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Research paper

Light deflection angle in General Relativity via Daftardar-Jafari method

V. K. Shchigolev *

Department of Theoretical Physics, Ulyanovsk State University, Ulyanovsk, 432000,
**Corresponding author E-mail: vkshch@yahoo.com*

Abstract

In this paper, the iterative method suggested by Daftardar and Jafari hereafter called Daftardar-Jafari method (DJM) is applied for studying the deflection of light in General Relativity. For this purpose, a brief review of the nonlinear geodesic equations in the spherical symmetry spacetime and the main ideas of DJM are given. As an illustrative example, the simple case of the Schwarzschild metric is considered for which the approximate solution to the null-geodesic equation and the deflection angle of light are obtained. We also compare the obtained result with some similar results presented earlier in the literature.

Keywords: *Daftardar-Jafari Iterative Method; Deflection of Light; General Relativity; Schwarzschild Metric.*

1. Introduction

The well-known effect of the light deflection in the gravitational fields of compact objects is one of the first and key predictions of General Relativity (GR) [1], [2], [3]. Although the study of light deflection have historically been associated with the Solar System [4]-[7], in the recent decades, much more attention has been paid to the study of light deflection in the gravitational fields of various compact astrophysical objects [8].

Basically, several approximate approaches are used to determine the deflection of light in the gravitational field of black holes. One of them is the standard parameterized post-Newtonian approach which applicable for b >> M, where b is the impact parameter of the unperturbed light ray and M is the mass of a black hole. Here and below, we use the units in which G = c = 1. As the next approach, one can note the standard weak-field approximate lens equation, which usually is called the classical lens equations [9]. However, these exact lens equations are also given in terms of elliptic integrals. Therefore, approximations of these exact solutions are also needed for a time-efficient data reduction. Several proposals for generalized lens equations have then appeared in the literature. One decisive advantage of the classical lens equation is its validity for arbitrarily small values of the impact distance b. A lens equation which allows an arbitrary large values of the deflection angle and used the deflection angle expression for the Schwarzschild metric is obtained in [10], [11].

As is known, weak gravitational lensing makes it possible to find the mass of astronomical objects without requiring information about their composition or dynamic states. Therefore, in recent years, many authors have proposed studies of gravitational lensing by various astrophysical objects using various methods. Let us mention just a few of the latest articles on this topic. For example, the equations of motion of the massive and massless particles in the Schwarzschild geometry is studied by using the Laplace-Adomian Decomposition Method in [12], that shows the obvious success of this method in obtaining series solutions to a wide range of strongly nonlinear differential equations.

A new method for calculating the angle of small deviation using the Gauss-Bonnet theorem was proposed by the authors of [13] and has found wide application in studies of this kind. This method was also applied to the study of light rays in a plasma medium in a static and spherically symmetric gravitational field and to the study of time-like geodesics followed for test massive particles in a spacetime with the same symmetries in Ref. [14]. The calculation of the bending angle using the trajectory equation based on geometric optics is also provided in [15], [16]. Because wormholes also cause gravitational lensing, this effect has been recently investigated in several papers (see, for example, [17], [18] and references therein).

The author of this article has recently proposed to use two well - known methods, such as the homotopy perturbation method and the variational iteration method [19, 20], to study the deflection of light and the perihelion precession in GR [21]-[24]. Here we use another efficient iterative method for approximating null - geodesics and finding the deflection angle of light. This iterative method has been proposed by Daftardar-Gejji and Jafari [25], and proved the effectiveness for solving many of the linear and nonlinear ordinary differential equations, partial differential equations and integral equations [26], [27]. The proposed DJM is very effective and reliable, and the solution is obtained in the series form with easily computed components [28]. The main aim of this paper is to apply DJM in the Schwarzschild metric to solve the null-geodesic equation and to find the approximate value of the light deflection angle. In addition, we compare the obtained result with the results previously known from the literature.

2. Null-geodesic equation in a spherically symmetric spacetime

According to General Relativity [1, 8], the line element of general static spherically symmetric spacetime can be represented by

$$ds^2 = -f(r)dt^2 + \frac{dr^2}{h(r)} + r^2(d\theta^2 + \sin^2\theta d\varphi^2). \tag{1}$$

The trajectories of photon in GR are usually considered as the null geodesics in spacetime. Therefore, we have to study the geodesics in the spherically symmetric spacetime (1) in the spherical coordinates $x^\mu = (t, r, \theta, \varphi)$ as $x^\mu(\tau)$, where τ is some affine parameter, and satisfies the geodesic equation:

$$\frac{d^2x^\mu}{d\tau^2} + \Gamma^\mu_{\nu\sigma}\frac{dx^\nu}{d\tau}\frac{dx^\sigma}{d\tau} = 0.$$

The geodesic trajectories can be obtained as the solutions of this equation. As is known, when deriving the geodesic equation, one can follow a simpler way if one takes into account the space-time symmetries of metric (1).
First of all, we should note that one component of the geodesic line can always be chosen as $\theta(\tau) = \pi/2$, which means that we can always choose a geodesic lying in the equatorial plane of spherically symmetric space-time. Then, taking into account the null geodesics condition $ds = 0$ in (1), we can obtain the following equation

$$f(r)\left(\frac{dt}{d\tau}\right)^2 - h^{-1}(r)\left(\frac{dr}{d\tau}\right)^2 - r^2\left(\frac{d\phi}{d\tau}\right)^2 = 0. \tag{2}$$

Since there are two conserved quantities along the geodesic in metric (1), the total energy $E = f(r)\frac{dt}{d\tau}$ and the angular momentum per unit mass $L = r^2\frac{d\varphi}{d\tau}$, we can substitute these constants into equation (2) to obtain

$$\left(\frac{dr}{d\tau}\right)^2 = \frac{h(r)}{f(r)}E^2 - h(r)\left(k + \frac{L^2}{r^2}\right). \tag{3}$$

This equation contains only one unknown function $r(\tau)$ and can in principle be solved. However, the deflection of light are usually related to the geodesics orbits, i.e. $r(\varphi)$. Therefore, with the help of $L = r^2\frac{d\varphi}{d\tau}$, one can rewrite equation (3) as follows

$$\left(\frac{dr}{d\phi}\right)^2\left(\frac{L}{r^2}\right)^2 = \frac{h(r)}{f(r)}E^2 - h(r)\left(k + \frac{L^2}{r^2}\right). \tag{4}$$

As the coordinate $u \equiv 1/r$ is more convenient than r in studying the geodesic equations in the spherically symmetric gravitational fields, equation (4) can be converting to the following one:

$$\left(\frac{du}{d\varphi}\right)^2 = \frac{h(u)}{f(u)}\left(\frac{E}{L}\right)^2 - u^2h(u).$$

Finally, differentiating this equation with respect to φ, we get the second-order geodesic equation in the following form:

$$\frac{d^2u}{d\varphi^2} = \frac{E^2}{2L^2}\frac{d}{du}\left[\frac{h(u)}{f(u)}\right] - uh(u) - \frac{1}{2}u^2\frac{dh(u)}{du}. \tag{5}$$

Thereafter, we are going to apply DJM for studying the propagation of light in metric (1). But first, we need to recall the main idea of the DJM and its implementation in solving the second-order ordinary differential equations.

3. Main ideas of DJM in brief

In Ref. [25], the authors consider the general functional equation:

$$u = N(u) + f, \tag{6}$$

Where N a nonlinear operator from a Banach is space $B \to B$ and f is a known function. Suppose that the solution u of Eq. (6) has the series form:

$$u = \sum_{i=0}^{\infty} u_i . \tag{7}$$

The nonlinear operator N can be decomposed as

$$N\left(\sum_{i=0}^{\infty} u_i\right) = N(u_0) + \sum_{i=0}^{\infty}\left[N\left(\sum_{j=0}^{i} u_j\right) - N\left(\sum_{j=0}^{i-1} u_j\right)\right] \tag{8}$$

As it follows from (7) and (8), Eq. (6) is equivalent to

$$\sum_{i=0}^{\infty} u_i = f + N(u_0) + \sum_{i=0}^{\infty}\left[N\left(\sum_{j=0}^{i} u_j\right) - N\left(\sum_{j=0}^{i-1} u_j\right)\right]. \tag{9}$$

One can suppose the following recurrence relation:

$$\begin{cases} u_0 = f, \\ u_1 = N(u_0), \\ u_{m+1} = N(u_0 + \ldots + u_m) - N(u_0 + \ldots + u_{m-1}), \end{cases} \tag{10}$$

Where $m = 1, 2, \ldots$. Then

$$(u_1 + \ldots + u_{m+1}) = N(u_0 + \ldots + u_m),\ m = 1, 2, \ldots, \tag{11}$$

And

$$u = f + \sum_{i=1}^{\infty} u_i . \tag{12}$$

The m-term approximate solution of Eq. (7) is given by $u = u_0 + u_1 + \ldots + u_{m-1}$. If N is a contraction, then the series $\Sigma_{i=1}^{\infty} u_i$ in (12) absolutely and uniformly converges to a solution of Eq. (6), which is unique, in view of the Banach fixed point theorem [25]. For more details about the convergence of DJM, we refer the reader to Ref. [28].

4. Application of DJM to a second order differential equation

Here our description mainly follows to Ref. [26]. Consider some non-linear ordinary differential equation of the second order,

$$u''(\varphi) + k_1 u'(\varphi) + k_2 u(\varphi) + \tilde{N}(u) = \tilde{f}(\varphi), \tag{13}$$

Where a prime stands for the derivative with respect to φ , k_1, k_2 are arbitrary constants, $\tilde{f}(\varphi)$ is a given continuous function, and $\tilde{N}(u)$ is a non-linear term. Besides, this equation must satisfy the initial condition:

$$u(0) = A,\ \ u'(0) = B. \tag{14}$$

Equation (13) can be written in an operator form as:

$$L_{\varphi\varphi} u(\varphi) + k_1 L_{\varphi} u(\varphi) + k_2 u(\varphi) + \tilde{N}(u) = \tilde{f}(\varphi), \tag{15}$$

Where $L_{\varphi} = \frac{d}{d\varphi}$ and $L_{\varphi\varphi} = \frac{d^2}{d\varphi^2}$. We assume that the inverse operators L_{φ}^{-1} and $L_{\varphi\varphi}^{-1}$ exist and can be taken as follows

$$L_{\varphi}^{-1}(.) = \int_0^{\varphi} (.) ds, \tag{16}$$

And

$$L_{\varphi\varphi}^{-1}(.) = \int_0^{\varphi} ds \int_0^{s} (.) dt = \int_0^{\varphi} (\varphi - s)(.) ds, \tag{17}$$

Where we have used the Cauchy formula for repeated integration:

$$\int_0^{\varphi}\int_0^{s_1}\dots\int_0^{s_{n-1}} v(s_n)ds_n\dots ds_2 ds_1 = \frac{1}{(n-1)!}\int_0^{\varphi}(\varphi-s)^{n-1}v(s)ds, \tag{18}$$

Then, applying the inverse operator $L_{\varphi\varphi}^{-1}$ to both sides of the equation (15) and taking into account the initial condition (14), we have

$$u(\varphi) = A + k_1 A\varphi + B\varphi + g(\varphi) - L_{\varphi}^{-1}k_1\, u(\varphi) - L_{\varphi\varphi}^{-1}[k_2\, u(\varphi) + \tilde{N}(u(\varphi))], \tag{19}$$

Where

$$g(\varphi) = \int_0^{\varphi} ds \int_0^{s} \tilde{f}(t)dt = \int_0^{\varphi}(\varphi - s)\,\tilde{f}(s)ds. \tag{20}$$

Therefore, by using equations (16)-(20), we can represent equation (14) in the form of equation (1) by setting

$$f(\varphi) = A + k_1 A\varphi + B\varphi + \int_0^{\varphi}(\varphi - s)\,\tilde{f}(s)ds, \tag{21}$$

And

$$N(u) = -L_{\varphi}^{-1}k_1 u(\varphi) - L_{\varphi\varphi}^{-1}[k_2\, u(\varphi) + \tilde{N}(u(\varphi))] = -\int_0^{\varphi}\Big(k_1 u(s) + (\varphi - s)[k_2\, u(s) + \tilde{N}(u(s))]\Big)ds. \tag{22}$$

By using expressions (21) and (22) in equation (6), we can follow the procedure (10) in order to obtain solution (12) of ODE (13), provided (14). Thereafter, we are going to apply DJM to the approximate solution of the null geodesic equation and finding the light deflection angle in Schwarzschild spacetime.

5. Light Deflection in Schwarzschild spacetime via DJM

The following example demonstrates the use of DJM for the analytical computation of the deflection angle of light in the simplest spherically symmetric spacetimes (1). In the absence of mass ($M = 0$), the obvious analytic solution for (5) is a straight line expressed in polar coordinates as

$$u_{s.l.}(\varphi) = \frac{\sin\varphi}{b}, \tag{23}$$

Where b is a constant impact parameter. Obviously, the term $3M\,u^2$ comes from the correction by GR. Therefore, we can consider (23) to be the null approximation for (5). To solve the problem of finding the subsequent approximations to the solution of equation (5) by the iterative method, we have to follow the procedure of DJM for solving the second order nonlinear differential equations described in Section 3. Let us consider the simplest case of metric (1), namely, the Schwarzschild spacetime describing the gravitational field of an uncharged non-rotating star. For the Schwarzschild solution, we have $f(r) = h(r) = 1 - \frac{2M}{r}$, or

$$f(u) = h(u) = 1 - 2Mu, \tag{24}$$

Where M is the mass of a star. Therefore, equation (5) for the null geodesic can be written as

$$\frac{d^2u}{d\varphi^2} + u = 3M\,u^2. \tag{25}$$

In the absence of mass ($M = 0$), the obvious analytic solution for (25) is a straight line expressed in polar coordinates (23). Thus, the term $3M\,u^2$ comes from the correction of path by GR. Taking into account that a trajectory of light due to equation (25) is started as the straight line (23) at φ , we have the following initial conditions for $u(\varphi)$:

$$u(0) = 0, \quad u'(0) = \frac{1}{b}. \tag{26}$$

We are going to solve the Cauchy problem (25), (26) for the nonlinear differential equation of the second order with a certain approximation using DJM. Therefore, comparing these equations with the corresponding equations (13), (14), we get the following equalities:

$$k_1 = 0, \quad k_2 = 1, \quad A = 0, \quad B = \frac{1}{b}, \tag{27}$$

And

$$\tilde{f} = 0, \quad \tilde{N}(\varphi) = -3M\,u^2. \tag{28}$$

According to equations (21), (22) and (16), (17), we get

$$f(\varphi) = \frac{\varphi}{b}, \quad N(u(\varphi)) = -\int_0^{\varphi} (\varphi - s)\left[u(s) - 3Mu^2(s)\right]ds, \tag{29}$$

And, therefore, the functional equation (6) becomes as follows

$$u(\varphi) = \frac{\varphi}{b} - \int_0^{\varphi} (\varphi - s)\left[u(s) - 3Mu^2(s)\right]ds. \tag{30}$$

So we can to use this equation to construct an approximate solution according to the procedure described by equation (10). The 0-term of the approximate solution (7), that is, the unperturbed motion, is described by $u_0(\varphi) = f(\varphi) = \varphi / b$, according to equations (10) and (29).
It is easy to verify that $u(\varphi)$, defined by equation (30), satisfies the initial conditions (26). Moreover, by differentiating this equation twice, we can verify that it is equivalent to the geodesic equation (25). Then, applying (10) to equation (30), one can obtain the following terms:

$$\begin{aligned} u_0(\varphi) &= \frac{\varphi}{b}, \\ N(u(\varphi)) &= -\int_0^{\varphi} (\varphi - s)\left[u(s) - 3Mu^2(s)\right]ds, \\ u_{m+1}(\varphi) &= N(u_0 + \ldots + u_m) - N(u_0 + \ldots + u_{m-1}), \end{aligned} \tag{31}$$

In the m -term approximate solution (7) of the following form: $u(\varphi) = \frac{\varphi}{b} + \sum_{i=1}^{m-1} u_i(\varphi)$. Thus, due to equations (10) and (31), we can obtain

$$u_1(\varphi) = N(u_0(\varphi)) = -\int_0^{\varphi} (\varphi - s)\left[u_0(s) - 3Mu_0^2(s)\right]ds = \frac{(3M\varphi - 2b)\varphi^3}{12b^2}. \tag{32}$$

Using equations (31) and (32), we get $u_2(\varphi) = N(u_0 + u_1) - N(u_0)$, that is

$$u_2(\varphi) = -\int_0^{\varphi} (\varphi - s)\left(u_0(s) + u_1(s) - 3M\left[u_0(s) + u_1(s)\right]^2\right)ds - u_1(\varphi). \tag{33}$$

As a result of calculating the integral in equation (33), the approximate solution $u = u_0 + u_1 + u_2$ can be obtained in the following form:

$$u(\varphi) = \frac{1}{b}\left(\varphi - \frac{\varphi^3}{3!} + \frac{\varphi^5}{5!}\right) + \frac{M}{b^2}\left(\frac{\varphi^4}{4} - \frac{\varphi^6}{24} + \frac{\varphi^8}{672}\right) + \frac{M^2}{b^3}\left(\frac{\varphi^7}{28} - \frac{\varphi^9}{288}\right) + \frac{M^3\varphi^{10}}{480b^4}. \tag{34}$$

Note that this expansion is quite different from the one given in [Shchigolev4] and does not seem very useful, since the coefficients are expressed as power polynomials in φ , although they usually contain trigonometric functions. However, some functions can be restored if we have enough terms in the expansion (12). Indeed, applying the power series of trigonometric functions,

$$\sin\varphi = \varphi - \frac{\varphi^3}{3!} + \frac{\varphi^5}{5!} - \ldots, \quad \cos\varphi = 1 - \frac{\varphi^2}{2!} + \frac{\varphi^4}{4!} - \ldots,$$

To the first and second terms of equation (34), we can represent the approximate solution in the following form:

$$u(\varphi) \approx \frac{\sin\varphi}{b} + \frac{M}{b^2}(1 - \cos\varphi)^2 + \frac{M^2}{b^3}\left(\frac{\varphi^7}{28} - \frac{\varphi^9}{288}\right) + \frac{M^3\varphi^{10}}{480b^4}. \tag{35}$$

Obviously, solution (35) satisfies the initial condition $u(0)=0$. Therefore, the deflection angle of light α can be obtained from the equation $u(\pi+\alpha)=0$, using the small angle approximation

$$\sin(\pi+\alpha)\approx-\alpha, \qquad \cos(\pi+\alpha)\approx-1, \tag{36}$$

And

$$(\pi+\alpha)^n\approx\pi^n+n\pi^{n-1}\alpha, \tag{37}$$

Due to the binomial formula $(\pi+\alpha)^n=\sum_{k=0}^{n}C_k^n\pi^{n-k}\alpha^k$, where $C_k^n=\frac{n!}{k!(n-k)!}$ are the binomial coefficients.Applying the formulas (36) and (37) to the null trajectory (35) results in the following small deflection angle in DJM approximation (up to the second order):

$$\alpha_{DJM}=\frac{4M}{b}+\frac{\pi^7}{28}\left(1-\frac{28\pi^2}{288}\right)\frac{M^2}{b^2}+\mathrm{O}\left(\frac{M^3}{b^3}\right). \tag{38}$$

Note that the angle of light deflection in the Schwarzschild metric obtained earlier (see, for example, [Virbhadra1]) and also derived using the homotopy perturbation method in [Shchigolev4] with the same accuracy is as follows

$$\alpha_{HPM}=\frac{4M}{b}+\frac{15\pi}{4}\frac{M^2}{b^2}+\mathrm{O}\left(\frac{M^3}{b^3}\right). \tag{39}$$

Graphs of deflection angles $\alpha_{DJM}(x)$ and $\alpha_{HPM}(x)$, where $x=b/M$ is a dimensionless impact parameter, are shown in Figure 1. It is known that a photon with an impact parameter of $b\le b_{cr}=r_S3\sqrt{3}/2$, where $r_S=2M$ is the Schwarzschild radius, may be captured by the central object of mass M [2]. Thus, we have to consider $x\ge3\sqrt{3}$.

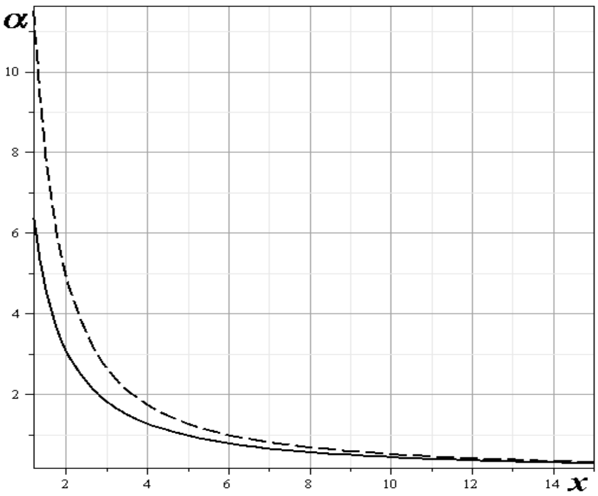

Fig. 1: Shows The Graphs of Deflection Angles $\alpha_{DJM}(x)$ (Continuous Line) and $\alpha_{HPM}(x)$ (Dashed Line) Versus the Dimensionless Impact Parameter $x=b/M$.

6. Conclusion

Thus, we have applied the iterative method called Daftardar-Jafari method for studying the deflection of light in General Relativity. First of all, we have represented a brief review of the nonlinear geodesic equations in the spherical symmetry spacetime and the main ideas of DJM. In order to approbate DJM in the problem of deflection of light and present the main steps in solving by this method, we have illustrated how DJM can be employed to obtain the approximate analytical solution of the null geodesic equation in the simple case of the Schwarzschild metric. For this metric, the approximate solutions to the null-geodesic equation and the deflection angle of light have been obtained. We also compared the obtained result with the similar result presented earlier in the literature.

An important advantage of DJM is the simplicity of obtaining approximate solutions by repeated applications of the iterative equations. The analytic and approximate solutions are obtained without any restrictive assumptions for nonlinear terms as required by some existing techniques. Moreover, by solving some examples, it is seems that the DJM appears to be very accurate to employ with reliable results. We used the Maple software for the calculations in this study.

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