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Research paper

Higher order resonance stability of triangular libration points for radiating primaries in ER3BP

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Abstract

The main aim of this paper is to study the existence of resonance and stability of the triangular equilibrium points in the framework of ER3BP when both the attracting bodies are sources of radiation at $\omega_1 = \omega_2, \omega_1 = 2\omega_2, \omega_1 = 3\omega_2$ in both circular and elliptical cases .A practical application of this model could be seen in the case of binary systems (Achird, Luyten, α Cen- AB, Kruger 60, Xi Bootis). The study is carried out both analytically and numerically by considering various values of radiation pressures and around binary systems .In both cases (CR3BP and ER3BP) it is found that $\omega_1 = \omega_2$ corresponds to the boundary region of the stability for the system, whereas the other two cases $\omega_1 = 2\omega_2, \omega_1 = 3\omega_2$ correspond to the resonant cases. In order to investigate the stability, the Hamiltonian is normalized up to the fourth order by using linear canonical transformation of variables. Then KAM theorem is applied to investigate the stability for different values of radiation pressures in general and around the binary systems in particular. Finally, simulation technique is applied to study the correlation between radiation pressures and mass ratio in circular case; mass ratio and eccentricity in elliptical case. It is found that all the binary systems considered are stable. Also, it is found that except for some values of the radiation pressure parameters and for $\mu \le \mu_c = 0.0385209;$ the triangular equilibrium points are stable.

Keywords: *ER3BP; Hamiltonian Functions; Triangular Libration Points; Resonance; KAM Theory.*

1. Introduction

The stability of the solar system is a longstanding problem. Three-body stability is fundamental to astrophysical processes on all length and mass scales from planetary systems to clusters of galaxies. Over the centuries, mathematicians and astronomers have spent large amount of energy providing stability theorems for dynamical systems closely related to the solar system. The study of the stability of an elliptical restricted three body problem of a Hamiltonian system is generally performed by using KAM theorem.

The nonlinear stability for resonance as well as for the non-resonance cases of the triangular libration points, taking one of the bodies as radiating was studied by Manju and Choudhary (1985). Kumar and Choudhary (1986) investigated the stability of the triangular libration points for non-resonance as well as resonance case; taking both the bodies as radiating in circular restricted three body problems in presence of the third and fourth order resonance. Bhatnagar (1994) discussed the nonlinear stability of the triangular equilibrium points in circular restricted three bodies, considering bigger primary as a source of radiation. The nonlinear stability of the triangular Lagrangian points, considering the bigger primary as oblate spheroid in circular case was examined by Markellos (1996).

The detailed description and the behavior of equilibrium points in ER3BP were given by Danby (1964), Bennet (1965), Szebehely (1967) Rabe (1970), Markeev (1978), Selaru Cucu-Dumitrescu (1994), Halan and Rana (2001). The influence of the eccentricity of the orbits of the primaries with or without radiation pressure (s) on the existence and stability of the equilibrium points was studied by Gyorgrey (1985), Grebenikov (1964), Kumar and Choudhary (1990), Markeellos (1992), Conxita (1995),Shaoo and Ishwar (2000), Roberts (2002), Floria (2004), Zimvoschikov and Thakai (2004), Ammar (2008), Erdi (2009), Kumar and Ishwar (2011). Singh and Umar (2012a), (2012b), Usha et. al. (2014), Narayan and Singh (2014 a, b, c,).

The present paper deals with the stability of the triangular points in non-resonance condition, considering both the primaries as radiating; satisfying the tions $\beta_1 + \beta_2 \le 1;$ $\beta_1 + \beta_2 \ge 1;$ and $0 \le \beta_1, \beta_2 \le 1;$ .The study is carried out for both circular case and elliptical case.

This paper has been organized in various sections: section 1 described the introduction Section 2 describes the equations of motion of the problem, section 3 deals with characteristics roots and first order stability of the triangular equilibrium points. The existence of resonance is discussed in section 4 and 5 while Section 6 deals with normalization and higher order stability of the libration points. Finally, section 7 summarizes the discussion and conclusion of the paper.

2. Equations of motion

The differential equations of the motion of the infinitesimal mass in elliptical restricted three body problem under radiating primaries in pulsating system is given as:
[Narayan and Singh (2014a)]

$$\bar{x}'' - 2\bar{y}' = \varphi \frac{\partial \Omega}{\partial \bar{x}};$$

$$\bar{y}'' + 2\bar{x}' = \varphi \frac{\partial \Omega}{\partial \bar{y}};$$

$$\bar{z}'' + \bar{z} = \varphi \frac{\partial \Omega}{\partial \bar{z}}, \tag{2.1}$$

where

$$\Omega = (1-\mu)\left[\frac{1}{2}\bar{r}_1^2 + \frac{(1-\beta_1)}{\bar{r}_1}\right] + \mu\left[\frac{1}{2}\bar{r}_2^2 + \frac{(1-\beta_2)}{\bar{r}_2}\right] - \frac{1}{2}\frac{\bar{z}^2}{r}; \tag{2.2}$$

$$\phi = r = (1+e\cos \mathrm{f})^{-1}; \tag{2.3}$$

$$\bar{r}_1^2 = (\bar{x}+\mu)^2 + \bar{y}^2 + \bar{z}^2;$$

$$\bar{r}_2^2 = (\bar{x}+\mu-1)^2 + \bar{y}^2 + \bar{z}^2. \tag{2.4}$$

where β_1, β_2 is the radiation pressure and f is a true anomaly of the primaries.
The coordinates of the triangular equilibrium points L_4 and L_5 in $(\bar{x}, \bar{y})$ *plane* are given by: [Narayan and Singh (2014)]

$$\bar{x} = \frac{1}{2} - \mu + \frac{\beta_2}{3} - \frac{\beta_1}{3};$$
$$\bar{y} = \pm\frac{\sqrt{3}}{2}\left[1 - \frac{2\beta_1}{9} - \frac{2\beta_2}{9}\right] \tag{2.5}$$

3. Characteristics roots and first order stability of the triangular equilibrium points

It may be easily verified that the system (2.1) describes the motion of dynamical system with Lagrangian given by:

$$L = \frac{(\dot{x})^2 + (\dot{y})^2}{2} + (\dot{y}\,\bar{x} - \dot{x}\,\bar{y}) + \frac{1}{1+e\cos f} \times \left[(1-\mu)\left(\frac{\bar{r}_1^2}{2} + \frac{(1-\beta_1)}{\bar{r}_1}\right) + \mu\left(\frac{\bar{r}_2^2}{2} + \frac{(1-\beta_2)}{\bar{r}_2}\right)\right] \tag{3.1}$$

where

$$p_{\bar{x}} = \frac{\partial L}{\partial \dot{\bar{x}}} = \dot{\bar{x}} - \bar{y}\;; \quad p_{\bar{y}} = \frac{\partial L}{\partial \dot{\bar{y}}} = \dot{\bar{y}} + \bar{x}\;.$$

Now, the expression for the Hamiltonian function of the problem is constructed using the formula given as follows:

$$H = -L + \frac{\partial L}{\partial \dot{\bar{x}}}\dot{\bar{x}} + \frac{\partial L}{\partial \dot{\bar{y}}}\dot{\bar{y}}\;;$$

$$= -L + p_{\bar{x}}\dot{\bar{x}} + p_{\bar{y}}\dot{\bar{y}}\;;$$

$$= -L + p_{\bar{x}}\left(p_{\bar{x}} + \bar{y}\right) + p_{\bar{y}}\left(p_{\bar{y}} - \bar{x}\right);$$

$$= -L + \left(p_{\bar{x}}^{\;2} + p_{\bar{y}}^{\;2}\right) + \left(p_{\bar{x}}\bar{y} - p_{\bar{y}}\bar{x}\right). \tag{3.2}$$

From equation (3.1) and (3.2), the perturbed Hamiltonian function of the problem is given by:

$$H = -\frac{(\dot{x})^2 + (\dot{y})^2}{2} - (\dot{y}\,\bar{x} - \dot{x}\,\bar{y}) - \frac{1}{1+e\cos f} \times \left[(1-\mu)\left(\frac{\bar{r}_1^2}{2} + \frac{(1-\beta_1)}{\bar{r}_1}\right) + \mu\left(\frac{\bar{r}_2^2}{2} + \frac{(1-\beta_2)}{\bar{r}_2}\right)\right] + (p_{\bar{x}}^2 + p_{\bar{y}}^2) + (p_{\bar{x}}\bar{y} - p_{\bar{y}}\bar{x}) \tag{3.3}$$

After further simplification the perturbed Hamiltonian function in the model of ER3BP considering the photogravitational effect of both the primaries having radiation pressure β_1 and β_2 is given as:

$$H = \frac{(p_{\bar{x}}^2 + p_{\bar{y}}^2)}{2} + (p_{\bar{x}}\bar{y} - p_{\bar{y}}\bar{x}) + \frac{(\bar{x})^2 + (\bar{y})^2}{2} - \frac{1}{1+e\cos f} \times \left[(1-\mu)\left(\frac{\bar{r}_1^2}{2} + \frac{(1-\beta_1)}{\bar{r}_1}\right) + \mu\left(\frac{\bar{r}_2^2}{2} + \frac{(1-\beta_2)}{\bar{r}_2}\right)\right] \tag{3.4}$$

Since the two triangular points are symmetrical, the nature of the motion near the two points is the same. So, the motion of the infinitesimal near one of the equilibrium points is considered. For this, we shift the origin at L_4. Let us use the change of the variables given by:

$$\bar{x} = \xi + q_1\;; \quad \bar{y} = \eta + q_2\;;$$

$$p_{\bar{x}} = p_{\xi} + p_1\;; \quad p_{\bar{y}} = p_{\eta} + p_2 \tag{3.5}$$

Where the displacement at and near L_4 is represented as follows:

$$\xi = \frac{1}{2} - \mu + \frac{\beta_2}{3} - \frac{\beta_1}{3}\;;$$

$$\eta = \frac{\sqrt{3}}{2}\left[1 - \frac{2\beta_1}{9} - \frac{2\beta_2}{9}\right];$$

$$p_{\xi} = -\frac{\sqrt{3}}{2}\left[1 - \frac{2\beta_1}{9} - \frac{2\beta_2}{9}\right]\;;$$

$$p_{\eta} = \frac{1}{2} - \mu + \frac{\beta_2}{3} - \frac{\beta_1}{3} \tag{3.6}$$

The solutions for equation (3.6) in the new variables are given by the equilibrium position:

$$q_1 = q_2 = p_1 = p_2 = 0 \tag{3.7}$$

Now expanding the Hamiltonian function given by equation (3.4) in the powers of p_i and q_i as:

$$H = \sum_{K=0}^{\infty} H_K = H_0 + H_1 + H_2 + H_3 + H_4 + H_5 + ... \tag{3.8}$$

where

$$H_0 = H\left(\xi, \eta, p_\xi, p_\eta\right) = \text{constant}; \; H_1 = 0 . \tag{3.9}$$

Now, the Hamiltonian given by equation (3.4) is expanded using equation (3.5). The terms independent of p_i , q_i and those having order more than one are neglected. Thus, the Hamiltonian is represented as follows:

$$\begin{aligned} H &= \frac{p_1^2 + p_2^2}{2} + (p_1 q_2 - p_2 q_1) + \\ & \frac{(q_1^2 + q_2^2) e\cos f}{2(1 + e\cos f\ )} - \frac{1}{1 + e\cos f} \times \\ & \left\{(1-\mu)(1-\beta_1)(\bar{r}_1)^{-1} + \mu(1-\beta_2)(\bar{r}_2)^{-1}\right\} \end{aligned} \tag{3.10}$$

After further mathematical manipulations and using Taylor's theorem the Hamiltonian is given as follows:

$$\begin{aligned} H &= \frac{p_1^2 + p_2^2}{2} + (p_1 q_2 - p_2 q_1) \\ & + \frac{(q_1^2 + q_2^2) e\cos f}{2(1 + e\cos f\ )} - \end{aligned}$$

$$\begin{aligned} &\left[\left\{q_1\left(\frac{-1}{2} - \frac{\beta_1}{6} - \frac{\beta_2}{3}\right) + q_2\left(\frac{-\sqrt{3}}{2} - \frac{7\beta_1}{6\sqrt{3}} + \frac{\beta_2}{3\sqrt{3}}\right)\right\} + \right. \\ & \frac{1}{2}\left\{q_1^{\,2}\left(\frac{-1}{4} - \frac{3\beta_1}{4} + \beta_2\right) + 2q_1q_2\left(\frac{3\sqrt{3}}{4} + \frac{7\beta_1}{4\sqrt{3}} + \frac{\beta_2}{\sqrt{3}}\right) \right. \\ & \left. + q_2^{\,2}\left(\frac{5}{4} + \frac{7\beta_1}{4} - \beta_2\right)\right\} + \frac{1}{6}\left\{q_1^3\left(\frac{21}{8} + \frac{31\beta_1}{8} - \frac{3\beta_2}{4}\right) + \right. \\ & + 3q_1^2q_2\left(\frac{-3\sqrt{3}}{8} + \frac{17\beta_1}{8\sqrt{3}} - \frac{58\sqrt{3}\beta_2}{8\sqrt{3}}\right) \\ & + 3q_1q_2^2\left(\frac{-33}{8} - \frac{43\beta_1}{8} - \frac{\beta_2}{4}\right) \\ & \left. + q_2^3\left(-\frac{9\sqrt{3}}{8} - \frac{23\sqrt{3}\beta_1}{8} + \frac{11\sqrt{3}\beta_2}{4}\right)\right\} \\ & + \frac{1}{24}\left\{q_1^4\left(\frac{-111}{16} - \frac{85\beta_1}{16} + \frac{25\beta_2}{2}\right)\right. \\ & + 4q_1^3q_2\left(\frac{-75}{16} - \frac{535\sqrt{3}\beta_1}{48} + \frac{20\sqrt{3}\beta_2}{3}\right) \\ & + 6q_1^2q_2^2\left(\frac{123}{16} + \frac{65\beta_1}{16} + \frac{35\beta_2}{2}\right) \\ & + 4q_1q_2^3\left(\frac{135\sqrt{3}}{16} - \frac{265\sqrt{3}\beta_1}{16} - 5\sqrt{3}\beta_2\right) \\ & \left.\left. + q_2^4\left(\frac{9}{16} + \frac{195\beta_1}{16} - \frac{45\beta_2}{2}\right)\right\}\right] \end{aligned}$$

$$-\frac{\mu(1-\beta_2)}{1 + e\cos f}\left(1 + \frac{\beta_2}{3}\right) - \frac{\mu(1-\beta_2)}{1 + e\cos f} \times$$

$$\begin{aligned} &\left[\left\{q_1\left(\frac{1}{2} + \frac{\beta_1}{3} + \frac{\beta_2}{6}\right) + q_2\left(\frac{-\sqrt{3}}{2} + \frac{\beta_1}{3\sqrt{3}} - \frac{7\beta_2}{6\sqrt{3}}\right)\right\} + \right. \\ & \frac{1}{2}\left\{q_1^{\,2}\left(\frac{-1}{4} + \beta_1 - \frac{3\beta_2}{4}\right) + \right. \\ & 2q_1q_2\left(\frac{-3\sqrt{3}}{4} - \frac{\beta_1}{\sqrt{3}} - \frac{7\beta_2}{4\sqrt{3}}\right) \\ & \left. + q_2^{\,2}\left(\frac{5}{4} - \beta_1 + \frac{7\beta_2}{4}\right)\right\} \\ & + \frac{1}{6}\left\{q_1^3\left(\frac{-21}{8} + \frac{3\beta_1}{4} - \frac{31\beta_2}{8}\right)\right. \\ & + 3q_1^2q_2\left(\frac{-3\sqrt{3}}{8} - \frac{58\beta_1}{8\sqrt{3}} + \frac{17\beta_2}{8\sqrt{3}}\right) + \\ & 3q_1q_2^2\left(\frac{33}{8} + \frac{\beta_1}{4} + \frac{43\beta_2}{8}\right) \\ & \left. + q_2^3\left(\frac{-9\sqrt{3}}{8} + \frac{11\sqrt{3}\beta_1}{4} - \frac{23\sqrt{3}\beta_2}{8}\right)\right\} \\ & + \frac{1}{24}\left\{q_1^4\left(\frac{-111}{16} - \frac{25\beta_1}{2} - \frac{85\beta_2}{16}\right) + \right. \\ & + 4q_1^3q_2\left(\frac{75\sqrt{3}}{16} - \frac{20\sqrt{3}\beta_1}{3} + \frac{535\sqrt{3}\beta_2}{48}\right) \\ & + 6q_1^2q_2^2\left(\frac{123}{16} + \frac{35\beta_1}{2} + \frac{65\beta_2}{16}\right) + \\ & + 4q_1q_2^3\left(\frac{-135\sqrt{3}}{16} + 5\sqrt{3}\beta_1 - \frac{265\sqrt{3}}{16}\beta_2\right) \\ & \left.\left. + q_2^4\left(\frac{9}{16} - \frac{45\beta_1}{2} + \frac{195\beta_2}{16}\right)\right\}\right] \end{aligned} \tag{3.11}$$

Equating the coefficient of 2nd, 3rd, 4th order we get:

$$\begin{aligned} H_2 &= \frac{p_1^2 + p_2^2}{2} + (p_1 q_2 - p_2 q_1) + \\ & \frac{(q_1^2 + q_2^2) e\cos f}{2(1 + e\cos f\ )} + \frac{1}{(1 + e\cos f\ )} \times \\ & \left[\left(\frac{1}{8} + A\right)q_1^2 - q_1q_2(K - B) - q_2^2\left(\frac{5}{8} + A\right)\right] \end{aligned} \tag{3.12}$$

where

$$A = \frac{\beta_1}{4} - \frac{\beta_2}{2} - \frac{3\mu\beta_1}{4} + \frac{3\mu\beta_2}{4}$$

$$B = \frac{3\sqrt{3}}{4}\left(\frac{2\beta_1}{9} - \frac{4\beta_2}{9} + \frac{2\mu\beta_1}{9} + \frac{2\mu\beta_2}{9}\right)$$

$$K = \frac{3\sqrt{3}(1 - 2\mu)}{4} \tag{3.13}$$

For sake of convenience, let us substitute:

$$H_{20} = \frac{1}{8} + A; \; H_{11} = K - B; \; H_{02} = \frac{5}{8} + A \tag{3.14}$$

The equation (3.12) reduces to the form:

$$H_2 = \frac{p_1^2 + p_2^2}{2} + (p_1 q_2 - p_2 q_1) + \frac{e\cos f}{2(1+e\cos f\,)} \times (q_1^2 + q_2^2) - \frac{1}{2(1+e\cos f\,)} \times (H_{20} q_1^2 + H_{11} q_1 q_2 + H_{02} q_2^2) \tag{3.15}$$

where,

$$H_{20} = \frac{1}{8} + \frac{\beta_1}{4} - \frac{\beta_2}{2} - \frac{3\mu\beta_1}{4} + \frac{3\mu\beta_2}{4}$$

$$H_{11} = \frac{3\sqrt{3}}{4} - \frac{\beta_1}{2\sqrt{3}} - \frac{\sqrt{3}\beta_2}{3} - \frac{3\sqrt{3}\mu}{2} - \frac{\sqrt{3}\mu\beta_1}{6} - \frac{\sqrt{3}\mu\beta_2}{6}$$

$$H_{02} = \frac{5}{8} + \frac{\beta_1}{4} - \frac{\beta_2}{2} - \frac{3\mu\beta_1}{4} + \frac{3\mu\beta_2}{4} \tag{3.16}$$

$$\begin{aligned} H_3 = {} & \frac{-1}{6(1+e\cos f\,)} \times \\ & \Big[q_1^3 \left(\frac{21}{8} + \frac{5\beta_1}{4} - \frac{3}{4}\beta_2 - \frac{21\mu}{4} - \frac{\mu\beta_1}{2} - \frac{\mu\beta_2}{2} \right) \\ & +3 q_1^2 q_2 \left(-\frac{3\sqrt{3}}{8} - \frac{26\sqrt{3}\beta_1}{8} - \frac{29\sqrt{3}\beta_2}{12} - \frac{138\sqrt{3}\mu\beta_1}{24} + \frac{84\sqrt{3}\mu\beta_2}{24} \right) \\ & +3 q_1 q_2^2 \left(-\frac{33}{8} - \frac{5\beta_1}{4} - \frac{\beta_2}{4} + \frac{33\mu}{4} + \frac{3\mu\beta_1}{2} + \frac{3\mu\beta_2}{2} \right) + \\ & q_2^3 \left(\frac{-9\sqrt{3}}{8} - \frac{14\sqrt{3}\beta_1}{8} + \frac{11\sqrt{3}}{12}\beta_2 + \frac{25\sqrt{3}\mu\beta_1}{4} - \frac{25\sqrt{3}\mu\beta_2}{4} \right) \Big] \end{aligned} \tag{3.17}$$

$$\begin{aligned} H_4 = {} & \frac{-1}{24(1+e\cos f\,)} \times \\ & \Big[q_1^4 \left(\frac{-111}{16} + \frac{13\beta_1}{8} + \frac{25\beta_2}{2} - \frac{113\mu\beta_1}{8} - \frac{87\mu\beta_2}{8} \right) \\ & +4 q_1^3 q_2 \left(-\frac{75\sqrt{3}}{16} - \frac{310\sqrt{3}\beta_1}{48} + \frac{20\sqrt{3}\beta_2}{3} + \frac{75\sqrt{3}\mu}{8} - \frac{10\sqrt{3}\mu\beta_1}{48} - \frac{10\sqrt{3}\mu\beta_2}{48} \right) \\ & +6 q_1^2 q_2^2 \left(\frac{123}{16} - \frac{29\beta_1}{16} + \frac{35\beta_2}{2} + \frac{169\mu\beta_1}{8} - \frac{169\mu\beta_2}{8} \right) + \\ & 4 q_1 q_2^3 \left(-\frac{135\sqrt{3}}{16} + \frac{130\sqrt{3}\beta_1}{16} - 5\sqrt{3}\beta_2 - \frac{135\sqrt{3}\mu}{8} - \frac{50\sqrt{3}\mu\beta_1}{16} - \frac{50\sqrt{3}\mu\beta_2}{16} \right) \\ & + q_2^4 \left(\frac{9}{16} + \frac{186\beta_1}{16} - \frac{45}{2}\beta_2 - \frac{273\mu\beta_1}{8} + \frac{273\mu\beta_2}{8} \right) \Big] \end{aligned} \tag{3.18}$$

4. Existence of resonance in circular case.

The existence of resonance for $\omega_1 = \omega_2, \omega_1 = 2\omega_2\,, \omega_1 = 3\omega_2$ has been analyzed in the model of elliptical restricted three body problem under the radiating primaries for the particular case, e=0. The Hamiltonian for the circular problem up to the second order term is obtained, which is expressed as follows:

$$H_2 = \frac{p_1^2 + p_2^2}{2} + (p_1 q_2 - p_2 q_1) + \left\{ H_{20} q_1^2 - q_1 q_2 H_{11} - H_{02} q_2^2 \right\} \tag{4.1}$$

where H_{20}, H_{11} and H_{02} are defined by the equations (3.16).

If H_2 is of positive definite form, then the equilibrium points are stable for all orders as given by Liapunov's (1956). Otherwise stability can be investigated by Arnold's theorem [1963(a, b)] .In order to apply Arnold's theorem, H2 is normalized using linear transformation of variables as given by Markeev (1978). Restricting to H_2 and using the canonical transformation the characteristics equation takes the following form:

$$\begin{vmatrix} \lambda^2 - (1 - 2H_{20}) & -2\lambda - H_{11} \\ 2\lambda - H_{11} & \lambda^2 - (1 + 2H_{02}) \end{vmatrix} = 0 \tag{4.2}$$

After further calculations the characteristics equation reduces to the form:

$$\lambda^4 + \lambda^2 + \left\{ \frac{27\mu(1-\mu)}{4} \right\} \left\{ 1 + \frac{2\beta_1}{9} + \frac{2\beta_2}{9} \right\} = 0 \tag{4.3}$$

If ω_1 and ω_2 be the frequencies then we put $\lambda^2 = -\omega^2$ and the equation satisfied by the frequencies is given as:

$$\omega^4 - \omega^2 + \left\{ \frac{27\mu(1-\mu)}{4} \right\} \left\{ 1 + \frac{2\beta_1}{9} + \frac{2\beta_2}{9} \right\} = 0 \tag{4.4}$$

If ω_1, ω_2 be the frequencies corresponding to the equation (4.4) then $\omega_1^2 = -\lambda_{1,2}^2$ and $\omega_2^2 = -\lambda_{3,4}^2$

$$\omega_1^2 = -\lambda_{1,2}^2 = \frac{1}{2}\left[1 - \left\{\frac{1-27\mu(1-\mu)\times}{\left(1+\frac{2\beta_1}{9}+\frac{2\beta_2}{9}\right)}\right\}^{1/2}\right]$$

$$\omega_1^2 = -\lambda_{3,4}^2 = \frac{1}{2}\left[1 + \left\{\frac{1-27\mu(1-\mu)\times}{\left(1+\frac{2\beta_1}{9}+\frac{2\beta_2}{9}\right)}\right\}^{1/2}\right] \quad (4.5)$$

We discussed the following cases:

Case1. Assuming $\omega_1 = \omega_2$ i.e. $\lambda_{1,2}^2 = \lambda_{3,4}^2$

I.e. roots of the characteristics equation are equal. Therefore, discriminant is equal to zero.i.e.

$$1 - 27\mu(1-\mu)(1+2\beta_1/9+2\beta_2/9) = 0 \quad (4.6)$$

i.e.

$$27\left(1+\frac{2\beta_1}{9}+\frac{2\beta_2}{9}\right)\mu^2 - 27\left(1+\frac{2\beta_1}{9}+\frac{2\beta_2}{9}\right)\mu + 1 = 0 \quad (4.7)$$

Since $\mu \le \frac{1}{2}$, the positive sign is inadmissible. Hence, the region of stability in the first approximation can be written as:

$$0 < \mu < \frac{\left\{9-\sqrt{69}\left(1+\frac{4\beta_1}{207}+\frac{4\beta_2}{207}\right)\right\}}{18} \quad (4.8)$$

Thus, the value of μ responsible for stable equilibrium points is given by:

$$\mu_{01} = 0.03852088 - 0.008917\beta_1 - 0.008917\beta_2 \quad (4.9)$$

When β_1, β_2 vanishes, $\mu_{critical} = 0.03852088$,

Case2. Assuming $\omega_1 = 2\omega_2$ i.e. $\lambda_{1,2}^2 = 4\lambda_{3,4}^2$

i.e.

$$675\left(1+\frac{2\beta_1}{9}+\frac{2\beta_2}{9}\right)\mu^2 - 675\left(1+\frac{2\beta_1}{9}+\frac{2\beta_2}{9}\right)\mu + 16 = 0 \quad (4.10)$$

Solving for μ , the resonance value is obtained as:

$$\mu_{02} = 0.02429396 - 0.00553648\beta_1 - 0.00553648\beta_2$$

Case3.If $\omega_1 = 3\omega_2$ i.e. $\lambda_{1,2}^2 = 9\lambda_{3,4}^2$

(4.11)

i.e.

$$675\left(1+\frac{2\beta_1}{9}+\frac{2\beta_2}{9}\right)\mu^2 - 675\left(1+\frac{2\beta_1}{9}+\frac{2\beta_2}{9}\right)\mu + 9 = 0 \quad (4.12)$$

Solving for μ, the resonance value is obtained as:

$$\mu_{03} = 0.0135167 - 0.0029629\beta_1 - 0.0029629\beta_2 \quad (4.13)$$

The critical value μ_c usually corresponds to the boundary of the region of stability of the linear system and hence the case is not being considered in our further investigation. Taking into consideration of the range of stability, it is found that the range which we defined for, $\mu_c < 0.0385208$ the condition $\omega_1 = \omega_2$, will not be satisfied .The condition of the range of stability is satisfied for $\omega_1 = \omega_2$,when $\mu_c > 0.0385208$

From equation (4.11) and (4.13) it is clear that resonance is possible for $\omega_1 = 2\omega_2, \omega_1 = 3\omega_2$.

Figs.1-5 are representing the correlation between μ and ω_1, ω_2 for e=0

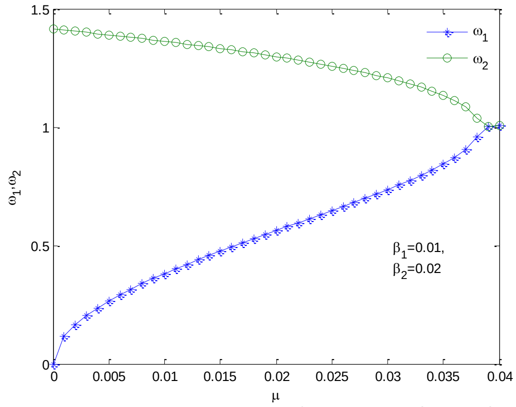

Fig. 1: Correlation between μ and ω_1, ω_2 for e=0 when β_1=.01, β_2=0.02.

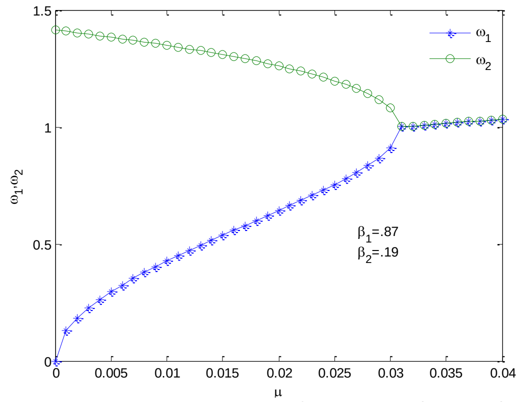

Fig. 2: Correlation between μ and ω_1, ω_2 for e=0 when β_1=0.87, β_2=0.19

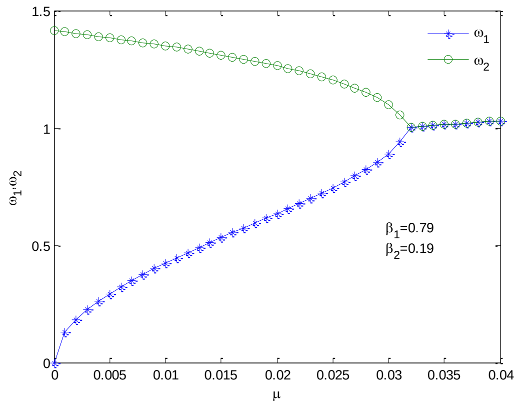

Fig. 3: Correlation between μ and ω_1, ω_2.for e=0 when β_1=0.79, β_2=0.29.

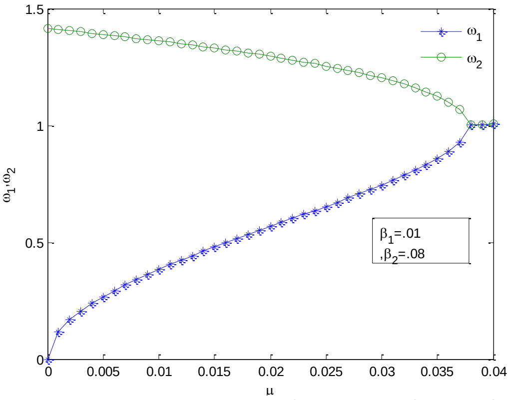

Fig. 4: Correlation between μ and ω_1, ω_2 for e=0 when β_1=0.01, β_2=0.08

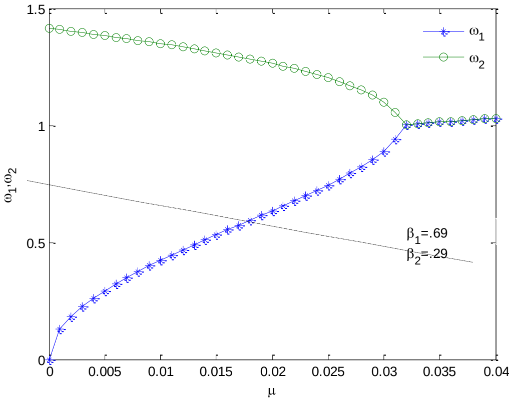

Fig. 5: Correlation between μ and ω_1, ω_2 for e=0 when β_1=0.69, β_2=0.29

The correlation between β_2 and μ for circular case for various values of radiation pressure is shown in fig. 6-13.

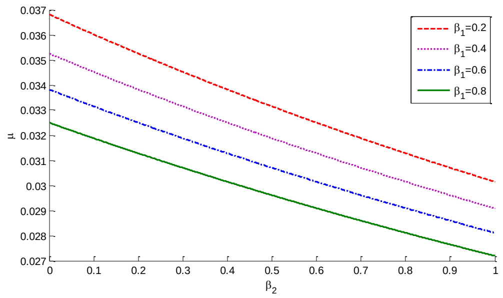

Fig. 6: Correlation between β_2 and μ when ω_1= ω_2 for e=0

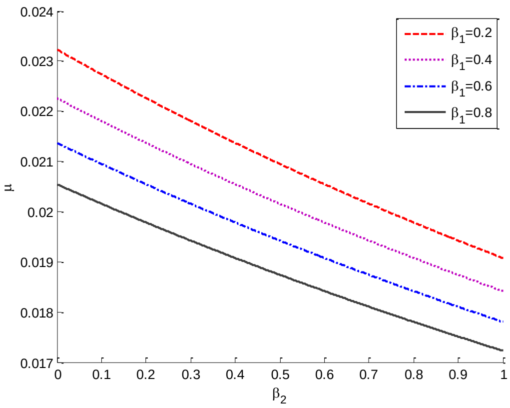

Fig. 7: Correlation between β_2 and μ when ω_1= $2\omega_2$ for e=0

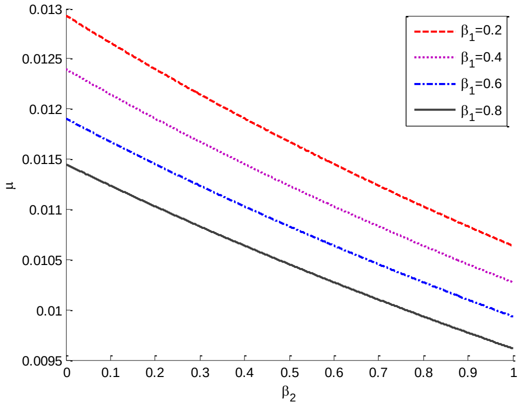

Fig. 8: Correlation between β_2 and μ when ω_1= $3\omega_2$ for e=0

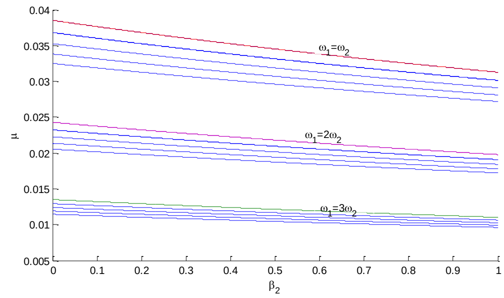

Fig. 9: Combined Graph of Fig .6, 7, 8

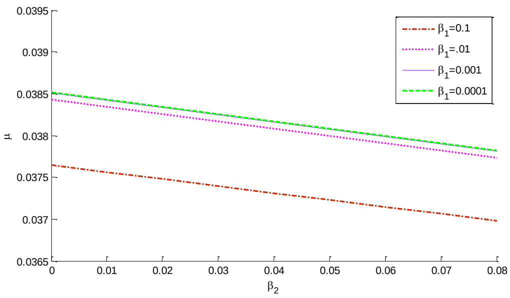

Fig. 10: Correlation between β_2 and μ when ω_1= ω_2 for e=0

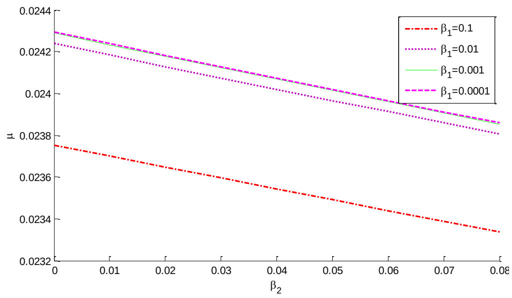

Fig. 11: Correlation between β_2 and μ when ω_1= $2\omega_2$ for e=0

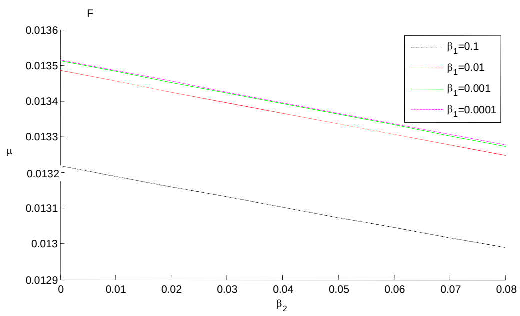

Fig. 12: Correlation between β_2 and μ when ω_1= $3\omega_2$ for e=0

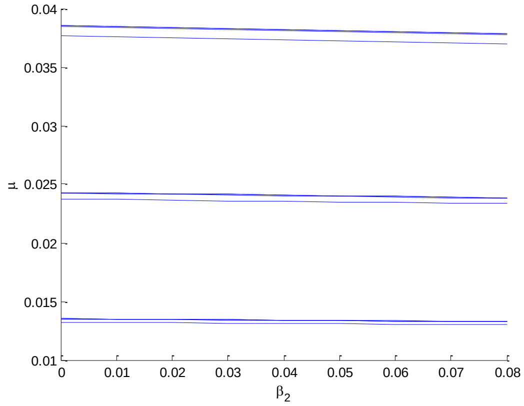

Fig. 13: Combined Graph of Fig 10, 11, 12

5. Existence of resonance in elliptical case.

Considering equation (3.5) and expanding the Hamiltonian function H_2 in powers of 'e' up to the first approximation we get:

$$H_2=\frac{p_1^2+p_2^2}{2}+\left(P_1q_2-P_2q_1\right)+\left[\left(\frac{1}{8}+A\right)q_1^2-(k-B)q_1q_2-\left(\frac{5}{8}+A\right)q_2^2\right]+(e\cos f)\times\left[\left(\frac{3}{8}-A\right)q_1^2+(k-B)q_1q_2+\left(\frac{9}{8}+A\right)q_2^2\right]\qquad(5.1)$$

The characteristics equation in this case is given by:

$$\lambda^4+\left(1+\frac{e^2}{2}\right)\lambda^2+\left\{\frac{27\mu(1-\mu)}{4}\right\}\times\left\{1+\frac{2\beta_1}{9}+\frac{2\beta_2}{9}\right\}(1+e^2)=0\qquad(5.2)$$

Proceeding in the same manner as in section 4, the roots of the characteristics equation are represented as:

$$\omega_1^2=-\lambda_{1,2}^2$$

$$\omega_1^2=\frac{1}{2}\left[\left(1+\frac{e^2}{2}\right)-\left\{\left(1+\frac{e^2}{2}\right)^2-27\mu(1-\mu)\times(1+e^2)\times\left(1+\frac{2\beta_1}{9}+\frac{2\beta_2}{9}\right)\right\}^{1/2}\right]$$

$$\omega_2^2=-\lambda_{3,4}^2$$

$$\omega_2^2=\frac{1}{2}\left[\left(1+\frac{e^2}{2}\right)+\left\{\left(1+\frac{e^2}{2}\right)^2-27\mu(1-\mu)\times(1+e^2)\times\left(1+\frac{2\beta_1}{9}+\frac{2\beta_2}{9}\right)\right\}^{1/2}\right]\qquad(5.3)$$

Again considering the three cases as follows:

Case 1.Assuming, $\omega_1=\omega_2$ i.e. $\lambda_{1,2}^2=\lambda_{3,4}^2$

Then, solving for μ, we get:

$$\mu_{01e}=0.03852088-0.008917\beta_1-0.008917\beta_2+e^4(0.03009-0.006687\beta_1-0.006687\beta_2)\qquad(5.4)$$

This value corresponds to the boundary of the stability region.

Case2.

Let $\omega_1=2\omega_2$ i.e. $\lambda_{1,2}^2=4\lambda_{3,4}^2$

Hence, the value of μ is given as:

$$\mu_{02e}=0.02429396-0.00553648\beta_1-0.00553648\beta_2+e^4(0.0046714-0.001038\beta_1-0.001038\beta_2)\qquad(5.5)$$

Case3.

Assuming $\omega_1=3\omega_2$ i.e. $\lambda_{1,2}^2=9\lambda_{3,4}^2$

Again, solving for μ we get:

$$\mu_{03e}=0.0135167-0.0029629\beta_1-0.0029629\beta_2+e^4(0.010277-0.002839\beta_1-0.002839\beta_2)\qquad(5.6)$$

Table-1and Table-2 (see appendices) shows the values of μ corresponding to $\omega_1=\omega_2,\omega_1=2\omega_2$ *and* $\omega_1=3\omega_2$ satisfying the conditions $\beta_1+\beta_2>1;\beta_1,\beta_2<1$. and $\beta_1+\beta_2<1;\beta_1,\beta_2<1$ respectively.

It is clear from Table 1 and Table2 that the corresponding values of μ are less than the critical value for every combination of the radiation pressures taken which states that the resonance condition exists for $\omega_1=2\omega_2,\omega_1=3\omega_2$. in both circular and elliptical cases.

Figs.14-18 show correlation between μ and ω_1, ω_2 for e=0.0489.

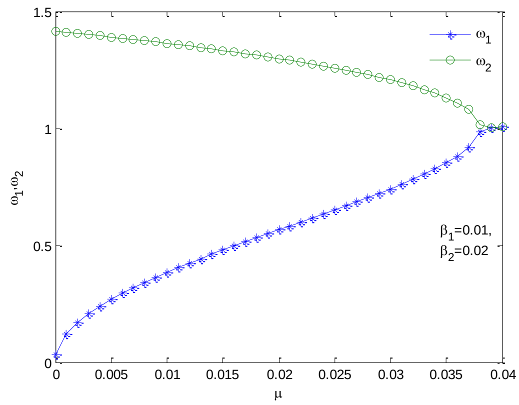

Fig. 14: Correlation between μ and ω_1, ω_2 for e=0.0489 when β_1=.01, β_2=0.02.

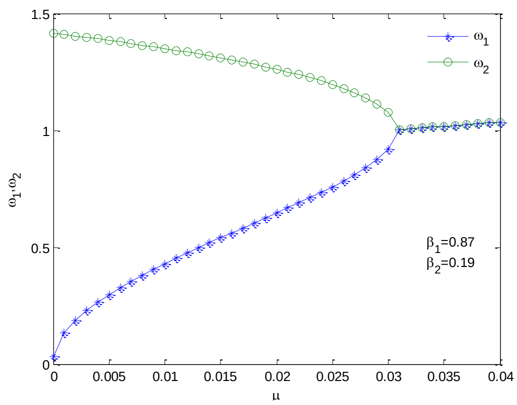

Fig. 15: Correlation between μ and ω_1, ω_2 for e=0.0489 when β_1=0.87, β_2=0.19.

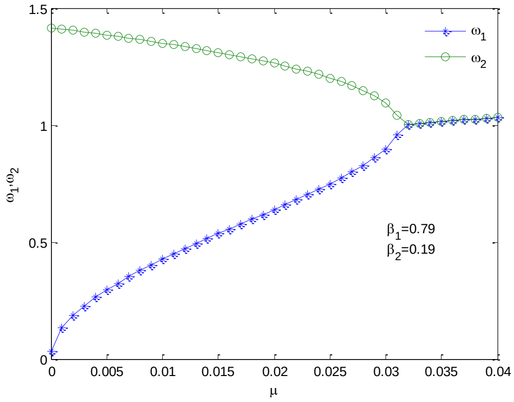

Fig. 16: Correlation between μ and ω_1, ω_2 for e=0.0489 when β_1=0.79, β_2=0.19.

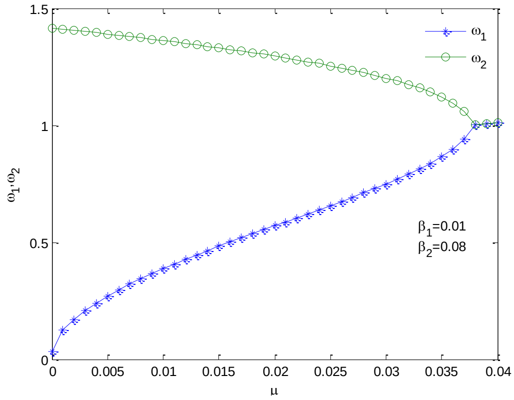

Fig. 17: Correlation between μ and ω_1, ω_2 for e=0.0489 when β_1=0.01, β_2=0.08.

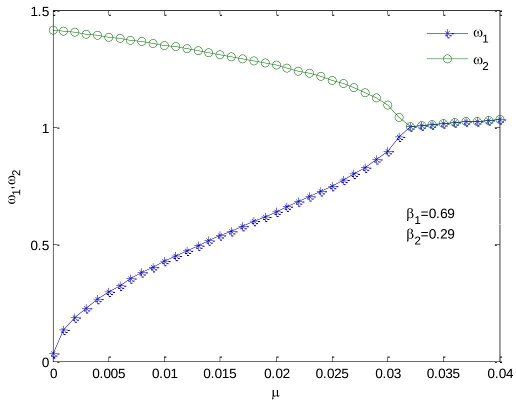

Fig. 18: Correlation between μ and ω_1, ω_2 for e=0.0489 when β_1=0.69, β_2=0.29.

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Figs .19 -22 are depicting correlation between e and μ for various values of radiation pressure.

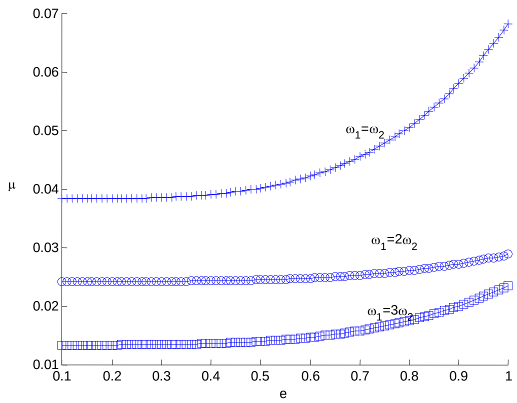

Fig. 19: Correlation between e and μ when β_1 =.01and β_2 =0.02.

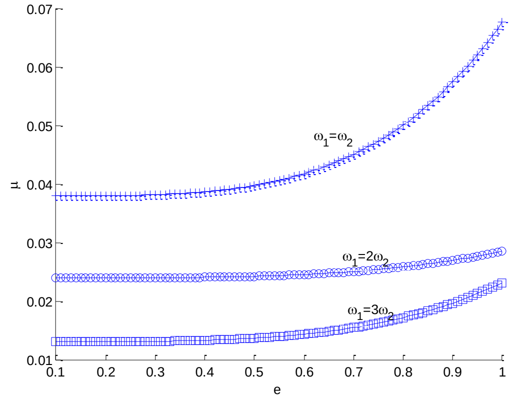

Fig .20: Correlation between e and μ when β_1 =.03 and β_2 =0.04.

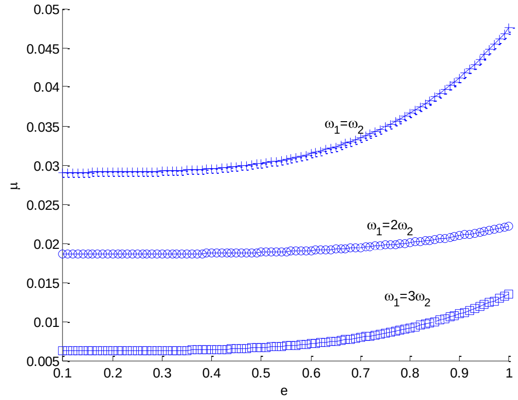

Fig .21: Correlation between e and μ when β_1 =0.87 and β_2 =0.19.

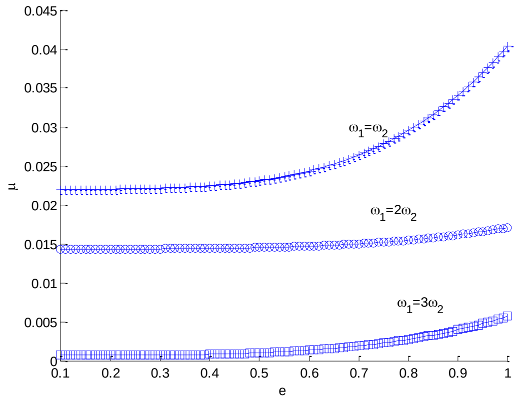

Fig .22: Correlation between e and μ when β_1 =0.87 and β_2 =0.99.

Figs 23-25 are representing correlation between β_2 and μ for e=0.0489 which is shown below.

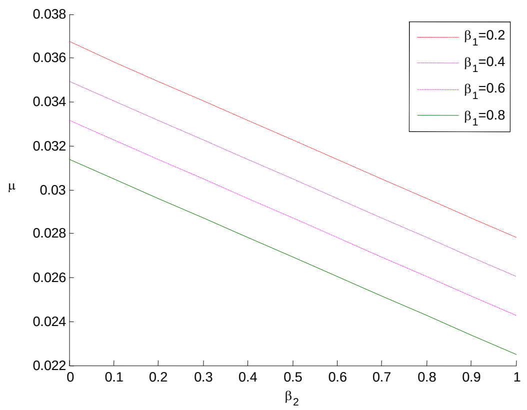

Fig. 23: Correlation between β_2 and μ when ω_1= ω_2 for e=0.0489.

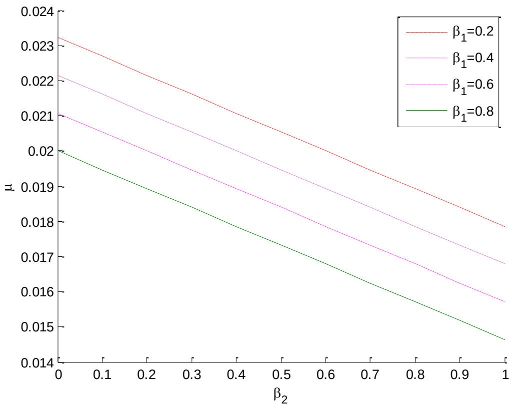

Fig. 24: Correlation between β_2 and μ when $\omega_1= 2\omega_2$ for e=0.0489.

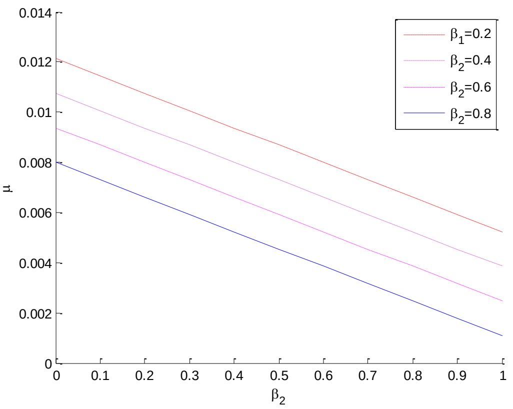

Fig. 25: Correlation between β_2 and μ when $\omega_1= 3\omega_2$ for e=0.0489.

6. Normalization and higher order stability of the libration points

In order to investigate the stability of the libration point, the Hamiltonian H is normalized by Birkhoff's method to the following form and then KAM theorem is applied.

$$H = \omega_1 r_1 - \omega_2 r_2 + c_{20} r_1^2 + c_{11} r_1 r_2 + c_{02} r_2^2 + o(r_1 + r_2)^{5/2} \tag{6.1}$$

In this section KAM theory is applied which can be stated as follows:

If the Hamiltonian of the perturbed motion is such that:

1) The characteristics equation of the system with H has purely imaginary roots,
2) $n_1\omega_1 + n_2\omega_2 \neq 0$, where n_1, n_2 are integers satisfying , $0 < |n_1| + |n_2| \leq 4$,
3) $D = c_{20}\omega_2^2 + c_{11}\omega_1\omega_2 + c_{02}\omega_1^2 \neq 0$,

Then, the equilibrium points are stable.

In order to analyse further we applied KAM theorem, for which linear canonical transformation of variations is used as given by:

$$(q_1, q_2, p_1, p_2) = (q'_1, q'_2, p'_1, p'_2)N \tag{6.2}$$

Where,

$$N = \begin{bmatrix} a_1 & a_1 c_1 & -a_1 c_1 & a_1\left(1-\omega_1^2 b_1\right) \\ a_2 & a_2 c_2 & -a_2 c_2 & a_2\left(1-\omega_2^2 b_2\right) \\ 0 & a_1 b_1 & a_1(1-b_1) & a_1 c_1 \\ 0 & -a_2 b_2 & -a_2(1-b_2) & -a_2 c_2 \end{bmatrix} \tag{6.3}$$

and

$$a_1 = \frac{1}{2}\left(\frac{2l_1}{\omega_1^2 - 1/2}\right)^{1/2}; a_2 = \frac{1}{2}\left(\frac{2l_2}{\omega_2^2 - 1/2}\right)^{1/2}$$
$$l_1 = 1 + \omega_1^2 + 2H_{02}; \quad l_2 = 1 + \omega_2^2 + 2H_{02}$$

$$b_1 = \frac{2}{l_1}; \quad b_2 = \frac{2}{l_2}$$

$$c_1 = \frac{-H_{11}}{l_1}; c_2 = \frac{-H_{11}}{l_2} \tag{6.4}$$

The transformation (6.2) reduces the Hamiltonian to the following form:

$$H = \frac{1}{2}\left(p_1'^2 + \omega_1^2 q_1'^2\right) - \frac{1}{2}\left(p_2'^2 + \omega_2^2 q_2'^2\right) + \sum_{\alpha+\delta=3}^{\infty} h_{\alpha_1,\alpha_2\delta_1,\delta_2} q_1'^{\alpha_1} q_2'^{\alpha_2} p_1'^{\delta_2} p_2'^{\delta_2} \tag{6.5}$$

where, $\alpha=\alpha_1+\alpha_2$; $\delta=\delta_1+\delta_2$.

The Hamiltonian H_3 can be expanded in the following form:

$$H_3 = H_{0003} p_2^3 + H_{0030} p_1^3 + H_{0300} q_2^3 + H_{3000} q_1^3 + H_{2100} q_1^2 q_2 + H_{2010} q_1^2 p_1 + \ldots\ldots\ldots\ldots + H_{1011} q_1 p_1 p_2 \tag{6.6}$$

Similarly H_4 can be expanded as:

$$H_4 = H_{0004} p_2^4 + H_{0040} p_1^4 + H_{0400} q_2^4 + H_{4000} q_1^4 + \ldots\ldots\ldots\ldots + H_{1111} q_1 q_2 p_1 p_2 \tag{6.7}$$

The coefficient of third and fourth order terms of $h'_{\alpha_1\alpha_2\delta_1\delta_2}$ and $h_{\alpha_1\alpha_2\delta_1\delta_2}$ can be given as:

$$h_{0030} = a_1^3 b_1^3 H_{0300};$$
$$h_{3000} = a_1^3 (H_{3000} + c_1 H_{2100} + c_1^2 H_{2100} + c_1^3 H_{0300});$$
$$h_{1020} = a_1^3 b_1^2 (H_{1200} + 3c_1 H_{0300});$$
$$h_{2010} = a_1^3 b_1 (H_{2100} + 2c_1 H_{1200} + 3c_1^2 H_{0300});$$

$$h_{0120} = a_1^2 a_2 b_1^2 (H_{1200} + 3c_2 H_{0300});$$
$$h_{2001} = -a_1^2 a_2 b_2 (H_{2100} + 2c_1 H_{1200} + 3c_1^2 H_{0300});$$

$$h_{1011} = -2a_1^2 a_2 b_1 b_2 (H_{1200} + 3c_1 H_{0300});$$
$$h_{1110} = 2a_1^2 a_2 b_1 [H_{2100} + (c_1 + c_2) H_{1200} + 3c_1 c_2 H_{0300}];$$

$$h_{0021} = -3a_1^2 b_1^2 b_2 H_{0300};$$
$$h_{2100} = a_1^2 a_2 \, [3H_{3000} + (2c_1 + c_2) H_{2100} + c_1(c_1 + 2c_2) H_{1200} + 3c_1^2 c_2 H_{0300}];$$

$$h_{1002} = a_1 b_1^2 a_2^2 (H_{1200} + 3c_1 H_{0300});$$

$$h_{0210} = a_1 a_2^2 b_1 (H_{2100} + 2c_2 H_{1200} + 3c_2^2 H_{0300});$$

$$h_{0012} = 3a_1 a_2^2 b_2^2 b_1 H_{0300};$$

$$h_{1200} = a_2^2 a_1 [3H_{3000} + (c_1 + 2c_2) H_{2100} + c_2 (2c_1 + c_2) H_{1200} + 3c_1 c_2^2 H_{0300}];$$

$$h_{0111} = -2a_1^2 a_2{}^2 b_1 b_2 (H_{1200} + 3c_2 H_{0300});$$

$$h_{1101} = -2a_2^2 a_1 b_2 [H_{2100} + (c_1 + c_2) H_{1200} + 3c_1 c_2 H_{0300}];$$

$$h_{0102} = a_2^3 b_{21}^2 (H_{1200} + 3c_2 H_{0300}); \; h_{0201} = -a_2^3 b_2 (H_{2100} + 2c_2 H_{1200} + 3c_2^2 H_{0300});$$

$$h_{0102} = a_2^3 b_2{}^2 (H_{1200} + 3c_2 H_{0300}); \; h_{0003} = -a_2^3 b_2{}^3 H_{0300};$$

$$h_{0300} = a_2^3 (H_{3000} + c_2 H_{2100} + c_2^2 H_{1200} + c_2^3 H_{0300}); \quad h_{0040} = a_1^4 b_1{}^4 H_{0400};$$

$$h_{4000} = a_1^4 (H_{4000} + c_1 H_{3100} + c_1^2 H_{2200} + c_1^3 H_{1300} + c_1^4 H_{0400});$$

$$h_{2020} = a_1^4 b_1^2 (H_{2200} + 3c_1 H_{1300} + 6c_1^2 H_{0400}); \; h_{0022} = 6a_1^2 a_2^2 b_1^2 b_2{}^2 H_{0400};$$

$$h_{2200} = a_1^2 a_2{}^2 [6H_{4000} + 3(c_1 + c_2) H_{3100} + (c_1{}^2 + 4c_1 c_2 + c_2^2) H_{2200} + 3c_1 c_2 (c_1 + c_2) H_{1300} + 6c_1^2 c_2^2 H_{0400}];$$

$$h_{2002} = a_1^2 a_2^2 b_2{}^2 (H_{2200} + 3c_1 H_{1300} + 6c_1^2 H_{0400});$$

$$h_{0220} = a_1^2 a_2^2 b_1{}^2 (H_{2200} + 3c_2 H_{1300} + 6c_2^2 H_{0400});$$

$$h_{0004} = a_2^4 b_2{}^4 H_{0400};$$

$$h_{0400} = a_2^4 (H_{4000} + c_2 H_{3100} + c_2^2 H_{2200} + c_2^3 H_{1300} + c_2^4 H_{0400});$$

$$h_{0202} = a_2^4 b_2{}^2 (H_{2200} + 3c_2 H_{1300} + 6c_2^2 H_{0400}); \; h_{0013} = -4a_1 a_2^3 b_1 b_2{}^3 H_{0400};$$

$$h_{1300} = a_1 a_2{}^3 [4H_{4000} + (c_1 + 3c_2) H_{3100} + 4c_1 c_2^3 H_{0400} + 2c_2 (c_1 + c_2) H_{2200} + c_2^2 (3c_1 + c_2) H_{1300}];$$

$$h_{1102} = a_1 a_2{}^3 b_2^2 [2H_{2200} + 3(c_1 + c_2) H_{1300} + 12c_1 c_2 H_{0400}];$$

$$h_{0211} = -2a_2^3 a_1 b_1 b_2 [H_{2200} + 3c_2 H_{1300} + 6c_2{}^2 H_{0400}];$$

$$h_{0112} = 3a_1 a_2^3 b_1 b_2{}^2 (H_{1300} + 4c_1 H_{0400});$$

$$h_{1003} = -a_1 a_2^3 b_2{}^3 (H_{1300} + 4c_1 H_{0400});$$

$$h_{1201} = -a_1 a_2{}^3 b_2 [3H_{3100} + 2(c_1 + 2c_2) H_{2200} + 3c_2 (2c_1 + c_2) H_{1300} + 12c_1 c_2^2 H_{0400}];$$

$$h_{0310} = a_1 a_2^3 b_1 (H_{3100} + 2c_2 H_{2200} + 3c_2^2 H_{1300} + 4c_2^2 H_{0400}); \tag{6.8}$$

Again the Hamiltonian is reduced to a more convenient form which is suitable for further investigation by using the following canonical transformation:

$$q'_1 = \frac{1}{2} q''_1 + \frac{i}{\omega_1} p''_1;$$

$$p'_1 = \frac{1}{2} i\,\omega_1 q''_1 + p''_1;$$

$$q'_2 = -\frac{i}{2} q''_2 + \frac{1}{\omega_2} p''_2;$$

$$p'_2 = -\frac{1}{2} \omega_2 q''_2 + i p''_2. \tag{6.9}$$

Thus, the Hamiltonian (6.5) may be written as:

$$H = i\,\omega_1 q_1'' p_1'' + i\,\omega_2 q_2'' p_2'' + \sum_{\alpha+\delta=3}^{\infty} h'_{\alpha_1,\alpha_2 \delta_1,\delta_2} q_1''^{\alpha_1} q_2''^{\alpha_2} p_1''^{\delta_1} p_2''^{\delta_2} \tag{6.10}$$

If $h'_{\alpha_1\alpha_2\delta_1\delta_2} = x_{\alpha_1\alpha_2\delta_1\delta_2} + i y_{\alpha_1\alpha_2\delta_1\delta_2}$ then,

$$x_{0030} = h_{0030} - \frac{1}{\omega_1^2} h_{2010}; \qquad y_{0030} = \frac{1}{\omega_1} h_{1020} - \frac{1}{\omega_1^3} h_{3000};$$

$$x_{1020} = -\frac{1}{2} h_{1020} - \frac{3}{2\omega_1^2} h_{3000}; \quad y_{1020} = \frac{3\omega_1}{2} h_{0030} + \frac{1}{2\omega_1} h_{2010};$$

$$x_{0120} = -\frac{\omega_2}{2} h_{0021} + \frac{1}{2\omega_1} h_{1110} + \frac{\omega_2}{2\omega_1{}^2} h_{2001};$$

$$y_{0120} = -\frac{1}{2} h_{0120} - \frac{\omega_2}{2\omega_1} h_{1011} + \frac{1}{2\omega_1{}^2} h_{2100};$$

$$x_{1011} = -\omega_1 h_{0021} - \frac{1}{\omega_1} h_{2001}; \quad y_{1011} = \frac{\omega_1}{\omega_2} h_{0120} + \frac{1}{\omega_1 \omega_2} h_{2100};$$

$$x_{1002} = -\frac{\omega_1}{2\omega_2} h_{0111} - \frac{1}{2} h_{1002} + \frac{1}{2\omega_2^2} h_{1200};$$

$$y_{1002} = -\frac{\omega_1}{2} h_{0012} + \frac{\omega_1}{2\omega_2^2} h_{0210} + \frac{1}{2\omega_2^2} h_{1101};$$

$$x_{0012} = -h_{0012} + \frac{1}{\omega_2^2} h_{0210} - \frac{1}{\omega_1 \omega_2} h_{1101};$$

$$y_{0012} = \frac{1}{\omega_2} h_{0111} - \frac{1}{\omega_1} h_{1002} + \frac{1}{\omega_1 \omega_2^2} h_{1200};$$

$$x_{0111}=\frac{\omega_2}{\omega_1}h_{1002}+\frac{1}{\omega_1\omega_2}h_{1200};\ y_{0111}=-\omega_2 h_{0012}-\frac{1}{\omega_2}h_{0210};$$

$$x_{0201}=\frac{-\omega_2}{4}h_{0102}-\frac{3}{4\omega_2}h_{0300};\ y_{0201}=\frac{3\omega_2^2}{4}h_{0003}+\frac{1}{4}h_{0201};$$

$$x_{0003}=\frac{-1}{\omega_2}h_{0102}+\frac{1}{\omega_2^{\,3}}h_{0300};$$

$$y_{0003}=-h_{0003}+\frac{1}{\omega_2^{\,2}}h_{0201}; \tag{6.11}$$

The other coefficients of third order terms are obtained by the formula:

$$h'_{\alpha_1\alpha_2\delta_1\delta_2}=(y_{\alpha_1\alpha_2\delta_1\delta_2}+ix_{\alpha_1\alpha_2\delta_1\delta_2})\times\left(\frac{-\omega_1}{2}\right)^{\delta_1-\alpha_1}\left(\frac{\omega_2}{2}\right)^{\delta_2-\alpha_2}$$

Using Birkhoff's transformation $(q_j^{''},p_j^{''})\to(q_j^{'''},p_j^{'''})$, all the third order terms from the Hamiltonian (6.10) is nullified provided the third order resonance does not occur. This transformation is introduced by means of the generating function which is given as follows:

$$s=q_1^{''}p_1^{'''}+q_2^{''}p_2^{'''}+s_3+s_4; \tag{6.12}$$

where, $q_i^{'''}=q_i^{''}+\dfrac{\partial s_3}{\partial p_i^{''}}+\dfrac{\partial s_4}{\partial p_i^{''}};$

$$p_i^{''}=p_i^{'''}+\frac{\partial s_3}{\partial q_i^{''}}+\frac{\partial s_4}{\partial q_i^{''}};\ (i=1,2) \tag{6.13}$$

Using equation (6.4) and (6.6), expanding and equating the terms of the same degree on the two sides, we obtain:

$$H'_2(q_1^{''},q_2^{''},p_1^{'''},p_2^{'''})=H_2((q_1^{''},q_2^{''},p_1^{''},p_2^{''})$$

$$\sum_{i=1}^{2}\left[-\frac{\partial s_3}{\partial p_i^{''}}\frac{\partial H'_2}{\partial p_i^{''}}+\frac{\partial s_3}{\partial q_i^{''}}\frac{\partial H_2}{\partial p_i^{''}}\right]+$$
$$H_3(q_1^{''},q_2^{''},p_1^{''},p_2^{''})=H'_3(q_1^{''},q_2^{''},p_1^{''},p_2^{''})=0;$$

$$\sum_{i=1}^{2}\left[-\frac{\partial s_4}{\partial p_i^{''}}\frac{\partial H'_2}{\partial q_i^{''}}+\frac{\partial s_4}{\partial q_i^{''}}\frac{\partial H_2}{\partial p_i^{''}}\right]+K_4=0;$$

$$H'_4=\sum_{i=1}^{2}\left[-\frac{\partial s_3}{\partial p_i^{'''}}\frac{\partial H'_3}{\partial q_i^{''}}+\frac{\partial s_3}{\partial q_i^{''}}\frac{\partial H_3}{\partial p_i^{'''}}\right]+H_4-K_4; \tag{6.14}$$

where, K_4 is the term other than the homogeneous ones in q_1, p_1 and q_2, p_2. In equation (6.14) the new variables $q_1^{''}$ *and* $p_2^{''}$ can be replaced by $q_1^{''}$ *and* $p_2^{''}$ by implicit function theorem.

Since our system is autonomous we have: $\dfrac{\partial s_3}{\partial t}=\dfrac{\partial s_4}{\partial t}=0$.

Again, if we put

$$H_3=\sum_{\alpha+\delta=3}^{\infty}h_{\alpha_1,\alpha_2\delta_1,\delta_2}q_1^{'''\alpha_1}q_2^{'''\alpha_2}p_1^{'''\delta_1}p_2^{'''\delta_2};\ s_3=\sum_{\alpha+\delta=3}^{\infty}g_{\alpha_1,\alpha_2\delta_1,\delta_2}q_1^{'''\alpha_1}q_2^{'''\alpha_2}p_1^{'''\delta_1}p_2^{'''\delta_2}$$

Using (6.14), we get:

$$g_{\alpha_1,\alpha_2\delta_1,\delta_2}=\frac{ih'_{\alpha_1,\alpha_2\delta_1,\delta_2}}{(\alpha_1-\delta_1)\omega_1+(\alpha_2-\delta_2)\omega_2}$$

With the help of the third equation of (6.14), the new Hamiltonian inclusive of the fourth order terms is given as:

$$H'=i\,\omega_1 q_1^{'''}p_1^{'''}+i\,\omega_2 q_2^{'''}p_2^{'''}-c_{20}(q_1^{'''}p_1^{'''})^2$$
$$+c_{11}(q_1^{'''}p_1^{'''})(q_2^{'''}p_2^{'''})-c_{02}(q_2^{'''}p_2^{'''})^2+...$$

where,

$$K_4=H_4(q_1^{'''},q_2^{'''},p_1^{'''},p_2^{'''})-h'_{2020}q_1^{'''2}p_1^{'''2}$$
$$-h'_{1111}(q_1^{'''}p_1^{'''})(q_2^{'''}p_2^{'''})-h'_{0202}q_2^{'''2}p_2^{'''2}; \tag{6.15}$$

$$c_{20}=-h'_{2020}-\frac{3\omega_1^2}{8}(x_{0030}^{\,2}+y_{0030}^{\,2})-\frac{3}{2}(x_{1020}^{\,2}+y_{1020}^{\,2})+\frac{1}{2}(x_{1011}^{\,2}+y_{1011}^{\,2})$$
$$-\frac{\omega_1^2}{2\omega_2(2\omega_1-\omega_2)}(x_{0120}^{\,2}+y_{0120}^{\,2})+\frac{\omega_1^2\omega_2}{8(2\omega_1+\omega_2)}(x_{0021}^{\,2}+y_{0021}^{\,2})$$

$$c_{11}=h'_{1111}-\frac{2\omega_2^2}{\omega_1(\omega_1-2\omega_2)}(x_{1002}^{\,2}+y_{1002}^{\,2})+\frac{\omega_1\omega_2^2}{2(\omega_1+2\omega_2)}(x_{0012}^{\,2}+y_{0012}^{\,2})$$
$$-\frac{\omega_2\omega_1^2}{2(2\omega_1+\omega_2)}\times(x_{0021}^{\,2}+y_{0021}^{\,2})-\frac{2\omega_1^2}{(2\omega_1-\omega_2)\omega_2}(x_{0120}^{\,2}+y_{0120}^{\,2})$$
$$+2(x_{0111}x_{1020}+y_{0111}y_{1020})-\frac{4}{\omega_2}(x_{0201}y_{1011}+x_{1011}y_{0201};)$$

$$c_{02}=h'_{0202}+\frac{3\omega_2^2}{8}(x_{0003}^{\,2}+y_{0003}^{\,2})+\frac{6}{\omega_2^2}(x_{0201}^{\,2}+y_{0201}^{\,2})-\frac{\omega_2^2}{2\omega_1(\omega_1-2\omega_2)}(x_{1002}^{\,2}+y_{1002}^{\,2})$$
$$-\frac{1}{2}(x_{0111}^{\,2}+y_{0111}^{\,2})-\frac{\omega_1\omega_2^2}{8(\omega_1+2\omega_2)}(x_{0012}^{\,2}+y_{0012}^{\,2});$$

$$h'_{2020}=-\frac{3}{2}\omega_1^2 h_{0040}-\frac{3}{2\omega_1^2}h_{4000}-\frac{1}{2}h_{2020};$$

$$h'_{1111}=\omega_1\omega_2 h_{0022}+\frac{1}{\omega_1\omega_2}h_{2200}+\frac{\omega_1}{\omega_2}h_{0220}+\frac{\omega_2}{\omega_1}h_{2002};$$

$$h'_{0202} = \frac{-3}{2}\omega_2^{\ 2} h_{0004} - \frac{3}{2\omega_2^{\ 2}} h_{0400} - \frac{1}{2} h_{0202};$$

(6.16)

Now, KAM theorem is applied where the values of $D = c_{20}\omega_2^2 + c_{11}\omega_1\omega_2 + c_{02}\omega_1^2 \neq 0$ is calculated. Tables 3 and 4 show different values of D satisfying $\beta_1+\beta_2<1$ and $\beta_1+\beta_2>1$ respectively. Table 5 shows the values of D for the binary systems (Achird, Luyten, α Cen- AB, Kruger 60, Xi Bootis) .It is found that D≠0, in all the cases of radiation pressures taken in general and in binary systems in particular. (See Appendices for table)

7. Discussion and conclusion

The existence of resonance and stability of the Lagrangian triangular equilibrium points in ER3BP is investigated considering both the body as radiating; under the non-resonance case using KAM theory in circular as well as elliptical cases. This work is generalization of the work of [Narayan and Singh (2014)] by considering the third and fourth order resonance for different values of radiation pressures satisfying both the conditions

$\beta_1+\beta_2>1; \beta_1+\beta_2<1, \beta_1, \beta_2<1.$ in general and binary systems in particular. It is noticed that the system experiences resonance at $\omega_1 = 2\omega_2, \omega_1 = 3\omega_2$ for different values of radiation pressures in circular and elliptical case. The case $\omega_1 = \omega_2$ corresponds to the boundary region of the stability for the system both in circular and elliptical cases .It is observed that resonance of the third and fourth order exists in ER3BP under the radiating primaries for all values of μ, β_1, β_2 taken.

Also it is found that in case of binary systems $\omega_1 = \omega_2$ occurs since $\mu \geq 0.0385209$.

It is clear from table (4) that for $\mu = 0.025, \beta_1 = 0.1, \beta_2 = 0.5, D = -5803.5$; and for $\mu = 0.025, \beta_1 = 0.1, \beta_2 = 0.05, D = 8846.7$ there is change in the sign of D for both circular and elliptical cases. This change in sign suggests that for $\mu = 0.025\, and\ \beta_1 = 0.1$ there exists $0.05 \leq \beta_2 \leq 0.5$ such that D vanishes. Similarly, it is observed that for $\mu = 0.025, \beta_1 = 0.1,$ there exists β_2 such that $0.04 \leq \beta_2 \leq 0.4$ for which D=0 in both the circular and elliptical cases. It is noticed that, expect for the two cases discussed, KAM theorem is applicable, hence showing that the triangular equilibrium points are stable.

It is seen that the binary systems (Achird, Luyten, α Cen- AB, Kruger 60, and Xi Bootis) are stable as D≠0. It is also found that except for some cases for all values of the radiation pressures and for $\mu < 0.0385209$ the triangular equilibrium points are stable.

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Appendices:

Table 1: Values of μ at $\omega_1=\omega_2$, $\omega_1=2\omega_2$, $\omega_1=3\omega_2$ when $\beta_1+\beta_2>1$.

β_1	β_2	e=0			e=0.0489		
		μ_{01}	μ_{02}	μ_{03}	μ_{01e}	μ_{02e}	μ_{03e}
0.87	0.29	0.0304	0.0192	0.0107	0.0282	0.0181	0.0055
0.87	0.34	0.0301	0.0190	0.0106	0.0277	0.0178	0.0052
0.87	0.54	0.0290	0.0184	0.0103	0.0259	0.0167	0.0038
0.87	0.59	0.0288	0.0182	0.0102	0.0255	0.0165	0.0034
0.92	0.19	0.0306	0.0194	0.0108	0.0286	0.0183	0.0059
0.92	0.24	0.0304	0.0192	0.0107	0.0282	.0181	0.0055
0.92	0.64	0.0283	0.0194	0.0108	0.0246	0.0159	0.0027
0.82	0.39	0.0301	0.190	0.0106	0.0277	0.0178	0.0052
0.82	0.49	0.0296	0.0187	0.0104	0.0268	0.0173	0.0045
0.82	0.69	0.0285	0.0181	0.0101	0.0251	0.0162	0.0031
0.82	0.79	0.0281	0.0178	0.0099	0.0242	0.0157	0.0024
0.92	0.9	0.0312	0.0197	0.011	0.0223	0.0148	0.0009
0.92	0.19	0.0306	0.0194	0.0108	0.0286	0.0183	0.0059
0.92	0.79	0.0276	0.0175	0.0098	0.0233	0.0151	0.0017
0.92	0.89	0.0272	0.0172	0.0096	0.0224	0.0146	0.0010
0.72	0.49	0.0301	0.0190	0.0106	0.0277	0.0178	0.0052
0.72	0.59	0.0296	0.0187	0.0104	0.0268	0.0173	0.0045
0.72	0.89	0.0281	0.0178	0.0099	0.0242	0.0157	0.0024
0.72	0.99	0.0276	0.0175	0.0098	0.0233	0.0151	0.0017
0.82	0.39	0.0301	0.0190	0.0106	0.0277	0.0178	0.0052
0.82	0.89	0.0276	0.0175	0.0098	0.0233	0.0151	0.0017
0.82	0.99	0.0272	0.0172	0.0096	0.0224	0.0146	0.0010
0.92	0.09	0.0312	0.0197	0.0110	0.0295	0.0189	0.0065
0.92	0.19	0.0301	0.0190	0.0106	0.0268	0.0183	0.0059
0.70	0.40	0.0307	0.0194	0.0108	0.0287	0.0184	0.0059
0.70	0.50	0.0301	0.0191	0.0106	0.0278	0.0179	0.0052
0.90	0.09	0.0313	0.0198	0.0111	0.0297	0.0190	0.0067
0.90	0.19	0.0308	0.0195	0.0109	0.0288	0.0184	0.0060
0.30	0.80	0.0307	0.0194	0.0108	0.0287	0.0184	0.0059
0.30	0.90	0.0301	0.0191	0.0106	0.0278	0.0179	0.0052
0.30	0.80	0.0307	0.0194	0.0100	0.0287	0.0184	0.0059
0.50	0.60	0.0307	0.0194	0.0108	0.0287	0.0184	0.0059
0.50	0.70	0.0301	0.0191	0.0106	0.0278	0.0179	0.0052
0.60	0.50	0.0296	0.0187	0.0105	0.0287	0.0184	0.0059
0.60	0.60	0.0301	0.0191	0.0106	0.0278	0.0179	0.0052
0.70	0.40	0.0307	0.0194	0.0108	0.0287	0.0184	0.0059
0.70	0.50	0.0301	0.0191	0.0106	0.0278	0.0179	0.0052
0.20	0.70	0.0319	0.0202	0.0112	0.0260	0.0168	0.0038
0.20	0.81	0.0313	0.0198	0.0110	0.0295	0.0189	0.0065
0.80	0.80	0.0281	0.0178	0.0099	0.0243	0.0157	0.0025
0.80	0.90	0.0276	0.0175	0.0098	0.0234	0.0152	0.0018
0.90	0.70	0.0281	0.0178	0.0099	0.0243	0.0157	0.0025
0.90	0.80	0.0276	0.0175	0.0098	0.0234	0.0152	0.0018

Table 2: Values of μ at $\omega_1=\omega_2$, $\omega_1=2\omega_2$, $\omega_1=3\omega_2$ when $\beta_1+\beta_2<1$.

β_1	β_2	e=0			e=0.0489		
		μ_{01}	μ_{02}	μ_{03}	μ_{01e}	μ_{02e}	μ_{03e}
0.01	0	0.038431	0.0242386	0.013485	0.0384	0.0242	0.0134
0.01	0.01	0.038343	0.0241836	0.013455	0.0383	0.0242	0.0134
0.01	0.02	0.038255	0.0241289	0.013425	0.0383	0.0241	0.0133
0.01	0.03	0.038167	0.0240744	0.013395	0.0382	0.0241	0.0132
0.01	0.04	0.038080	0.0240201	0.013365	0.0381	0.0240	0.0132
0.01	0.05	0.037993	0.0239661	0.013335	0.0380	0.0240	0.0131
0.01	0.06	0.037906	0.0291243	0.013330	0.0379	0.0239	0.0130
0.01	0.07	0.037820	0.0238589	0.013270	0.0378	0.0239	0.0130
0.01	0.08	0.037734	0.0238056	0.013247	0.0377	0.0238	0.0129
0.001	0.01	0.038423	0.0242331	0.013482	0.0384	0.0242	0.0134
0.001	0.02	0.038334	0.0241781	0.013452	0.0383	0.0242	0.0134
0.001	0.03	0.038426	0.0241234	0.013422	0.0382	0.0241	0.0133
0.001	0.04	0.038158	0.0240690	0.013392	0.0382	0.0241	0.0132
0.001	0.05	0.038071	0.0240101	0.013362	0.0381	0.0240	0.0132
0.001	0.06	0.037984	0.0239608	0.133327	0.0380	0.0240	0.0131
0.001	0.07	0.037898	0.0239070	0.013303	0.0379	0.0239	0.0130
0.001	0.08	0.037811	0.0238535	0.013273	0.0378	0.0239	0.0130
0.1	0.01	0.037564	0.0236990	0.013189	0.0375	0.0237	0.0128
0.1	0.02	0.037479	0.0236472	0.013160	0.0375	0.0237	0.0127
0.1	0.03	0.037395	0.0235948	0.013131	0.0374	0.0236	0.0126
0.1	0.04	0.037311	0.0235427	0.013102	0.0373	0.0235	0.0126
0.1	0.05	0.037228	0.0234908	0.013074	0.0372	0.0235	0.0125
0.1	0.06	0.037145	0.0234392	0.013045	0.0371	0.0234	0.0124
0.1	0.07	0.037062	0.0233878	0.013017	0.0370	0.0234	0.0123
0.1	0.08	0.036980	0.0233366	0.012989	0.0369	0.0233	0.0123

Table 3: Values of D when $\beta_1+\beta_2>1$

μ	β_1	β_2	Values of D at e=0.0489	Values of D at e=0
0.001	0.87	0.29	-1.9573×10^7	-1.7809×10^7
0.001	0.87	0.34	-1.8077×10^7	-1.6448×10^7
0.001	0.87	0.54	-1.3725×10^7	-1.2489×10^7
0.001	0.87	0.59	-1.2980×10^7	-1.1812×10^7
0.001	0.92	0.19	-2.3668×10^7	-2.1534×10^7
0.001	0.92	0.24	-2.1707×10^7	-1.9750×10^7
0.001	0.92	0.59	-1.3136×10^7	-1.1953×10^7
0.001	0.92	0.64	-1.2480×10^7	-1.1357×10^7
0.005	0.82	0.39	-1.8242×10^5	$--1.3089\times10^5$
0.005	0.82	0.49	-1.6578×10^5	-1.5125×10^5
0.005	0.82	0.69	-1.4942×10^5	-1.3658×10^5
0.005	0.82	0.79	-1.4890×10^5	-1.3626×10^5
0.005	0.92	0.09	-1.5074×10^5	-2.6245×10^5
0.005	0.92	0.19	-2.4571×10^5	-2.2368×10^5
0.005	0.92	0.79	-1.4705×10^5	-1.3442×10^5
0.005	0.92	0.89	-1.5015×10^5	-1.3737×10^5
0.010	0.72	0.49	-3.4213×10^5	-1.4939×10^5
0.010	0.72	0.59	-3.3800×10^4	-1.4107×10^4
0.010	0.72	0.89	-3.9107×10^4	-3.6163×10^4
0.010	0.72	0.99	-4.3317×10^4	-4.0109×10^4
0.010	0.82	0.39	-3.7221×10^4	-3.2987×10^4
0.010	0.82	0.89	-3.7221×10^4	-3.4341×10^4
0.010	0.82	0.99	-4.0951×10^4	-3.7835×10^4
0.010	0.92	0.09	-5.2180×10^4	-4.7519×10^4
0.015	0.70	0.40	-2.0233×10^4	-1.8562×10^4
0.015	0.70	0.50	-2.1860×10^4	-1.8523×10^4
0.015	0.80	0.30	-2.1816×10^4	-1.9943×10^4
0.015	0.90	0.19	-2.5133×10^4	-2.2915×10^4
0.015	0.30	0.80	-2.7543×10^4	-2.5753×10^4
0.015	0.30	0.90	-3.1119×10^4	-2.9130×10^4
0.015	0.30	0.80	-2.7543×10^4	-2.5753×10^4
0.025	0.50	0.60	-3.8796×10^3	-3.7364×10^3
0.025	0.50	0.70	-5.0133×10^3	-4.8078×10^3
0.025	0.50	0.50	-2.2754×10^3	-2.7278×10^3
0.025	0.60	0.60	-3.2053×10^3	-3.0871×10^3
0.025	0.70	0.40	-1.0171×10^3	-1.0183×10^3
0.025	0.70	0.50	-1.8149×10^3	-1.7656×10^3
0.025	0.70	0.70	-3.4763×10^3	-3.3318×10^3
0.025	0.70	0.80	-4.4176×10^3	-4.2228×10^3
0.025	0.80	0.80	-3.6712×10^3	-3.5071×10^3
0.030	0.50	0.60	-4.100×10^3	-3.9019×10^3
0.030	0.50	0.70	-4.9224×10^3	-4.6827×10^3
0.030	0.50	0.80	-5.8530×10^3	$-5.5678\times10^3 +64.951i$
0.030	0.50	0.90	-6.8962×10^3	$-6.5618\times10^3+83.781i$
0.030	0.60	0.70	-4.2184×10^3	$-4.008\times10^3 +80.513i$
0.035	0.40	0.80	-5.8297×10^3	$-5.5488\times10^3 +1.064\times10^2 i$
0.035	0.40	0.70	-4.9641×10^3	$-4.7236\times10^3 +1.275\times10^2 i$
0.035	0.40	0.80	-5.829×10^3	$-5.5488\times10^3 +1.1064\times10^2 i$
0.035	0.80	0.90	-6.8488×10^3	$-3.3612\times10^3 +1.2392\times10^3 i$

Table 4: Values of D when $\beta_1+\beta_2<1$

μ	β_1	β_2	Values of D at e=0.0489	Values of D at e =0
0.001	0.01	0.02	-2.2480×10^7	-2.0457×10^7
0.001	0.01	0.03	-2.2083×10^7	-2.0097×10^7
0.001	0.01	0.04	-2.1696×10^7	-1.9744×10^7
0.001	0.01	0.05	-2.1317×10^7	-1.9400×10^7
0.005	0.01	0.002	-2.4321×10^5	-2.2218×10^7
0.005	0.01	0.003	-2.4321×10^5	-2.2187×10^7
0.005	0.01	0.004	-2.4253×10^5	-2.2156×10^7
0.005	0.01	0.005	-2.4218×10^5	-2.2125×10^7
0.010	0.001	0.002	-2.4269×10^5	-4.2750×10^4
0.010	0.001	0.003	-2.4235×10^5	-4.2715×10^4
0.010	0.001	0.004	-2.4200×10^5	-4.2680×10^4
0.010	0.001	0.005	-2.4166×10^5	-4.2645×10^4
0.015	0.10	0.20	-2.0880×10^4	-1.9354×10^4
0.015	0.10	0.30	-2.1141×10^4	-1.9657×10^4
0.015	0.10	0.40	-2.1961×10^4	-2.0437×10^4
0.015	0.10	0.50	-2.3345×10^4	-2.1777×10^4
0.025	0.01	0.02	1.9584×10^4	1.7642×10^4
0.025	0.01	0.03	1.7619×10^4	1.5852×10^4
0.025	0.01	0.04	1.5882×10^4	$1. .4268\times10^4$
0.025	0.01	0.05	1.4330×10^4	1.2856×10^4
0.025	0.10	0.50	-5.8035×10^3	-6.4762×10^3
0.025	0.10	0.05	8.8467×10^3	7.8920×10^3
0.025	0.10	0.005	1.3043×10^3	1.1721×10^3
0.025	0.10	0.40	-3.8410×10^3	-3.7766×10^3
0.025	0.10	0.04	9.6539×10^3	8.6290×10^3
0.030	0.10	0.50	-6.4125×10^3	-6.1125×10^3
0.030	0.10	0.05	-2.1105×10^3	-2.0514×10^3
0.030	0.10	0.40	-5.3322×10^3	-5.0878×10^3
0.030	0.10	0.04	-2.0190×10^3	-1.9659×10^3
0.035	0.10	0.04	-2.6500×10^3	-2.5216×10^3
0.035	0.10	0.004	-2.4638×10^3	-2.3454×10^3
0.035	0.10	0.20	-3.5610×10^3	-3.3861×10^3
0.035	0.10	0.02	-2.5459×10^3	-2.4231×10^3

Table 5: Values of D for Binary Systems

Binaries system	μ	β_1	β_2	eccentricity	Values of D
Achird	0.3949	0.0029	0.0003	0.4970	136.7−298.2i
Luyten	0.5	0.00002	0.000001	0.6200	9.4138−4.2464i
α Cen AB	0.4519	0.0029	0.15	0.5179	-62.9098−3.8063i
Kruger-60	0.3937	0.00008	0.00004	0.4100	-1311.6−2624i
Xi-bootis	0.4231	0.0012	0.0012	0.5117	-99.4820−3.3311i