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Research paper

Analytical theory in terms of J_2, J_3, J_4 with eccentric anomaly for short-term orbit predictions using uniformly regular KS canonical elements

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Abstract

A new non-singular analytical theory with respect to the Earth's zonal harmonic terms J_2, J_3, J_4 has been developed for short-periodic motion, by analytically integrating the uniformly regular KS canonical equations of motion using generalized eccentric anomaly 'E' as the independent variable. Only one of the eight equations need to be integrated analytically to generate the state vector, due to the symmetry in the equations of motion, and the computation for the other equations is done by changing the initial conditions. King-Hele's expression for radial distance 'r' with J_2 is also considered in generating the solution. The results obtained from the analytical expressions in a single step during half a revolution match quite well with numerically integrated values. Numerical results also indicate that the solution is reasonably accurate for a wide range of orbital elements during half a revolution and is an improvement over Sharon et al. [17] theory, which is generated in terms of KS regular elements. It can be used for studying the short-term relative motion of two or more space objects and in collision avoidance studies of space objects. It can be also useful for onboard computation in the navigation and guidance packages.

Keywords: *Hamilton's Equations of Motion; Uniformly Regular KS Canonical Elements; Earth's Oblateness; Short-Term Orbit Predictions; Analytical Integration.*

1. Introduction

In the artificial satellite theory, the motion of a satellite under the effect of Earth's oblateness, namely the second zonal harmonic J_2 in the gravitational potential field is known as the main problem. Any Earth satellite mission requires precise orbit computation under the influence of this dominating perturbation. The non-integrability dynamics of the J_2 problem [2] allows the avenue for analytical theories to be developed. In the past, several authors had treated this problem to obtain closed form solutions using different methods. Several analytical theories for the motion of Earth's satellite under the effect of Earth's first few zonal harmonic terms are available in the literature. Some of the notable are by [10], [12], [6], [3], [1], [11] and [7].

The KS transformation regularizes the non-linear equations of motion and converts into linear differential equations of a harmonic oscillator. KS formulation was used by [5] and [9] for short-term orbit predictions with J_2 effect.

The KS uniform regular canonical equations of motion [19] are a particular canonical form where all the ten elements are constant for unperturbed two-body problem and are applicable to elliptic, parabolic and hyperbolic orbital motion. In [13] these equations were numerically integrated to obtain accurate orbits under the effect of Earth's oblateness with zonal harmonic terms up to J_{36}. Analytical theory in terms of KS elements with J_2 [14] and [16], and with J_3 and J_4 [15] was developed for short-term orbit predictions. [8] analytically integrated the uniformly regular KS canonical elements with Earth's zonal harmonics J_2, J_3 and J_4. The independent variable, fictitious time 's' given by $dt/ds = r$ with t and r being the physical time and radial distance, respectively, and used for analytical integration, resulted in complicated integrals. Because of the complexity of the integrals in evaluation for practical problems, the utility of the analytical solution was limited for operational purposes.

{18] developed a new non-singular analytical solution with J_2 in close form in eccentricity 'e' for short-term orbit predictions by analytically integrating the uniformly regular KS canonical equations of motion, using the generalized eccentric anomaly 'E' as the independent variable. The integrals are found to be much simpler than obtained in [8].

In this paper, the analytical solution of [18] is improved by using King-Hele's expression [10] for radial distance r as function of J_2. Further, new non-singular analytical solutions with J_3 and J_4 in close form in eccentricity and inclination for short-term orbit predictions by analytically integrating the uniformly regular KS canonical equations of motion, using the generalized eccentric anomaly 'E' as the independent variable are developed. Numerical study has been carried out for a wide range of orbital parameters. The theory is found to provide reasonably accurate results over half a revolution.

The solutions can have number of applications. It can be used for studying the short-term relative motion of two or more space objects and in collision avoidance studies of space objects and generation of mean orbital elements. It can be also useful for onboard computation in the navigation and guidance packages, where the modeling of J_2 effect becomes necessary.

2. Equations of motion

The K-S uniformly regular canonical equations of motion in terms of fictitious time s are [13], [18]

$$\frac{d\alpha_i}{ds} = -\frac{\partial \overline{H}}{\partial \beta_i}, \frac{d\beta_i}{ds} = \frac{\partial \overline{H}}{\partial \alpha_i}, \text{for } i = 0,1,2,3,4 \tag{1}$$

The relation between s and E is given by

$$E = 2\sqrt{\alpha_0}s \tag{2}$$

Equation (1) in terms of E can be written as

$$\frac{d\beta_i}{dE} = \frac{\partial \overline{H}}{\partial \alpha_i}\left(\frac{ds}{dE}\right), \frac{d\alpha_i}{dE} = -\frac{\partial \overline{H}}{\partial \beta_i}\left(\frac{ds}{dE}\right), \tag{3}$$

Where

$$\frac{ds}{dE} = \frac{1}{2\sqrt{\alpha_0}}.$$

$$\overline{H} = \frac{1}{4}\left\{\sum_{k=1}^{4} u_k^2(\alpha_i\beta_i)\right\}V(\alpha_i\beta_i) - \frac{K^2}{4} \tag{4}$$

When the perturbation due to Earth's oblateness J_2 is considered:

$$\overline{H} = \frac{1}{4}(rV - K^2) \tag{5}$$

Perturbing potential,

$$V(\alpha_i\beta_i) = \frac{K^2}{r}\sum_{n=2}^{\infty} J_n\left(\frac{R}{r}\right)^n P_n(\cos\upsilon),$$

Where

$$\cos\upsilon = \frac{x_3}{r}.$$

In particular

$$V_2 = \frac{K^2R^2J_2}{2\,r^3}\left[-1 + 3\cos^2\upsilon\right],$$

$$V_3 = \frac{K^2R^3J_3}{2\,r^4}\left[-3\cos\upsilon + 5\cos^3\upsilon\right],$$

$$V_4 = \frac{K^2R^4J_4}{8\,r^5}\left[3 - 30\cos^2\upsilon + 35\cos^4\upsilon\right],$$

$$h = 2\alpha_0 = \left(\frac{K^2}{r}\right) - \left(\frac{\sqrt{(\dot{x}^2+\dot{y}^2+\dot{z}^2)}}{2}\right) - V,$$

$$\left(u, \frac{\partial V}{\partial u}\right) = -2(n+1)V,$$

With

$$u_i = \left(\frac{\beta_i}{\sqrt{\alpha_0}}\right)\sin\left(\frac{E}{2}\right) - \alpha_i\cos\left(\frac{E}{2}\right), w_i = \left(\alpha_i\sqrt{\alpha_0}\right)\sin\left(\frac{E}{2}\right) + \beta_i\cos\left(\frac{E}{2}\right),$$

$$\alpha_i = \left(\frac{w_i}{\sqrt{\alpha_0}}\right)\sin\left(\frac{E}{2}\right) - u_i\cos\left(\frac{E}{2}\right),$$

$$\beta_i = \left(u_i\sqrt{\alpha_0}\right)\sin\left(\frac{E}{2}\right) + w_i\cos\left(\frac{E}{2}\right),$$

$$\frac{dh}{ds} = 0\,, r = \frac{dt}{ds} = {u_1}^2 + {u_2}^2 + {u_3}^2 + {u_4}^2,$$

$$(x, y, z) = L(u)\,u, (\dot{x}, \dot{y}, \dot{z}) = 2L(u)w/r$$

$$L(u) = \begin{pmatrix} u_1 & -u_2 & -u_3 & u_4 \\ u_2 & u_1 & -u_4 & -u_3 \\ u_3 & u_4 & u_1 & u_2 \\ u_4 & -u_3 & u_2 & -u_1 \end{pmatrix},$$

$$x = (x_1, x_2, x_3) = L(u)u,$$

$$r = \sqrt{(x_1^2 + x_2^2 + x_3^2)} = u_1^2 + u_2^2 + u_3^2 + u_4^2,$$

Where h, K^2, R, E, r, J_n are total energy, gravitational parameter, Earth's equatorial radius, eccentric anomaly, radial distance and nth zonal harmonic term of Earth, respectively.

2.1. Initial conditions

$$u_1^2 + u_4^2 = (r + x_1)/2,$$

$$u_2 = (x_2u_1 + x_3u_4)/(r + x_1),$$

$$u_3 = (x_3u_1 - x_2u_4)/(r + x_1).$$

For $x_1 \geq 0$,

$$u_2^2 + u_3^2 = (r - x_1)/2,$$

$$u_1 = (x_2u_2 + x_3u_3)/(r - x_1),$$

$$u_4 = (x_3u_2 - x_2u_3)/(r - x_1).$$

Furthermore,

$$w_1 = (u_1\dot{x}_1 + u_2\dot{x}_2 + u_3\dot{x}_3)/2,$$

$$w_2 = (-u_2\dot{x}_1 + u_1\dot{x}_2 + u_4\dot{x}_3)/2,$$

$$w_3 = (-u_3\dot{x}_1 - u_4\dot{x}_2 + u_1\dot{x}_3)/2,$$

$$w_4 = (u_4\dot{x}_1 - u_3\dot{x}_2 + u_2\dot{x}_3)/2.$$

3. Analytical integration

The right-hand side of the equation for J_2 can be written as

$$\frac{\partial\bar{H}}{\partial\alpha_i}\frac{ds}{dE} = \frac{K^2R^2J_2}{8\sqrt{\alpha_o}}\left(\frac{1}{r^3}\frac{\partial r}{\partial\alpha_i} + \frac{3x_3}{r^4}\frac{\partial x_3}{\partial\alpha_i} - \frac{6x_3^2}{r^5}\frac{\partial r}{\partial\alpha_i}\right), \tag{6}$$

$$\frac{\partial\bar{H}}{\partial\beta_i}\frac{ds}{dE} = \frac{K^2R^2J_2}{8\sqrt{\alpha_o}}\left(\frac{1}{r^3}\frac{\partial r}{\partial\beta_i} + \frac{3x_3}{r^4}\frac{\partial x_3}{\partial\beta_i} - \frac{6x_3^2}{r^5}\frac{\partial r}{\partial\beta_i}\right) \tag{7}$$

For J_3:

$$\frac{\partial\bar{H}}{\partial\alpha_i}\frac{ds}{dE} = \frac{3J_3K^2R^3}{16\sqrt{\alpha_o}}\left[-\frac{1}{r^4}\frac{\partial x_3}{\partial\alpha_i} + \frac{4x_3}{r^5}\frac{\partial r}{\partial\alpha_i} + \frac{5x_3^2}{r^6}\frac{\partial x_3}{\partial\alpha_i} - \frac{10x_3^3}{r^7}\frac{\partial r}{\partial\alpha_i}\right], \tag{8}$$

$$\frac{\partial\bar{H}}{\partial\beta_i}\frac{ds}{dE} = \frac{3J_3K^2R^3}{16\sqrt{\alpha_o}}\left[-\frac{1}{r^4}\frac{\partial x_3}{\partial\beta_i} + \frac{4x_3}{r^5}\frac{\partial r}{\partial\beta_i} + \frac{5x_3^2}{r^6}\frac{\partial x_3}{\partial\beta_i} - \frac{10x_3^3}{r^7}\frac{\partial r}{\partial\beta_i}\right]. \tag{9}$$

For J_4 :

$$\frac{\partial\bar{H}}{\partial\alpha_i}\frac{ds}{dE} = \frac{J_4K^2R^4}{16\sqrt{\alpha_o}}\left[-\frac{3}{r^5}\frac{\partial r}{\partial\alpha_i} - \frac{15x_3}{r^6}\frac{\partial x_3}{\partial\alpha_i} + \frac{45x_3^2}{r^7}\frac{\partial r}{\partial\alpha_i} + \frac{35x_3^3}{r^8}\frac{\partial x_3}{\partial\alpha_i} - \frac{70x_3^4}{r^9}\frac{\partial r}{\partial\alpha_i}\right] \tag{10}$$

$$\frac{\partial\bar{H}}{\partial\beta_i}\frac{ds}{dE} = \frac{J_4K^2R^4}{16\sqrt{\alpha_o}}\left[-\frac{3}{r^5}\frac{\partial r}{\partial\beta_i} - \frac{15x_3}{r^6}\frac{\partial x_3}{\partial\beta_i} + \frac{45x_3^2}{r^7}\frac{\partial r}{\partial\beta_i} + \frac{35x_3^3}{r^8}\frac{\partial x_3}{\partial\beta_i} - \frac{70x_3^4}{r^9}\frac{\partial r}{\partial\beta_i}\right] \tag{11}$$

Where

$$x_3 = a_0 + a_1\cos E + a_2\sin E,$$

$$x_3^2 = b_0 + b_1\cos E + b_2\cos^2 E + b_3\sin E + b_4\sin E\cos E,$$

$$x_3^3 = c_0 + c_1\cos E + c_2\cos^2 E + c_3\cos^3 E + c_4\sin E + c_5\sin E\cos E + c_6\sin E\cos^2 E$$

$$x_3^4 = d_0 + d_1\cos E + d_2\cos^2 E + d_3\cos^3 E + d_4\cos^4 E + d_5\sin E + d_6\sin E\cos E + d_7\sin E\cos^2 E + d_8\sin E\cos^3 E$$

$$a_0 = \alpha_1\alpha_3 + \alpha_2\alpha_4 + \frac{1}{\alpha_0}(\beta_1\beta_3 + \beta_2\beta_4)$$

$$a_1 = \alpha_1\alpha_3 + \alpha_2\alpha_4 - \frac{1}{\alpha_0}(\beta_1\beta_3 + \beta_2\beta_4)$$

$$a_2 = \frac{-1}{\sqrt{\alpha_0}}(\alpha_1\beta_3 + \beta_1\alpha_3 + \alpha_2\beta_4 + \beta_2\alpha_4)$$

$$b_0 = {a_0}^2 + {a_2}^2$$

$$b_1 = 2a_0a_1$$

$$b_2 = a_1^2 - a_2^2$$

$$b_3 = 2a_0a_2$$

$$b_4 = 2a_1a_2$$

$$c_0 = a_0b_0 + a_2b_3$$

$$c_1 = a_0b_1 + a_1b_0 + a_2b_4$$

$$c_2 = a_0b_2 + a_1b_1 - a_2b_3$$

$$c_3 = a_1b_2 - a_2b_4$$

$$c_4 = a_0b_3 + a_2b_0$$

$$c_5 = a_0b_4 + a_1b_3 + a_2b_1$$

$$c_6 = a_1b_4 + a_2b_2$$

$$d_0 = {b_0}^2 + {b_3}^2$$

$$d_1 = 2(b_0b_1 + b_3b_4)$$

$$d_2 = {b_1}^2 - {b_3}^2 + {b_4}^2 + 2b_0b_2$$

$$d_3 = 2(b_1b_2 - b_3b_4)$$

$$d_4 = {b_2}^2 - {b_4}^2$$

$$d_5 = 2b_0b_3$$

$$d_6 = 2(b_0b_4 + b_1b_3)$$

$$d_7 = 2(b_1b_4 + b_2b_3)$$

$$d_8 = 2b_2b_4$$

Substituting the values of x_3, x_3^2 ,$x_3^3$and x_3^4 into the equations (6), (8) and (10), we get

$$\frac{d\alpha_i}{dE} = \frac{J_2K^2R^2}{8\sqrt{\alpha_0}}\Big[\frac{1}{r^3}\Big\{q_0^{(I)} + q_1^{(I)}\cos E + q_2^{(I)}\sin E\Big\} + \frac{3}{r^4}\Big\{g_0^{(k)} + g_1^{(k)}\cos E + g_2^{(k)}\cos^2 E + g_3^{(k)}\sin E + g_4^{(k)}\sin E\cos E\Big\} - \frac{6}{r^5}\Big\{f_0^{(i)} + f_1^{(i)}\cos E + f_2^{(i)}\cos^2 E + f_3^{(i)}\cos^3 E + f_4^{(i)}\sin E + f_5^{(i)}\sin E\cos E + f_6^{(i)}\sin E\cos^2 E\Big\}\Big] \tag{12}$$

$$\frac{d\alpha_i}{dE} = \frac{3J_3K^2R^3}{16\sqrt{\alpha_0}}\Big[-\frac{1}{r^4}\Big\{q_0^{(k)} + q_1^{(k)}\cos E + q_2^{(k)}\sin E\Big\} + \frac{4}{r^5}\Big\{g_0^{(i)} + g_1^{(i)}\cos E + g_2^{(i)}\cos^2 E + g_3^{(i)}\sin E + g_4^{(i)}\sin E\cos E\Big\} + \frac{5}{r^6}\Big\{f_0^{(k)} + f_1^{(k)}\cos E + f_2^{(k)}\cos^2 E + f_3^{(k)}\cos^3 E + f_4^{(k)}\sin E + f_5^{(k)}\sin E\cos E + f_6^{(k)}\sin E\cos^2 E\Big\} - \frac{10}{r^7}\Big\{h_0^{(i)} + h_1^{(i)}\cos E + h_2^{(i)}\cos^2 E + h_3^{(i)}\cos^3 E + h_4^{(i)}\cos^4 E + h_5^{(i)}\sin E + h_6^{(i)}\sin E\cos E + h_7^{(i)}\sin E\cos^2 E + h_8^{(i)}\sin E\cos^3 E\Big\}\Big] \tag{13}$$

$$\frac{d\alpha_i}{dE} = \frac{J_4K^2R^4}{16\sqrt{\alpha_0}}\Big[-\frac{3}{r^5}\Big\{q_0^{(i)} + q_1^{(i)}\cos E + q_2^{(i)}\sin E\Big\} - \frac{15}{r^6}\Big\{g_0^{(k)} + g_1^{(k)}\cos E + g_2^{(k)}\cos^2 E + g_3^{(k)}\sin E + g_4^{(k)}\sin E\cos E\Big\} + \frac{45}{r^7}\Big\{f_0^{(i)} + f_1^{(i)}\cos E + f_2^{(i)}\cos^2 E + f_3^{(i)}\cos^3 E + f_4^{(i)}\sin E + f_5^{(i)}\sin E\cos E + f_6^{(i)}\sin E\cos^2 E\Big\} + \frac{35}{r^8}\Big\{h_0^{(k)} + h_1^{(k)}\cos E + h_2^{(k)}\cos^2 E + h_3^{(k)}\cos^3 E +$$

$h_4^{(k)}\cos^4 E + h_5^{(k)}\sin E + h_6^{(k)}\sin E\cos E + h_7^{(k)}\sin E\cos^2 E + h_8^{(k)}\sin E\cos^3 E\Big\} - \frac{70}{r^9}\Big\{l_0^{(i)} + l_1^{(i)}\cos E + l_2^{(i)}\cos^2 E + l_3^{(i)}\cos^3 E +$
$l_4^{(i)}\cos^4 E + l_5^{(i)}\cos^5 E + l_6^{(i)}\sin E + l_7^{(i)}\sin E\cos E + l_8^{(i)}\sin E\cos^2 E + l_9^{(i)}\sin E\cos^3 E + l_{10}^{(i)}\sin E\cos^4 E\Big\}\Big],$ (14)

Where in equations (6), (8) and (10), we have

$$q_0^{(i)} = \frac{\beta_i}{\alpha_0}, q_1^{(i)} = \frac{-\beta_i}{\alpha_0}, q_2^{(i)} = \frac{-\alpha_i}{\sqrt{\alpha_0}},$$

And in equations (7), (9), (11), we have

$$q_0^{(i)} = \alpha_i, q_1^{(i)} = \alpha_i, q_2^{(i)} = \frac{-\beta_i}{\sqrt{\alpha_0}}$$

Also, k = i+2

$$g_0^{(i)} = a_0 q_0^{(i)} + a_2 q_2^{(i)},$$

$$g_1^{(i)} = a_1 q_0^{(i)} + a_0 q_1^{(i)},$$

$$g_2^{(i)} = a_1 q_1^{(i)} - a_2 q_2^{(i)},$$

$$g_3^{(i)} = a_2 q_0^{(i)} + a_0 q_2^{(i)},$$

$$g_4^{(i)} = a_1 q_2^{(i)} + a_2 q_1^{(i)},$$

$$f_0^{(i)} = b_0 q_0^{(i)} + b_3 q_2^{(i)},$$

$$f_1^{(i)} = b_1 q_0^{(i)} + b_0 q_1^{(i)} + b_4 q_2^{(i)},$$

$$f_2^{(i)} = b_2 q_0^{(i)} + b_1 q_1^{(i)} - b_3 q_2^{(i)},$$

$$f_3^{(i)} = b_2 q_1^{(i)} - b_4 q_2^{(i)},$$

$$f_4^{(i)} = b_3 q_0^{(i)} + b_0 q_2^{(i)},$$

$$f_5^{(i)} = b_4 q_0^{(i)} + b_3 q_1^{(i)} + b_1 q_2^{(i)},$$

$$f_6^{(i)} = b_4 q_1^{(i)} + b_2 q_2^{(i)},$$

$$h_0^{(i)} = c_0 q_0^{(i)} + c_4 q_2^{(i)}$$

$$h_1^{(i)} = c_1 q_0^{(i)} + c_0 q_1^{(i)} + c_5 q_2^{(i)}$$

$$h_2^{(i)} = c_2 q_0^{(i)} + c_1 q_1^{(i)} + (c_6 - c_4) q_2^{(i)}$$

$$h_3^{(i)} = c_3 q_0^{(i)} + c_2 q_1^{(i)} - c_5 q_2^{(i)}$$

$$h_4^{(i)} = c_3 q_1^{(i)} - c_6 q_2^{(i)}$$

$$h_5^{(i)} = c_4 q_0^{(i)} + c_0 q_2^{(i)}$$

$$h_6^{(i)} = c_5 q_0^{(i)} + c_4 q_1^{(i)} + c_1 q_2^{(i)}$$

$$h_7^{(i)} = c_6 q_0^{(i)} + c_5 q_1^{(i)} + c_2 q_2^{(i)}$$

$$h_8^{(i)} = c_6 q_1^{(i)} + c_3 q_2^{(i)}$$

$$l_0^{(i)} = d_0 q_0^{(i)} + d_5 q_2^{i(i)}$$

$$l_1^{(i)} = d_1 q_0^{(i)} + d_0 q_1^{(i)} + d_6 q_2^{(i)}$$

$$l_2^{(i)} = d_2 q_0^{(i)} + d_1 q_1^{(i)} - d_5 q_2^{(i)} + d_7 q_2^{(i)}$$

$$l_3^{(i)} = d_3 q_0^{(i)} + d_2 q_1^{(i)} - d_6 q_2^{(i)} + d_8 q_2^{(i)}$$

$$l_4^{(i)} = d_4 q_0^{(i)} + d_3 q_1^{(i)} - d_7 q_2^{(i)}$$

$$l_5^{(i)} = d_4 q_1^{(i)} - d_8 q_2^{(i)}$$

$$l_6^{(i)} = d_0 q_2^{(i)} + d_5 q_0^{(i)}$$

$$l_7^{(i)} = d_6 q_0^{(i)} + d_5 q_1^{(i)} + d_1 q_2^{(i)}$$

$$l_8^{(i)} = d_7 q_0^{(i)} + d_6 q_1^{(i)} + d_2 q_2^{(i)}$$

$$l_9^{(i)} = d_8 q_0^{(i)} + d_7 q_1^{(i)} + d_3 q_2^{(i)}$$

$$l_{10}^{(i)} = d_8 q_1^{(i)} + d_4 q_2^{(i)}$$

On substituting $r = a(1 - e\cos E)$ into equations (12), (13), (14) and integrating it analytically, we get

$$\Delta\alpha_i = \frac{K^2 R^2 J_2}{8a^3\sqrt{\alpha_0}}\Big[q_0^{(i)}\Lambda_3^{00} + q_1^{(i)}\Lambda_3^{10} + q_2^{(i)}\Lambda_3^{01} + \frac{3}{a}\Big\{g_0^{(k)}\Lambda_4^{00} + g_1^{(k)}\Lambda_4^{10} + g_2^{(k)}\Lambda_4^{20} + g_3^{(k)}\Lambda_4^{01} + g_4^{(k)}\Lambda_4^{11}\Big\} - \frac{6}{a^2}\Big\{f_0^{(i)}\Lambda_5^{00} + f_1^{(i)}\Lambda_5^{10} + f_2^{(i)}\Lambda_5^{20} + f_3^{(i)}\Lambda_5^{30} + f_4^{(i)}\Lambda_5^{01} + f_5^{(i)}\Lambda_5^{11} + f_6^{(i)}\Lambda_5^{21}\Big\}\Big] \tag{15}$$

$$\Delta\alpha_i = \frac{3J_3 k^2 R^3}{16\sqrt{\alpha_0}a^4}\Big[-\Big\{q_0^{(k)}\Lambda_4^{00} + q_1^{(k)}\Lambda_4^{10} + q_2^{(k)}\Lambda_4^{01}\Big\} + \frac{4}{a}\Big\{g_0^{(i)}\Lambda_5^{00} + g_1^{(i)}\Lambda_5^{10} + g_2^{(i)}\Lambda_5^{20} + g_3^{(i)}\Lambda_5^{01} + g_4^{(i)}\Lambda_5^{11}\Big\} + \frac{5}{a^2}\Big\{f_0^{(k)}\Lambda_6^{00} + f_1^{(k)}\Lambda_6^{10} + f_2^{(k)}\Lambda_6^{20} + f_3^{(k)}\Lambda_6^{30} + f_4^{(k)}\Lambda_6^{01} + f_5^{(k)}\Lambda_6^{11} + f_6^{(k)}\Lambda_6^{21}\Big\} - \frac{10}{a^3}\Big\{h_0^{(i)}\Lambda_7^{00} + h_1^{(i)}\Lambda_7^{10} + h_2^{(i)}\Lambda_7^{20} + h_3^{(i)}\Lambda_7^{30} + h_4^{(i)}\Lambda_7^{40} + h_5^{(i)}\Lambda_7^{01} + h_6^{(i)}\Lambda_7^{11} + h_7^{(i)}\Lambda_7^{21} + h_8^{(i)}\Lambda_7^{31}\Big\}\Big], \tag{16}$$

$$\Delta\alpha_i = \frac{J_4 k^2 R^4}{16\sqrt{\alpha_0}a^5}\Big[-3\Big\{q_0^{(k)}\Lambda_5^{00} + q_1^{(k)}\Lambda_5^{10} + q_2^{(k)}\Lambda_5^{01}\Big\} - \frac{15}{a}\Big\{g_0^{(i)}\Lambda_6^{00} + g_1^{(i)}\Lambda_6^{10} + g_2^{(i)}\Lambda_6^{20} + g_3^{(i)}\Lambda_6^{01} + g_4^{(i)}\Lambda_6^{11}\Big\} + \frac{45}{a^2}\Big\{f_0^{(k)}\Lambda_7^{00} + f_1^{(k)}\Lambda_7^{10} + f_2^{(k)}\Lambda_7^{20} + f_3^{(k)}\Lambda_7^{30} + f_4^{(k)}\Lambda_7^{01} + f_5^{(k)}\Lambda_7^{11} + f_6^{(k)}\Lambda_7^{21}\Big\} + \frac{35}{a^3}\Big\{h_0^{(i)}\Lambda_8^{00} + h_1^{(i)}\Lambda_8^{10} + h_2^{(i)}\Lambda_8^{20} + h_3^{(i)}\Lambda_8^{30} + h_4^{(i)}\Lambda_8^{40} + h_5^{(i)}\Lambda_8^{01} + h_6^{(i)}\Lambda_8^{11} + h_7^{(i)}\Lambda_8^{21} + h_8^{(i)}\Lambda_8^{31}\Big\} - \frac{70}{a^4}\Big\{l_0^{(i)}\Lambda_9^{00} + l_1^{(i)}\Lambda_9^{10} + l_2^{(i)}\Lambda_9^{20} + l_3^{(i)}\Lambda_9^{30} + l_4^{(i)}\Lambda_9^{40} + l_5^{(i)}\Lambda_9^{50} + l_6^{(i)}\Lambda_9^{01} + l_7^{(i)}\Lambda_9^{11} + l_8^{(i)}\Lambda_9^{21} + l_9^{(i)}\Lambda_9^{31} + l_{10}^{(i)}\Lambda_9^{41}\Big\}\Big]. \tag{17}$$

4. Expression for radial distance 'r' in terms of J_2

From King-Hele [10]

$$\frac{1}{r} = L\left[1 + e\cos\theta + \frac{3}{2}J_2 V^*\right]. \tag{18}$$

$$V^* = L^2 R^2\left[\frac{2 - 5\sin^2 i}{2} - \frac{\sin^2 i}{6}\cos(2\theta + 2\omega)\right]. \tag{19}$$

V^* is the change in the radial distance. On simplification, we get

$$V^* = \frac{L^2 R^2}{2}\left[2 - \frac{16\sin^2 i}{3} + \frac{2}{3}\left(\frac{z}{r}\right)^2\right], \tag{20}$$

where

$$\left(\frac{z}{r}\right) = \sin i \sin(\theta + \omega),$$

$$L = \frac{1}{a(1 - e^2)},$$

$$(1 + e\cos\theta) = \frac{1 - e^2}{1 - e\cos E}.$$

To avoid the error in the position of the satellite, the value of V^* at $\theta = 0°$ is subtracted from the value of V^* obtained. Similarly, to avoid the error in the position of the satellite, the value of w^*, at $\theta = 0°$ is subtracted from the value of w^*. Therefore, the equation

$$V^* = -\frac{L^2 R^2}{3}\left[\sin^2 i \sin^2\omega - \left(\frac{z}{r}\right)^2\right]. \tag{21}$$

Substituting the value of V^*, we get

$$\frac{1}{r^3} = L^3(1+e\cos\theta)^3\left[1-\frac{3J_2L^2R^2}{2(1-e^2)}(1-e\cos E)\left\{sin^2\, i\, sin^2\,\omega - \left(\frac{z}{r}\right)^2\right\}\right], \tag{22}$$

$$\frac{1}{r^4} = L^4(1+e\cos\theta)^4\left[1-\frac{2J_2L^2R^2}{(1-e^2)}(1-e\cos E)\left\{sin^2\, i\, sin^2\,\omega - \left(\frac{z}{r}\right)^2\right\}\right], \tag{23}$$

$$\frac{1}{r^5} = L^5(1+e\cos\theta)^5\left[1-\frac{5J_2L^2R^2}{2(1-e^2)}(1-e\cos E)\left\{sin^2\, i\, sin^2\,\omega - \left(\frac{z}{r}\right)^2\right\}\right]. \tag{24}$$

Substituting equations (22), (23) and (24) in J_2 expression in equation (15), we get

$$\Delta\alpha_i = \frac{K^2J_2R^2}{8\sqrt{\alpha_0}a^3}\Bigg[q_{0_i}\Lambda_3^{00}+q_{1_i}\Lambda_3^{10}+q_{2_i}\Lambda_3^{01}-\frac{3J_2R^2}{2(1-e^2)^3a^2}\left\{\frac{A(q_{0_i}\Lambda_2^{00}+q_{1_i}\Lambda_2^{10}+q_{2_i}\Lambda_2^{01})}{1}-\frac{(f_{0_i}\Lambda_4^{00}+f_{1_i}\Lambda_4^{10}+f_{2_i}\Lambda_4^{20}+f_{3_i}\Lambda_4^{30}+f_{4_i}\Lambda_4^{01}+f_{5_i}\Lambda_4^{11}+f_{6_i}\Lambda_4^{21})}{a^2}\right\}+$$
$$\frac{3}{a}\Bigg\{\left\{g_{0_k}\Lambda_4^{00}+g_{1_k}\Lambda_4^{10}+g_{2_k}\Lambda_4^{20}+g_{3_k}\Lambda_4^{01}+g_{4_k}\Lambda_4^{11}\right\}+\frac{2J_2R^2}{(1-e^2)^3a^2}\Bigg(A\left\{g_{0_k}\Lambda_3^{00}+g_{1_k}\Lambda_3^{10}+g_{2_k}\Lambda_3^{20}+g_{3_k}\Lambda_3^{01}+g_{4_k}\Lambda_3^{11}\right\}-$$
$$\frac{\left(h_{0_i}\Lambda_5^{00}+h_{1_i}\Lambda_5^{10}+h_{2_i}\Lambda_5^{20}+h_{3_i}\Lambda_5^{30}+h_4\Lambda_5^{40}+h_{5_i}\Lambda_5^{01}+h_{6_i}\Lambda_5^{11}+h_{7_i}\Lambda_5^{21}+h_{8_i}\Lambda_5^{31}\right)}{a^2}\Bigg)\Bigg\}-\frac{6}{a^2}\Bigg\{\left\{f_{0_i}\Lambda_5^{00}+f_{1_i}\Lambda_5^{10}+f_{2_i}\Lambda_5^{20}+f_{3_i}\Lambda_5^{30}+f_{4_i}\Lambda_5^{01}+f_{5_i}\Lambda_5^{11}+f_{6_i}\Lambda_5^{21}\right\}+$$
$$\frac{5J_2R^2}{2(1-e^2)^3a^2}\Bigg(A\left\{f_{0_i}\Lambda_4^{00}+f_{1_i}\Lambda_4^{10}+f_{2_i}\Lambda_4^{20}+f_{3_i}\Lambda_4^{30}+f_{4_i}\Lambda_4^{01}+f_{5_i}\Lambda_4^{11}+f_{6_i}\Lambda_4^{21}\right\}-\frac{\left(m_{0_i}\Lambda_6^{00}+m_{1_i}\Lambda_6^{10}+m_{2_i}\Lambda_6^{20}+m_{3_i}\Lambda_6^{30}+m_{4_i}\Lambda_6^{40}+m_{5_i}\Lambda_6^{50}\right)}{a^2}-$$
$$\frac{\left(m_{6_i}\Lambda_6^{01}+m_{7_i}\Lambda_6^{11}+m_{8_i}\Lambda_6^{21}+m_{9_i}\Lambda_6^{31}+m_{10_i}\Lambda_6^{41}\right)}{3a^2}\Bigg)\Bigg\}\Bigg],\ , \tag{25}$$

where

$$\Lambda_q^{ps} = \int\frac{cos^p\,E\,sin^s\,E}{(1-e\cos E)^q}dE$$

$$\Lambda_n^{00} = \frac{1}{(n-1)\eta}\left[\frac{e\sin E}{\varphi^{n-1}}+(2n-3)\,\Lambda_{n-1}^{00}-(n-2)\,\Lambda_{n-2}^{00}\right]\ n>1$$

$$A = 2-\frac{16\,sin^2\,i}{3}$$

$$\Lambda_0^{00} = E,\ \Lambda_1^{00} = \frac{2}{\eta^{\frac{1}{2}}}tan^{-1}\left[\left(\frac{1+e}{1-e}\right)^{\frac{1}{2}}tan\frac{E}{2}\right]$$

$$\Lambda_n^{01} = -\frac{1}{(n-1)e\wedge^{n-1}}\ n>1$$

$$\Lambda_n^{11} = \frac{1}{e}(\Lambda_n^{01}-\Lambda_{n-1}^{01})\ n>2$$

$$\Lambda_5^{21} = \frac{1}{e^2}[\Lambda_5^{01}-2\,\Lambda_4^{01}+\Lambda_3^{01}]$$

$$\Lambda_n^{m0} = \frac{1}{(-e)^m}\sum_{k=0}^{m}\binom{m}{k}(-1)^{m-k}\,\Lambda_{n-k}^{00}$$

$$\Lambda_5^{31} = \frac{\frac{1}{1-e\cos E}-\frac{1.5}{(1-e\cos E)^2}+\frac{1}{(1-e\cos E)^3}-\frac{0.25}{(1-e\cos E)^4}}{e^4}$$

$$\Lambda_6^{21} = \frac{-\frac{0.2}{(1-e\cos E)^5}+\frac{0.5}{(1-e\cos E)^4}-\frac{1}{3(1-e\cos E)^3}}{e^3}$$

$$\Lambda_6^{31} = \frac{\frac{-0.2}{(1-e\cos E)^5}+\frac{3}{4(1-e\cos E)^4}-\frac{1}{(1-e\cos E)^3}+\frac{0.5}{(1-e\cos E)^2}}{e^4}$$

$$\Lambda_6^{41} = -\frac{cos^5\,E}{E}-e\,cos^6\,E-3e^2\,cos^7\,E-7e^3\,cos^8\,E$$

with

$$\varphi = \frac{r}{a}\ \eta = 1-e^2$$

$$L = \frac{1}{a(1-e^2)}$$

$$(1-e\cos\theta) = \frac{1-e^2}{1-e\cos E}$$

$$h_{0_k} = b_3 g_{3_k} + b_0 g_{0_k}$$

$$h_{1_k} = b_3 g_{4_k} + b_4 g_{3_k} + b_0 g_{1_k} + b_1 g_{0_k}$$

$$h_{2_k} = b_4 g_{4_k} - b_3 g_{3_k} + b_0 g_{2_k} + b_1 g_{1_k} + b_2 g_{0_k}$$

$$h_{3_k} = -b_3 g_{4_k} - b_4 g_{3_k} + b_1 g_{2_k} + b_2 g_{1_k}$$

$$h_{4_k} = b_2 g_{2_k} - b_4 g_{4_k}$$

$$h_{5_k} = b_0 g_{3_k} + b_3 g_{0_k}$$

$$h_{6_k} = b_0 g_{4_k} + b_1 g_{3_k} + b_3 g_{1_k} + b_4 g_{0_k}$$

$$h_{7_k} = b_1 g_{4_k} + b_2 g_{3_k} + b_3 g_{2_k} + b_4 g_{1_k}$$

$$h_{8_k} = b_2 g_{4_k} + b_4 g_{2_k}$$

$$m_{0_i} = b_0 f_{0_i} + b_3 f_{4_i}$$

$$m_{1_i} = b_1 f_{0_i} + b_3 f_{5_i} + b_4 f_{4_i} + b_0 f_{1_i}$$

$$m_{2_i} = b_2 f_{0_i} + b_3 f_{6_i} + b_4 f_{5_i} - b_3 f_{4_i} + b_2 f_{0_i} + b_1 f_{1_i}$$

$$m_{3_i} = b_4 f_{6_i} - b_3 f_{5_i} - b_4 f_{4_i} + b_0 f_{3_i} + b_1 f_{2_i} + b_2 f_{1_i}$$

$$m_{4_i} = -b_3 f_{6_i} - b_4 \mathrm{f}_{5_\mathrm{i}} + \mathrm{b}_1 \mathrm{f}_{3_\mathrm{i}} + \mathrm{b}_2 \mathrm{f}_{2_\mathrm{i}}$$

$$\mathrm{m}_{5_\mathrm{i}} = \mathrm{b}_2 \mathrm{f}_{3_\mathrm{i}} - \mathrm{b}_4 \mathrm{f}_{6_\mathrm{i}}$$

$$\mathrm{m}_{6_\mathrm{i}} = \mathrm{b}_3 \mathrm{f}_{0_\mathrm{i}} - \mathrm{b}_0 \mathrm{f}_{4_\mathrm{i}}$$

$$\mathrm{m}_{7_\mathrm{i}} = \mathrm{b}_4 \mathrm{f}_{0_\mathrm{i}} + \mathrm{b}_0 \mathrm{f}_{5_\mathrm{i}} + \mathrm{b}_1 \mathrm{f}_{4_\mathrm{i}} + \mathrm{b}_3 \mathrm{f}_{1_\mathrm{i}}$$

$$\mathrm{m}_{8_\mathrm{i}} = \mathrm{b}_0 \mathrm{f}_{6_\mathrm{i}} + \mathrm{b}_1 \mathrm{f}_{5_\mathrm{i}} + \mathrm{b}_2 \mathrm{f}_{4_\mathrm{i}} + \mathrm{b}_3 \mathrm{f}_{2_\mathrm{i}} + \mathrm{b}_4 \mathrm{f}_{1_\mathrm{i}}$$

$$\mathrm{m}_{9_\mathrm{i}} = \mathrm{b}_1 \mathrm{f}_{6_\mathrm{i}} + \mathrm{b}_2 \mathrm{f}_{5_\mathrm{i}} + \mathrm{b}_3 \mathrm{f}_{3_\mathrm{i}} + \mathrm{b}_4 \mathrm{f}_{2_\mathrm{i}}$$

$$\mathrm{m}_{10_\mathrm{i}} = \mathrm{b}_2 \mathrm{f}_{6_\mathrm{i}} + \mathrm{b}_4 \mathrm{f}_{3_\mathrm{i}}$$

5. Numerical results

For computing the results with J_2, three test cases at an inclination of 85° for eccentricities 0.01, 0.1 and 0.2, having perigee height of 200 km are chosen to show the effectiveness of the present theory. The other initial conditions for the orbit are given as right ascension of ascending node (Ω) = 60° and argument of perigee (ω) = 0°. The results were generated to validate the improvement obtained with the addition of King Hele's [10] expression with J_2. The difference between the numerically integrated and analytically computed values with the modified theory (ANAL1) and existing theory (ANAL2) by Sharon et al. [17] during half a revolution with a single analytical step size are given in Table 1. It may be noted that the modified theory provides more accurate values of the important orbital parameter 'semi-major axis' during half a revolution than Sharon et al. theory [17].

Table 1: Variation in Semi-Major Axis during Half a Revolution with J_2

Parameter	Case	Method	ANAL steps in E (deg) 30	60	90	150	165	180
a (m)	A	NUM-ANAL1	-0.03	-2.58	-19.1	-2.35	20.3	42.2
		NUM-ANAL2	-0.05	-3.6	-20.1	-3.8	21.9	43.8
	B	NUM-ANAL1	-0.4	-7	-22.8	9.27	26.4	39.6
		NUM-ANAL2	-0.6	-8.6	-23.7	10.2	27.5	40.5
	C	NUM-ANAL1	-0.9	-13.08	-20.4	19.5	31.3	38.8
		NUM-ANAL2	-1.2	-14	-21.4	22.4	33.4	40.9

To generate the results with J_3 and J_4, three test cases A, B and C having eccentricities of 0.03791, 0.17524 and 0.53964 with inclination of 30° are chosen to show the effectiveness of the present theory for very small to very high eccentricity orbits. Details of the initial state vector x, ẋ along with the resulting orbital elements are provided in Table 2. As may be seen from Tables 3 and 4, the error is found to be less than 0.55 % with J_3 and less than 0.85% with J_4 during half a revolution in all the numerical simulations carried out.

Table 2: Initial Conditions (Position, Velocity and Osculating Orbital Elements)

Variables	Case A	B	C
x_1(km)	0	0	0
x_2(km)	-5888.97	-5888.97	-5888.97
x_3(km)	-3400	-3400	-3400
$\dot{x}_1$(km s^{-1})	7.8	8.3	9.5
$\dot{x}_2$(km s^{-1})	0	0	0
$\dot{x}_3$(km s^{-1})	0	0	0
a (km)	7067.95	8244.833	14770.91
e	0.0379	0.1752	0.5396
i (deg)	30	30	30
Ω(deg)	0	0	0
ω(deg)	270	270	270
M(deg)	0	0	0

Table 3: Variation in Semi-Major Axis (Metres) During Half a Revolution with J_3

Parameter	Case	Method	ANAL steps(deg) 10	30	60	90	120	150	180
		NUM	5.375	-72.483	-433.56	-1420.3	-2200.88	-2380.3	-2338.8
	A	NUM-ANAL	0.0009	0.2209	-2.3419	-5.653	-8.5128	10.6554	7.399
		% ERROR	0.016	-0.304	0.54	0.39	0.386	-0.447	-0.316
		NUM	-8.458	-177.531	-1124.1	-2081.9	-2372.83	-2322.7	-2275.8
a(m)	B	NUM-ANAL	-0.0013	0.1109	-1.1123	-2.8921	1.5878	3.4844	2.5173
		% ERROR	0.015	-0.062	0.099	0.139	-0.067	-0.15	-0.1106
		NUM	-309.27	-2968.05	-5959.5	-6087.8	-5975.98	-5931	-5920.3
	C	NUM-ANAL	-0.0198	-6.363	-15.577	-5.5506	-2.9294	-2.3412	-3.7742
		% ERROR	0.006	0.214	0.261	0.091	0.049	0.039	0.064

Table 4: Variation in Semi-Major Axis (Metres) During Half a Revolution with J_4

Parameter	Case	Method	ANAL steps(deg) 10	30	60	90	120	150	180
		NUM	23.627	221.057	782.392	1031.82	674.507	277.971	157.485
	A	NUM-ANAL	-0.0003	-0.2299	-1.486	2.597	3.8459	2.3254	1.317
		% ERROR	-0.001	-0.104	-0.189	0.251	0.57	0.836	0.837
		NUM	50.1117	420.851	1049.42	990.416	711.421	587.036	563.68
a(m)	B	NUM-ANAL	-0.0049	-0.4986	-1.1123	1.8921	1.9985	1.5602	1.4974
		% ERROR	-0.009	-0.001	-0.106	0.191	0.281	0.266	0.266
		NUM	460.139	2401.4	2457.91	2196.87	2171.15	2171.47	2172.11
	C	NUM-ANAL	-0.092	-3.358	0.4164	0.5717	0.3529	0.2825	0.3141
		% ERROR	-0.019	0.139	0.016	0.026	0.016	0.013	0.014

6. Conclusion

K-S uniformly regular canonical equations of motion with generalized eccentric anomaly provide a very efficient and accurate analytical integration method for short-term orbit computation with Earth's oblateness for short-term motion. Only one of the eight equations need to be integrated analytically to generate the state vectors, because of symmetry in the equations of motion. Numerical results indicate that the solution is quite accurate for a wide range of eccentricity. The solution obtained using King-Hele's expression for radial distance as function of J_2 is an improvement over the existing theory of Sharon et al. [17] which uses KS regular elements. The percentage error is found to be less than 0.55% for J_3 and less than 0.85% for J_4. The solutions can have number of applications. It can be used for studying the short-term relative motion of two or more space objects and in collision avoidance studies of space objects. It can be also useful for onboard computation in the navigation and guidance packages and generation of mean orbital elements.

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