(n-1)!e^{-nF(t)}, \tag{7}$$

where $s(n,k)$ and $S(n,k)$ stand for the Stirling numbers of the first and second kinds.

2. Proof of Theorem 1

The famous Faà di Bruno formula reads that

$$\frac{\mathrm{d}^n}{\mathrm{d}t^n}f\circ h(t)=\sum_{k=1}^{n}f^{(k)}(h(t))\,\mathrm{B}_{n,k}\left(h'(t),h''(t),\dots,h^{(n-k+1)}(t)\right) \tag{8}$$

for $n\in\mathbb{N}$, where the Bell polynomials of the second kind $\mathrm{B}_{n,k}(x_1,x_2,\dots,x_{n-k+1})$ for $n\ge k\ge 0$ are defined [1, p. 134, Theorem A] and [1, p. 139, Theorem C] by

$$\mathrm{B}_{n,k}(x_1,x_2,\dots,x_{n-k+1})=\sum_{\substack{1\le i\le n,\ell_i\in\{0\}\cup\mathbb{N}\\ \sum_{i=1}^{n}i\ell_i=n\\ \sum_{i=1}^{n}\ell_i=k}}\frac{n!}{\prod_{i=1}^{n-k+1}\ell_i!}\prod_{i=1}^{n-k+1}\left(\frac{x_i}{i!}\right)^{\ell_i}.$$

Applying $u=h(t)=\frac{\ln(1+\lambda t)}{\lambda}$ and $f(u)=\ln(1+u)$ to (8) gives

$$F^{(n)}(t)=\sum_{k=1}^{n}\frac{\mathrm{d}^k\ln(1+u)}{\mathrm{d}u^k}\mathrm{B}_{n,k}\left(\frac{1}{1+\lambda t},-\frac{\lambda}{(1+\lambda t)^2},\dots,\frac{(-1)^{n-k}\lambda^{n-k}(n-k)!}{(1+\lambda t)^{n-k+1}}\right).$$

In light of two identities

$$\mathrm{B}_{n,k}\left(abx_1,ab^2x_2,\dots,ab^{n-k+1}x_{n-k+1}\right)=a^kb^n\mathrm{B}_{n,k}(x_1,x_2,\dots,x_{n-k+1})$$

and

$$\mathrm{B}_{n,k}(0!,1!,2!,\dots,(n-k)!)=(-1)^{n-k}s(n,k)$$

in [1, p. 135], we acquire

$$F^{(n)}(t)=\sum_{k=1}^{n}\frac{(-1)^{k-1}(k-1)!}{(1+u)^k}\frac{(-\lambda)^{n-k}}{(1+\lambda t)^n}\mathrm{B}_{n,k}(0!,1!,\dots,(n-k)!)=\frac{\lambda^n}{(1+\lambda t)^n}\sum_{k=1}^{n}\frac{(-1)^{k-1}(k-1)!}{[\lambda+\ln(1+\lambda t)]^k}s(n,k).$$

The formula (6) is thus proved.
The inversion theorem [31, p. 171, Theorem 12.1] reads that

$$a_n=\sum_{\alpha=0}^{n}S(n,\alpha)b_\alpha \quad \text{if and only if} \quad b_n=\sum_{k=0}^{n}s(n,k)a_k. \tag{9}$$

Combining (9) with (6) arrives at

$$\frac{(-1)^{n-1}(n-1)!}{[\lambda+\ln(1+\lambda t)]^n}=\sum_{k=1}^{n}S(n,k)\frac{(1+\lambda t)^k}{\lambda^k}F^{(k)}(t)$$

which can be rewritten as (7). The proof of Theorem 1 is complete.

3. Remarks

Finally, we list several remarks on our main results and closely related things.

Remark 1. Comparing (1) and (2) with (7) and (6) reveals that

$$a_k(n)=S(n,k)$$

and

$$H_{n-1,k-1}=(-1)^{n-k}\frac{s(n,k)}{(n-1)!} \tag{10}$$

for $n\ge k\ge 1$. These two expressions are simpler, more meaningful, and more significant than the forms in (4) and (5). The expression (10) was also found in [18, 20].

Remark 2. We note that [3, Theorem 2.1] has also been discussed in [27, Thorem 2].

Remark 3. The motivations in the papers [4, 5, 6, 7, 9, 10, 12, 11, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 32] are same as the one in this paper.

Remark 4. This paper is a slightly modified version of the preprint [8].

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