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Research Paper

Effect of elliptic angle φ on the existence and stability of libration points in restricted three-body problem in earth-moon system considering earth as an ellipsoid

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Abstract

This paper deals with the existence and the stability of the earth-moon libration points in the restricted three-body problem. In this paper we have considered the bigger primary as an ellipsoid while the smaller one as a point-mass. This is observed that the collinear and non-collinear libration points exist only in the interval $0^\circ < \varphi < 45^\circ$. There exist three collinear libration points and the non-collinear libration points are forming a right triangle with the primaries. Further observed that the libration points either collinear or non-collinear all are unstable in $0^\circ < \varphi < 45^\circ$.

Keywords:*Restricted Three-Body Problem; Libration Points; Linear Stability; Elliptic Integrals; Newton-Raphson Method.*

1. Introduction

Restricted three-body problem is a well-known problem possesses five libration points; three are collinear and two are non-collinear. The non-collinear libration points form an equilateral triangle with the primaries and stable for a critical value of mass parameter μ_c such that $\mu < \mu_c = 0.0385208965\ldots$ while the collinear libration points are unstable for all values of μ. It has been studied by many scientists. In recent time, there are lots of papers in restricted three-body problem. Khanna and Bhatnagar (1999) have discussed the stationary solutions of the planar restricted three-body problem when the smaller primary is a triaxial rigid body with one of the axes as the axis of symmetry and its equatorial plane coinciding with the plane of motion. The bigger primary is taken as an oblate spheroid and its equatorial plane is also coinciding with the plane of motion. They have shown that there exist five libration points, two triangular and three collinear. The collinear points are unstable, while the triangular points are stable for the mass parameter $0 \le \mu < \mu_{crit}$ (the critical mass parameter) and the triangular points have long or short periodic elliptical orbits in the same range of μ. Raheem and Singh (2006) have investigated the stability of equilibrium points under the influence of small perturbations in the coriolis and centrifugal forces, together with the effects of oblateness and radiation pressures of the primaries. They have found that the collinear points remain unstable while the triangular points are stable for $0 \le \mu < \mu_c$ and unstable for $\mu_c \le \mu \le ½$, where μ_c is the critical mass parameter depends upon the coriolis force, centrifugal force, oblateness and radiation pressure of the primaries.

Narayan and Kumar (2011) have studied the effects of the oblateness and the photogravitational of the bigger primary and the oblateness of the smaller primary on the location of the triangular Lagrangian equilibrium points in the elliptical restricted three-body problem. They observed that the range of stability decreases as the oblateness and the radiation pressure parameter increases. Idrisi and Taqvi (2013) have studied the restricted three-body problem in terms of elliptic integrals in which smaller primary is an ellipsoid and bigger one a point mass. They have determined the equations of motion of the infinitesimal mass and then investigated the collinear and non collinear libration points and their stability in the linear sense. They have also proven in their work that there exist five collinear libration points L_i (i =1, 2,….5) and the non-collinear libration points lie on the arc of the unit circle centered at the bigger primary and both the collinear and non-collinear libration points are unstable for$0 < \mu \le ½$ and $0 \le \varphi \le \pi/2$. Singh et. al. (2013) have investigated the motion of an infinitesimal body in the generalized restricted three-body problem. It is generalized in the sense that both primaries are radiating, oblate bodies, together with the effect of gravitational potential from a belt. They have been found that, in addition to the usual five equilibrium points, there appear two new collinear points L_{n1}, L_{n2} due to the potential from the belt, and in the presence of all these perturbations, the equilibrium points L_1, L_3 come nearer to the primaries; while L_2, L_4, L_5, L_{n1} move towards the less massive primary and L_{n2} moves away from it. The collinear equilibrium points remain unstable, while the triangular points are stable for $0 < \mu < \mu_c$ and unstable for $\mu_c \le \mu \le ½$. Abouelmagdet. al. (2013) have studied the generalized restricted three-body problem, generalized in the sense that the effects of oblateness of the three participating bodies as well as the small perturbations in the coriolis and centrifugal forces are considered. They have shown the existence of equilibrium points, their linear stability and the periodic orbits around these points under these effects. They have found that the positions of the collinear points and y-coordinate of the triangular points are not affected by the small perturbations in the coriolis force while x-coordinate of the triangular points is neither affected by the small perturbations in the coriolis force nor the oblateness of the third body. Furthermore, the critical mass value and the elements of periodic orbits around the equilibrium points such as the semi-major and the semi-minor axes, the angular frequencies and corresponding periods may change by all the parameters of oblateness as well as the small perturbations in the coriolis and centrifugal forces. Reena Kumari et. al. (2014) have extended the basic model of the restricted four-body problem introducing two bigger dominant primaries m_1 and m_2 as oblate spheroids when masses of the two primary bodies (m_2 and m_3) are equal. The aim of this study is to investigate the use of zero velocity surfaces and the Poincaré surfaces of section to determine the possible allowed boundary regions and the stability orbit of the equilibrium points. According to different values of Jacobi constant C, we can determine a boundary region where the particle can move in possible permitted zones. The stability regions of the equilibrium points expanded due to presence of oblateness coefficient and various values of C, whereas for a certain range of t ($100 \le t \le 200$), orbits form a shape of cote's spiral. For different values of oblateness parameters A_1 ($0 < A_1 < 1$) and A_2 ($0 < A_2 < 1$), they obtained two collinear and six non-collinear equilibrium points. The non-collinear equilibrium points are stable when the mass parameter μ lies in the interval (0.0190637, 0.647603). However, basins of attraction are constructed with the help of Newton Raphson method to demonstrate the convergence as well as divergence region of the equilibrium points. The nature of basins of attraction of the equilibrium points are less affected in presence and absence of oblateness coefficients A_1 and A_2 respectively in the proposed model. Idrisi and Taqvi (2014) have investigated the location and stability of the non-collinear libration points when both the primaries are ellipsoids and found that the non-collinear libration points exist only in the interval $52^\circ < \varphi < 90^\circ$ and form an isosceles triangle with the primaries. Further they observed that the non collinear libration points are unstable in $52^\circ < \varphi < 90^\circ$. Idrisi (2014) has found the location of the collinear and non-collinear libration points and their linear stability when smaller primary is an oblate spheroid. He has found that there exist three collinear libration points and the non-collinear libration points are lying on the circumference of the unit circle whose center is the bigger primary. The collinear and non-collinear libration points all are unstable in his case.

In this paper we have considered the smaller primary as a point mass and the bigger one an ellipsoid. Our aim is to show the existence of collinear and non-collinear libration points and then check their stability in linear sense for Earth-Moon system. There are 5 Sections in this paper. In Section 2, the equations of motion of the infinitesimal mass m_3 have been determined. In Section 3, location of the collinear and non-collinear libration points have been investigated. In section 4, we have checked the stability of the collinear and non-collinear libration points. In last section, we have discussed all the results.

2. Equations of motion

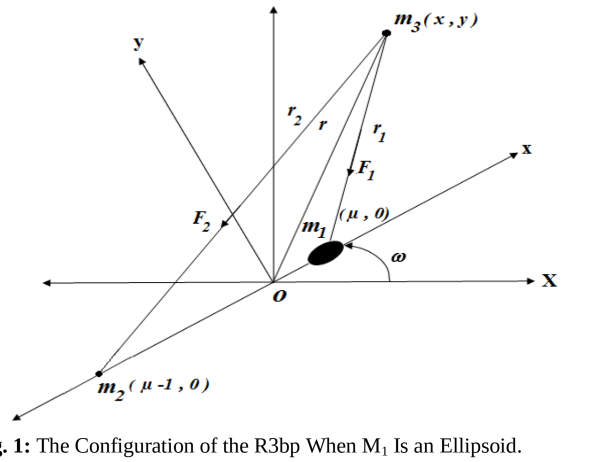

Fig. 1: The Configuration of the R3bp When M_1 Is an Ellipsoid.

Let m_1 be an ellipsoid whose axes are a, b and c (a > b > c) and m_2 a point mass ($m_1 > m_2$), are moving in the circular orbits around their center of mass O. An infinitesimal mass m_3 is moving in the plane of motion of m_1 and m_2. The distances of m_3 from m_1, m_2 and O are $\mathbf{r_1}$, $\mathbf{r_2}$ and $\mathbf{r}$ respectively. The principal axes of ellipsoid remains parallel to the synodic axes Oxyz throughout the motion and the equatorial plane of m_1 is coincided with the plane of motion of m_1 and m_2. Let the line joining m_1 and m_2 be taken as X –axis and O their center of mass as origin. Let the line passing through O and perpendicular to OX and lying in the plane of motion m_1 and m_2 be the Y –axis. Let us consider a synodic system of co-ordinates Oxyz initially coincide with the inertial system OXYZ, rotating with angular velocity ω about Z –axis (the z –axis is coincide with Z –axis). We wish to find the equations of motion of m_3 using the terminology of Szebehely (1967) in the synodic co-ordinate system and dimensionless variables i.e. the distance between the primaries is unity; the unit of time t is such that the gravitational constant G = 1 and the sum of the masses of the primaries is unity i.e. $m_1 + m_2 = 1$.

The potential of a homogeneous Ellipsoid at an external point P(x, y, z) is given by (Byrd and Friedman, 1954)

$$V = \frac{2\pi\rho\alpha\beta\gamma}{\sqrt{\alpha^2-\gamma^2}}\left[\left(1-\frac{x^2}{\alpha^2-\beta^2}+\frac{y^2}{\alpha^2-\beta^2}\right)F(\varphi,k) + \left(\frac{x^2}{\alpha^2-\beta^2}+\frac{(\gamma^2-\alpha^2)y^2}{(\alpha^2-\beta^2)(\beta^2-\gamma^2)}+\frac{z^2}{\beta^2-\gamma^2}\right)E(\varphi,k) + \left(\frac{\gamma^2+\gamma_1}{\beta^2-\gamma^2}y^2-\frac{\beta^2+\gamma_1}{\beta^2-\gamma^2}z^2\right)\sqrt{\frac{\alpha^2-\gamma^2}{(\alpha^2+\gamma_1)(\beta^2+\gamma_1)(\gamma^2+\gamma_1)}}\right]$$

$\alpha, \beta, \gamma =$ semi-axes of the ellipsoid,

$\rho =$ the density of the ellipsoid,

$\gamma_1 =$ largest root of the confocal ellipsoid

$$\frac{x^2}{\alpha^2+\gamma_1}+\frac{y^2}{\beta^2+\gamma_1}+\frac{z^2}{\gamma^2+\gamma_1}=1,$$

Where

$$F(\varphi,k) = \text{Elliptic Integral of First Kind} = \int_0^{\varphi}\frac{d\theta}{\sqrt{1-k^2\sin^2\theta}},$$

$$E(\varphi,k) = \text{Elliptic Integral of Second Kind} = \int_0^{\varphi}\sqrt{1-k^2\sin^2\theta}\,d\theta,$$

$$\varphi = \sin^{-1}\sqrt{\frac{\alpha^2-\gamma^2}{\gamma_1+\alpha^2}},\ 0 \le \varphi \le \frac{\pi}{2}$$

and

$$k = \sqrt{\frac{\alpha^2-\beta^2}{\alpha^2-\gamma^2}},\ 0 \le k^2 \le 1\ i.e. -1 \le k \le 1.$$

The equations of motion of the infinitesimal mass m_3 in terms of elliptic integrals of first and second kind are given by:

$$\ddot{x}-2n\dot{y}=n^2x-3(1-\mu)(x-\mu)\left[\frac{E(\varphi,k)-F(\varphi,k)}{p_6p_8}\right.$$
$$-\left\{1-k^2\sin^2\varphi\frac{(x-\mu)^2}{p_6}+\left(\frac{1}{p_6}+\frac{1-k^2\sin^2\varphi}{p_9}\right)y^2\right\}$$
$$\times\frac{(\lambda+p_3)}{2(\lambda+a^2)(2\lambda+p_1-r_1^2)\sqrt{\lambda+c^2}\sqrt{1-k^2\sin^2\varphi}} \qquad (1)$$
$$\left.-\frac{(\lambda^2+2c^2\lambda+p_{11})(\lambda+p_3)y^2}{2p_7(2\lambda+p_1-r_1^2)\sqrt{\lambda+c^2}(\lambda^2+p_5\lambda+p_{10})^{\frac{3}{2}}}\right]$$
$$-\frac{\mu}{r_2^3}(x+1-\mu)$$

and

$$\ddot{y}+2n\dot{x}=n^2y-3(1-\mu)y\left[\frac{1}{p_8}\left(\frac{E(\varphi,k)}{p_9}+\frac{F(\varphi,k)}{p_6}\right)\right.$$
$$-\left\{1-k^2\sin^2\varphi\frac{(x-\mu)^2}{p_6}+\left(\frac{1}{p_6}+\frac{1-k^2\sin^2\varphi}{p_9}\right)y^2\right\}$$
$$\times\frac{(\lambda+p_4)}{2(\lambda+a^2)(2\lambda+p_1-r_1^2)\sqrt{\lambda+c^2}\sqrt{1-k^2\sin^2\varphi}} \qquad (2)$$
$$-\frac{(\lambda^2+2c^2\lambda+p_{11})(\lambda+p_4)y^2}{2p_7(2\lambda+p_1-r_1^2)\sqrt{\lambda+c^2}(\lambda^2+p_5\lambda+p_{10})^{\frac{3}{2}}}$$
$$\left.+\frac{\sqrt{\lambda+c^2}}{p_7\sqrt{\lambda^2+p_5\lambda+p_{10}}}\right]-\frac{\mu}{r_2^3}y$$

where

$\alpha=a,\beta=b,\gamma=c,$

$m_2=\frac{4}{3}\pi abc\sigma,$

$\lambda=$ largest root of the confocal ellipsoid $\frac{x^2}{\alpha^2+\lambda}+\frac{y^2}{\beta^2+\lambda}+\frac{z^2}{\gamma^2+\lambda}=1,$

$$i.e.\ \lambda=\frac{r_1^2-p_1+\sqrt{\left(r_1^2-p_1\right)^2+4\left(p_3x^2+p_4y^2-p_2\right)}}{2},$$

$\mu=\frac{m_2}{m_1+m_2},$

$p_1=a^2+b^2+c^2,$

$p_2=a^2b^2+b^2c^2+c^2a^2,$

$p_3=b^2+c^2,$

$p_4=c^2+a^2,$

$p_5=a^2+b^2,$

$p_6=a^2-b^2,$

$p_7=b^2-c^2,$

$p_8=\sqrt{a^2-c^2},$

$p_9=\frac{(a^2-b^2)(b^2-c^2)}{(c^2-a^2)},$

$p_{10}=a^2b^2,$

$p_{11}=p_{10}-c^2p_5,$

$\varphi=\sin^{-1}\sqrt{\frac{a^2-c^2}{\lambda+a^2}},0\le\varphi\le\frac{\pi}{2},$

$k=\sqrt{\frac{a^2-b^2}{a^2-c^2}},\ 0\le k^2\le 1\ i.e.-1\le k\le 1,$

$r_1=\sqrt{(x-\mu)^2+y^2},$

$r_2=\sqrt{(x+1-\mu)^2+y^2}.$

The mean-motion of the primaries is given by

$$n=\left[3\left\{\left(1-\frac{k^2\sin^2\varphi'}{p_6}\right)\frac{(\lambda'+p_3)}{2(\lambda'+a^2)(2\lambda'+p_1-1)\sqrt{\lambda'+c^2}\sqrt{1-k^2\sin^2\varphi'}}-\frac{E(\varphi',k)-F(\varphi',k)}{p_6p_8}\right\}\right]^{\frac{1}{2}} \qquad (3)$$

where

$$\lambda'=\frac{1-p_1+\sqrt{\left(1-p_1\right)^2+4\left(p_3-p_2\right)}}{2},$$

$$\varphi'=\sin^{-1}\sqrt{\frac{a^2-c^2}{\lambda'+a^2}},\ 0\le\varphi'\le\frac{\pi}{2}.$$

From the astrophysical data:
Mass of Moon (m_2) = 7.3477× 10^{22} kg,
Mass of Earth (m_1) = 5.9742 × 10^{24} kg,
Axes of the Earth: a = 6378.140 km, b = 6368 km and c = 6356.755 km,
Mean distance of Moon from the Earth = 384,400 km.
In dimensionless system:

m_1+m_2 =1 unit i.e. 1 kg = 1.65353 × 10^{-25} unit.

Therefore,

$$\mu=\frac{m_2}{m_1+m_2}=0.0121496$$

Distance between the primaries = 1 unit i.e. 1 km = 2.60146×10^{-6} unit,
Thus, a = 0.0165925, b = 0.0165661 and c = 0.0165368.
From Eqn. (3), mean-motion of the primaries is n = 1.20494.

$$k=\sqrt{\frac{a^2-b^2}{a^2-c^2}}=0.688757,$$

$$\varphi=\sin^{-1}\sqrt{\frac{a^2-c^2}{\lambda+a^2}}\Rightarrow\lambda=a^2(\operatorname{cosec}^2\varphi-1)-a^2,0\le\varphi\le\frac{\pi}{2}.$$

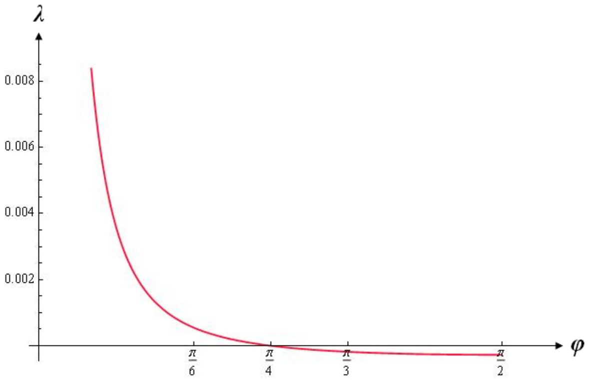

Fig. 2: φ versus λ in Earth-Moon System.

From the Fig. 2, this is observed that as $\varphi\to\pi/4$, $\lambda\to0$, as $\varphi\to0$, $\lambda\to\infty$ and in the interval $\pi/4\le\varphi\le\pi/2$, λ is negative. Thus, in Earth-Moon system φ is valid only in the interval $0<\varphi<\pi/4$.

3. Libration points

At the libration points $\dot{x}=0,\dot{y}=0,\ddot{x}=0,\ddot{y}=0$ therefore Eqns. (1) and (2) become

$$\begin{aligned}
&n^2x-3(1-\mu)(x-\mu)\Bigg[\frac{E(\varphi,k)-F(\varphi,k)}{p_6\,p_8}\\
&-\left\{1-k^2\sin^2\varphi\frac{(x-\mu)^2}{p_6}+\left(\frac{1}{p_6}+\frac{1-k^2\sin^2\varphi}{p_9}\right)y^2\right\}\\
&\times\frac{(\lambda+p_3)}{2(\lambda+a^2)(2\lambda+p_1-r_1^2)\sqrt{\lambda+c^2}\sqrt{1-k^2\sin^2\varphi}}\\
&-\frac{(\lambda^2+2c^2\lambda+p_{11})(\lambda+p_3)y^2}{2p_7(2\lambda+p_1-r_1^2)\sqrt{\lambda+c^2}(\lambda^2+p_5\lambda+p_{10})^{\frac{3}{2}}}\Bigg]\\
&-\frac{\mu}{r_2^3}(x+1-\mu)=0
\end{aligned}\tag{4}$$

and

$$\begin{aligned}
&n^2y-3(1-\mu)y\Bigg[\frac{1}{p_8}\left(\frac{E(\varphi,k)}{p_9}+\frac{F(\varphi,k)}{p_6}\right)\\
&-\left\{1-k^2\sin^2\varphi\frac{(x-\mu)^2}{p_6}+\left(\frac{1}{p_6}+\frac{1-k^2\sin^2\varphi}{p_9}\right)y^2\right\}\\
&\times\frac{(\lambda+p_4)}{2(\lambda+a^2)(2\lambda+p_1-r_1^2)\sqrt{\lambda+c^2}\sqrt{1-k^2\sin^2\varphi}}\\
&-\frac{(\lambda^2+2c^2\lambda+p_{11})(\lambda+p_4)y^2}{2p_7(2\lambda+p_1-r_1^2)\sqrt{\lambda+c^2}(\lambda^2+p_5\lambda+p_{10})^{\frac{3}{2}}}\\
&+\frac{\sqrt{\lambda+c^2}}{p_7\sqrt{\lambda^2+p_5\lambda+p_{10}}}\Bigg]-\frac{\mu}{r_2^3}y=0
\end{aligned}\tag{5}$$

3.1. Collinear libration points

The collinear libration points are the solution of the Equations (5) and (6) when y = 0 i.e.

$$\begin{aligned}
&n^2x-3(1-\mu)(x-\mu)\Bigg[\frac{E(\varphi,k)-F(\varphi,k)}{p_6\,p_8}\\
&-\left\{1-k^2\sin^2\varphi\frac{(x-\mu)^2}{p_6}\right\}\\
&\times\frac{(\lambda+p_3)}{2(\lambda+a^2)(2\lambda+p_1-(x-\mu)^2)\sqrt{\lambda+c^2}\sqrt{1-k^2\sin^2\varphi}}\Bigg]\\
&-\frac{\mu}{(x+1-\mu)^2}=0
\end{aligned}\tag{6}$$

The Eqn. (6) is a fifth degree equation in x, depending upon φ. On solving Eqn. (6) for a = 0.0165925, b = 0.0165661, c = 0.0165368 and μ = 0.0121496 in $0<\varphi<\pi/4$ we get three values of x, these values are given in Table 1.
As shown in the Fig. 3, there exist three collinear libration points L_i (i = 1, 2, 3). The first libration point L_1 lies between m_2 and the center of mass O, as $\varphi\to9^\circ$, $L_1\to$ O and in $9^\circ\le\varphi<45^\circ$ it moves away from the center of mass. The center of mass of bigger primary m_1 i.e. ellipsoid behaves as a second libration point L_2 and remains stationary in $0^\circ<\varphi<45^\circ$, the third libration point L_3 lies at the right of m_1 and move towards the center of mass in $0^\circ<\varphi<45^\circ$.

Table 1: Collinear Libration Points inEarth-Moon System

φ°	$L_1(x)$	$L_2(x)$	$L_3(x)$	φ°	$L_1(x)$	$L_2(x)$	$L_3(x)$
1	-0.0832616	0.0121484	0.1075610	23	0.0073625	0.0121496	0.0169367
2	-0.0360226	0.0121495	0.0603218	24	0.0075515	0.0121496	0.0167477
3	-0.0204022	0.0121496	0.0447014	25	0.0077255	0.0121496	0.0165737
4	-0.0126271	0.0121496	0.0369263	26	0.0078860	0.0121496	0.0164132
5	-0.0079555	0.0121496	0.0322547	27	0.0080346	0.0121496	0.0162646
6	-0.0048199	0.0121496	0.0291191	28	0.0081725	0.0121496	0.0161267
7	-0.0025573	0.0121496	0.0268565	29	0.0083008	0.0121496	0.0159984
8	-0.0008403	0.0121496	0.0251394	30	0.0084205	0.0121496	0.0158787
9	0.00051114	0.0121496	0.0237881	31	0.0085323	0.0121496	0.0157669
10	0.00160438	0.0121496	0.0226948	32	0.0086369	0.0121496	0.0156622
11	0.00250789	0.0121496	0.0217913	33	0.0087352	0.0121496	0.0155641
12	0.00326751	0.0121496	0.0210317	34	0.0088275	0.0121496	0.0154717
13	0.00391519	0.0121496	0.0203840	35	0.0089143	0.0121496	0.0153849
14	0.00447397	0.0121496	0.0198252	36	0.0089962	0.0121496	0.0153031
15	0.00496093	0.0121496	0.0193383	37	0.0090735	0.0121496	0.0152257
16	0.0053889	0.0121496	0.018911	38	0.0091466	0.0121496	0.0151526
17	0.0057681	0.0121496	0.018531	39	0.0092158	0.0121496	0.0150834
18	0.0061062	0.0121496	0.018193	40	0.0092814	0.0121496	0.0150178
19	0.0064095	0.0121496	0.017889	41	0.0093437	0.0121496	0.0149555
20	0.0066829	0.0121496	0.017616	42	0.0094028	0.0121496	0.0148964
21	0.0069308	0.0121496	0.017368	43	0.0094591	0.0121496	0.0148401
22	0.0071564	0.0121496	0.017143	44	0.0095127	0.0121496	0.0147865

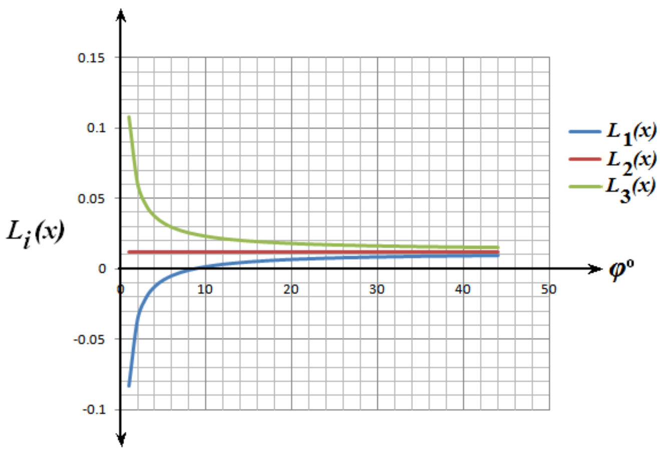

Fig. 3: Φ versus L_i(X), I = 1, 2, 3.

3.2.Non-collinear libration points

The non-collinear libration points are the solution of the Eqns. (1) and (2) when y ≠ 0 i.e.

$$n^2 x - 3(1-\mu)(x-\mu)\left[\frac{E(\varphi,k)-F(\varphi,k)}{p_6 p_8} - \left\{1-k^2\sin^2\varphi\,\frac{(x-\mu)^2}{p_6}+\left(\frac{1}{p_6}+\frac{1-k^2\sin^2\varphi}{p_9}\right)y^2\right\} \times \frac{(\lambda+p_3)}{2(\lambda+a^2)(2\lambda+p_1-r_1^2)\sqrt{\lambda+c^2}\sqrt{1-k^2\sin^2\varphi}} - \frac{(\lambda^2+2c^2\lambda+p_{11})(\lambda+p_3)y^2}{2p_7(2\lambda+p_1-r_1^2)\sqrt{\lambda+c^2}(\lambda^2+p_5\lambda+p_{10})^{\frac{3}{2}}}\right] - \frac{\mu}{r_2^3}(x+1-\mu)=0 \qquad (7)$$

and

$$n^2 - 3(1-\mu)\left[\frac{1}{p_8}\left(\frac{E(\varphi,k)}{p_9}+\frac{F(\varphi,k)}{p_6}\right) - \left\{1-k^2\sin^2\varphi\,\frac{(x-\mu)^2}{p_6}+\left(\frac{1}{p_6}+\frac{1-k^2\sin^2\varphi}{p_9}\right)y^2\right\} \times \frac{(\lambda+p_4)}{2(\lambda+a^2)(2\lambda+p_1-r_1^2)\sqrt{\lambda+c^2}\sqrt{1-k^2\sin^2\varphi}} - \frac{(\lambda^2+2c^2\lambda+p_{11})(\lambda+p_4)y^2}{2p_7(2\lambda+p_1-r_1^2)\sqrt{\lambda+c^2}(\lambda^2+p_5\lambda+p_{10})^{\frac{3}{2}}} + \frac{\sqrt{\lambda+c^2}}{p_7\sqrt{\lambda^2+p_5\lambda+p_{10}}}\right] - \frac{\mu}{r_2^3} = 0 \qquad (8)$$

On solving the Eqns. (7) and (8) simultaneously for 0° < φ < 45° by Newton-Raphson Method, taking the initial conditions: $x_0 = \mu - \frac{1}{2}$ and $y_0 = \pm\sqrt{3}/2$, we get the co-ordinates of non-collinear libration points $L_{4,5}$ in Earth-Moon system given in Table 2.

Table 2: Non-Collinear Libration Points in Earth-Moon System

φ°	x	± y	φ°	x	± y
1	0.0121472	∓ 0.0181461	23	0.0121496	∓ 0.0000330987
2	0.0121493	∓ 0.00457585	24	0.0121496	∓ 0.0000304438
3	0.0121495	∓ 0.00201824	25	0.0121496	∓ 0.0000281027
4	0.0121496	∓ 0.00112455	26	0.0121496	∓ 0.0000260278
5	0.0121496	∓ 0.000713569	27	0.0121496	∓ 0.0000241803
6	0.0121496	∓ 0.000492075	28	0.0121496	∓ 0.0000225283
7	0.0121496	∓ 0.000359556	29	0.0121496	∓ 0.0000210451
8	0.0121496	∓ 0.000274146	30	0.0121496	∓ 0.0000197084
9	0.0121496	∓ 0.000215941	31	0.0121496	∓ 0.0000184998
10	0.0121496	∓ 0.000174517	32	0.0121496	∓ 0.0000174035
11	0.0121496	∓ 0.000143999	33	0.0121496	∓ 0.0000164059
12	0.0121496	∓ 0.000120869	34	0.0121496	∓ 0.0000154957
13	0.0121496	∓ 0.000102922	35	0.0121496	∓ 0.0000146631
14	0.0121496	∓ 0.0000887176	36	0.0121496	∓ 0.0000138994
15	0.0121496	∓ 0.0000772824	37	0.0121496	∓ 0.0000131974
16	0.0121496	∓ 0.0000679405	38	0.0121496	∓ 0.0000125506
17	0.0121496	∓ 0.0000602102	39	0.0121496	∓ 0.0000119535
18	0.0121496	∓ 0.0000537409	40	0.0121496	∓ 0.0000114013
19	0.0121496	∓ 0.0000482725	41	0.0121496	∓ 0.0000108895
20	0.0121496	∓ 0.0000436085	42	0.0121496	∓ 0.0000104144
21	0.0121496	∓ 0.0000395986	43	0.0121496	∓ 0.0000099726
22	0.0121496	∓ 0.0000361259	44	0.0121496	∓ 0.0000095612

From the Table 2, it is observed the non-collinear libration points are forming a right angled-triangle with the primaries, right angled at bigger primary (Earth) for $0^\circ < \varphi < 45^\circ$ and as φ increases the shape of right triangle reduces. Therefore, on the basis of the results obtained we can say that the non-collinear libration points in Earth-Moon system lie in the neighborhood of Earth (Fig. 4).

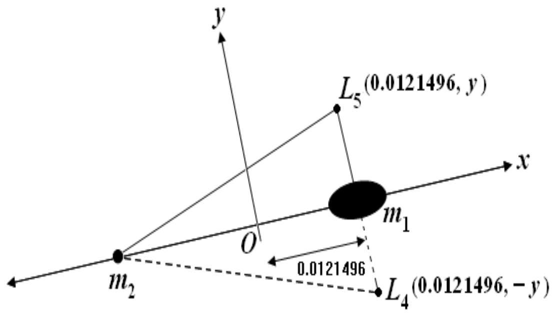

Fig. 4: Location of Non-Collinear Libration Points in Earth-Moon System.

4. Stability of the libration points

The equations of the motion of the infinitesimal mass are

$$\ddot{x} - 2n\dot{y} = \Omega_x \qquad \ddot{y} + 2n\dot{x} = \Omega_y \tag{9}$$

To study the possible motion of the infinitesimal mass around the libration points let the coordinates of these points are (x_0, y_0). If we give small displacement (α, β) to (x_0, y_0), the variation α and β can be written as: $\alpha = x - x_0$ and $\beta = y - y_0$ and the equations of the motion become

$$\ddot{\alpha} - 2n\dot{\beta} = \Omega_x(x_0+\alpha, y_0+\beta) = \alpha\overset{o}{\Omega}_{xx} + \beta\overset{o}{\Omega}_{xy}, \qquad \ddot{\beta} + 2n\dot{\alpha} = \Omega_y(x_0+\alpha, y_0+\beta) = \alpha\overset{o}{\Omega}_{yx} + \beta\overset{o}{\Omega}_{yy}, \tag{10}$$

where 'o' indicates that the partial derivatives are to be calculated at the libration points under consideration.
The characteristic equation of the Equations (8) is given by

$$\xi^4 + (4n^2 - \overset{o}{\Omega}_{xx} - \overset{o}{\Omega}_{yy})\xi^2 + \overset{o}{\Omega}_{xx}\overset{o}{\Omega}_{yy} - (\overset{o}{\Omega}_{xy})^2 = 0, \tag{11}$$

which is a fourth degree equation in ξ. The libration point (x_0, y_0) is said to be stable if all the four roots of the Equation (11) are either negative real numbers or pure imaginary.
Note: The values of Ω_{xx}, Ω_{xy} and Ω_{yy} are given in Appendix I.

4.1. Stability of collinear libration points

At the collinear libration points y = 0. Therefore, $A_5 = A_6 = A_7 = A_8 = 0$ i.e. $\overset{o}{\Omega}_{xy} = 0$. Thus the roots of the characteristic Eqn. (11) i.e. ξ_i (i =1, 2, 3, 4) for L_i (i =1, 2, 3) are given in Table 3, 4 and 5 respectively.

Table 3: Stability of L_1

φ°	$L_1(x)$	Ω^o_{xx}	Ω^o_{yy}	$\xi_{1,2}$	$\xi_{3,4}$
1	-0.0832616	73971.1	-2321.14	±271.966	±48.18i
2	-0.0360226	502793	-18575.4	±709.075	±136.292i
3	-0.0204022	1.37909×10^6	-62715.1	±1174.35	±250.43i
4	-0.0126271	2.66487×10^6	-148697.1	±1632.44	±385.612i
5	-0.0079555	4.36679×10^6	-290447.1	±2089.69	±538.932i
6	-0.0048199	6.55878×10^6	-501827.1	±2561.01	±708.398i
7	-0.0025573	9.35025×10^6	-796700.1	±3057.82	±892.581i
8	-0.0008403	1.28589×10^7	-1.18877×10^6	±3585.94	±1090.31i
9	0.00051114	1.72034×10^7	-1.69163×10^6	±4147.71	±1300.63i
10	0.00160438	2.2504×10^7	-2.31892×10^6	±4743.83	±1522.8i
11	0.00250789	2.8878×10^7	-3.08407×10^6	±5373.83	±1756.15i
12	0.00326751	3.64431×10^7	-4.00039×10^6	±6036.81	±2000.1i
13	0.00391519	4.53182×10^7	-5.0811×10^6	±6731.88	±2254.13i
14	0.00447397	5.56237×10^7	-6.33927×10^6	±7458.13	±2517.79i
15	0.00496093	6.74806×10^7	-7.78774×10^6	±8214.66	±2790.65i
16	0.0053889	8.10249×10^7	-9.44005×10^6	±9001.38	±3072.47i
17	0.0057681	9.63575×10^7	-1.13069×10^7	±9816.18	±3362.58i
18	0.0061062	1.13628×10^8	-1.3402×10^7	±10659.6	±3660.88i
19	0.0064095	1.32962×10^8	-1.57371×10^7	±11530.9	±3967.01i
20	0.0066829	1.54535×10^8	-1.83266×10^7	±12431.2	±4280.96i
21	0.0069308	1.78416×10^8	-2.11771×10^7	±13357.2	±4601.85i
22	0.0071564	2.04798×10^8	-2.43033×10^7	±14310.8	±4929.84i
23	0.0073625	2.3385×10^8	-2.77178×10^7	±15292.2	±5264.77i
24	0.0075515	2.65722×10^8	-3.14309×10^7	±16301.1	±5606.32i
25	0.0077255	3.00518×10^8	-3.54488×10^7	±17335.4	±5953.89i
26	0.0078860	3.38531×10^8	-3.97909×10^7	±18399.2	±6308.01i
27	0.0080346	3.79853×10^8	-4.44607×10^7	±19489.8	±6667.89i
28	0.0081725	4.24712×10^8	-4.94711×10^7	±20608.5	±7033.57i

29	0.0083008	4.73294×10^8	-5.48305×10^7	±21755.3	±7404.76i
30	0.0084205	5.25732×10^8	-6.05427×10^7	±22928.8	±7780.92i
31	0.0085323	5.82401×10^8	-6.66277×10^7	±24133.1	±8162.58i
32	0.0086369	6.43612×10^8	-7.30996×10^7	±25369.5	±8549.83i
33	0.0087352	7.09162×10^8	-7.99336×10^7	±26630.1	±8940.56i
34	0.0088275	7.7977×10^8	-8.71705×10^7	±27924.4	±9336.51i
35	0.0089143	8.55785×10^8	-9.48224×10^7	±29253.8	±9737.68i
36	0.0089962	9.37166×10^8	-1.02871×10^8	±30613.2	±10142.5i
37	0.0090735	1.02449×10^9	-1.11342×10^8	±32007.6	±10551.9i
38	0.0091466	1.11799×10^9	-1.20232×10^8	±33436.3	±10965.1i
39	0.0092158	1.21813×10^9	-1.29554×10^8	±34901.7	±11382.2i
40	0.0092814	1.32531×10^9	-1.3931×10^8	±36404.8	±11803.1i
41	0.0093437	1.43979×10^9	-1.49495×10^8	±37944.6	±12226.8i
42	0.0094028	1.56277×10^9	-1.60161×10^8	±39531.8	±12655.5i
43	0.0094591	1.69386×10^9	-1.71249×10^8	±41156.6	±13086.2i
44	0.0095127	1.83418×10^9	-1.82797×10^8	±42827.3	±13520.2i

Table 4: Stability of L_2

φ^o	$L_2(x)$	Ω^o_{xx}	Ω^o_{yy}	$\xi_{1,2}$	$\xi_{3,4}$
1	0.0121484	58.6428	2328.98	±48.1977	±7.66767
2	0.0121495	468.207	19393.1	±139.238	±21.6414
3-44	0.0121496	1621.67	69142.3	±262.938	±40.2717

Table 5: Stability of L_3

φ^o	$L_3(x)$	Ω^o_{xx}	Ω^o_{yy}	$\xi_{1,2}$	$\xi_{3,4}$
1	0.1075610	73975.3	-2321.2	±271.974	±48.1807i
2	0.0603218	502793.1	-18575.4	±709.075	±136.292i
3	0.0447014	1.37909×10^6	-62715	±1174.35	±250.43i
4	0.0369263	2.66487×10^6	-148697	±1632.44	±385.612i
5	0.0322547	4.36679×10^6	-290447	±2089.69	±538.932i
6	0.0291191	6.55878×10^6	-501827	±2561.01	±708.398i
7	0.0268565	9.35025×10^6	-796700	±3057.82	±892.581i
8	0.0251394	1.28577×10^7	-1.1887×10^6	±3585.76	±1090.28i
9	0.0237881	1.72041×10^7	-1.69167×10^6	±4147.78	±1300.64i
10	0.0226948	2.25035×10^7	-2.31889×10^6	±4743.78	±1522.79i
11	0.0217913	2.88777×10^7	-3.08405×10^6	±5373.8	±1756.15i
12	0.0210317	3.64435×10^7	-4.00042×10^6	±6036.84	±2000.11i
13	0.0203840	4.53176×10^7	-5.08106×10^6	±6731.84	±2254.12i
14	0.0198252	5.56215×10^7	-6.33912×10^6	±7457.98	±2517.76i
15	0.0193383	6.74835×10^7	-7.78794×10^6	±8214.83	±2790.69i
16	0.018911	8.11086×10^7	-9.44601×10^6	±9006.03	±3073.44i
17	0.018531	9.63425×10^7	-1.13058×10^7	±9815.42	±3362.42i
18	0.018193	1.13628×10^8	-1.34020×10^7	±10659.6	±3660.88i
19	0.017889	1.32805×10^8	-1.57256×10^7	±11524.1	±3965.55i
20	0.017616	1.54453×10^8	-1.83206×10^7	±12427.9	±4280.26i
21	0.017368	1.78285×10^8	-2.11674×10^7	±13352.3	±4600.81i
22	0.017143	2.04877×10^8	-2.43091×10^7	±14313.5	±4930.43i
23	0.0169367	2.33850×10^8	-2.77178×10^7	±15292.2	±5264.77i
24	0.0167477	2.65722×10^8	-3.14309×10^7	±16301.1	±5606.32i
25	0.0165737	3.00518×10^8	-3.54488×10^7	±17335.4	±5953.89i
26	0.0164132	3.38531×10^8	-3.97909×10^7	±18399.2	±6308.01i
27	0.0162646	3.79853×10^8	-4.44607×10^7	±19489.8	±6667.89i
28	0.0161267	4.24712×10^8	-4.94711×10^7	±20608.5	±7033.57i
29	0.0159984	4.73294×10^8	-5.48305×10^7	±21755.3	±7404.76i
30	0.0158787	5.25732×10^8	-6.05427×10^7	±22928.8	±7780.92i
31	0.0157669	5.82401×10^8	-6.66277×10^7	±24133.1	±8162.58i
32	0.0156622	6.43440×10^8	-7.30871×10^7	±25366.1	±8549.10i
33	0.0155641	7.09356×10^8	-7.99477×10^7	±26633.7	±8941.35i
34	0.0154717	7.79770×10^8	-8.71705×10^7	±27924.4	±9336.51i
35	0.0153849	8.55785×10^8	-9.48224×10^7	±29253.8	±9737.68i
36	0.0153031	9.37443×10^8	-1.02891×10^8	±30617.7	±10143.5i
37	0.0152257	1.02449×10^9	-1.11342×10^8	±32007.6	±10551.9i
38	0.0151526	1.11799×10^9	-1.20232×10^8	±33436.3	±10965.1i
39	0.0150834	1.21813×10^9	-1.29554×10^8	±34901.7	±11382.2i
40	0.0150178	1.32531×10^9	-1.39310×10^8	±36404.8	±11803.1i
41	0.0149555	1.43979×10^9	-1.49495×10^8	±37944.6	±12226.8i
42	0.0148964	1.56277×10^9	-1.60161×10^8	±39531.8	±12655.5i
43	0.0148401	1.69386×10^9	-1.71249×10^8	±41156.6	±13086.2i
44	0.0147865	1.83418×10^9	-1.82797×10^8	±42827.3	±13520.2i

From the nature of the roots of characteristic Eqn. (11) as mentioned in Table 3, 4 and 5, it is observed that all the collinear libration points are unstable in $0^\circ < \varphi < 45^\circ$.

4.2. Stability of non-collinear libration points

The roots of the characteristic Eqn. (11) i.e. ξ_i(i =1, 2, 3, 4) for L_4 are given in Table 6.

Table 6: Stability of L_4

φ°	x	y	Ω^o_{xx}	Ω^o_{yy}	Ω^o_{xy}	$\xi_{1,2}$	$\xi_{3,4}$
1	0.0121472	−0.0181461	2320.87	-4445427	1.88592	±48.1751	±667.408i
2	0.0121493	−0.00457585	18566.4	-319586	-2.37467	±136.257	±565.324i
3	0.0121495	−0.00201824	62657.1	-343633	-6.837	±250.312	±586.206i
4	0.0121496	−0.00112455	148495.01	-513798	-4.09884×10^{-5}	±385.349	±716.8i
5	0.0121496	−0.000713569	289949	-859493	-2.60087×10^{-5}	±538.468	±927.091i
6	0.0121496	−0.000492075	500843	-1.42622×10^{6}	-1.79355×10^{-5}	±707.702	±1194.25i
7	0.0121496	−0.000359556	794936	-2.26431×10^{6}	-1.31054×10^{-5}	±891.591	±1504.76i
8	0.0121496	−0.000274146	1.18591×10^{6}	-3.42512×10^{6}	-9.99229×10^{-6}	±1088.99	±1850.71i
9	0.0121496	−0.000215941	1.68737×10^{6}	-4.96004×10^{6}	-7.87079×10^{-6}	±1298.99	±2227.12i
10	0.0121496	−0.000174517	2.31286×10^{6}	-6.92016×10^{6}	-6.36093×10^{-6}	±1520.81	±2630.62i
11	0.0121496	−0.000143999	3.07571×10^{6}	-9.35701×10^{6}	-5.24859×10^{-6}	±1753.77	±3058.92i
12	0.0121496	−0.000120869	3.98932×10^{6}	-1.23219×10^{7}	-4.40553×10^{-6}	±1997.33	±3510.25i
13	0.0121496	−0.000102922	5.06679×10^{6}	-1.58672×10^{7}	-3.75138×10^{-6}	±2250.95	±3983.36i
14	0.0121496	−0.0000887176	6.32108×10^{6}	-2.00461×10^{7}	-3.23365×10^{-6}	±2514.17	±4477.28i
15	0.0121496	−0.0000772824	7.76512×10^{6}	-2.49123×10^{7}	-2.81685×10^{-6}	±2786.6	±4991.23i
16	0.0121496	−0.0000679405	9.41159×10^{6}	-3.05212×10^{7}	-2.47635×10^{-6}	±3067.83	±5524.6i
17	0.0121496	−0.0000602102	1.12731×10^{7}	-3.69289×10^{7}	-2.19459×10^{-6}	±3357.54	±6076.92i
18	0.0121496	−0.0000537409	1.33618×10^{7}	-4.41933×10^{7}	-1.95879×10^{-6}	±3655.38	±6647.8i
19	0.0121496	−0.0000482725	1.56901×10^{7}	-5.23738×10^{7}	-1.75947×10^{-6}	±3961.06	±7236.97i
20	0.0121496	−0.0000436085	1.82697×10^{7}	-6.15308×10^{7}	-1.58948×10^{-6}	±4274.31	±7844.16i
21	0.0121496	−0.0000395986	2.11125×10^{7}	-7.17275×10^{7}	-1.44332×10^{-6}	±4594.84	±8469.21i
22	0.0121496	−0.0000361259	2.42301×10^{7}	-8.30281×10^{7}	-1.31675×10^{-6}	±4922.41	±9111.97i
23	0.0121496	−0.0000330987	2.76332×10^{7}	-9.54999×10^{7}	-1.20641×10^{-6}	±5256.73	±9772.41i
24	0.0121496	−0.0000304438	3.13336×10^{7}	-1.0921×10^{8}	-1.10964×10^{-6}	±5597.64	±10450.4i
25	0.0121496	−0.0000281027	3.53415×10^{7}	-1.24232×10^{8}	-1.02431×10^{-6}	±5944.87	±11145.9i
26	0.0121496	−0.0000260278	3.96678×10^{7}	-1.40636×10^{8}	-9.48682×10^{-7}	±6298.24	±11859.1i
27	0.0121496	−0.0000241803	4.43226×10^{7}	-1.585×10^{8}	-8.81343×10^{-7}	±6657.52	±12589.7i
28	0.0121496	−0.0000225283	4.93151×10^{7}	-1.77906×10^{8}	-8.21129×10^{-7}	±7022.47	±13338.1i
29	0.0121496	−0.0000210451	5.46559×10^{7}	-1.98931×10^{8}	-7.67069×10^{-7}	±7392.97	±14104.3i
30	0.0121496	−0.0000197084	6.03558×10^{7}	-2.21657×10^{8}	-7.18348×10^{-7}	±7768.91	±14888.1i
31	0.0121496	−0.0000184998	6.64214×10^{7}	-2.46179×10^{8}	-6.74296×10^{-7}	±8149.93	±15690.1i
32	0.0121496	−0.0000174035	7.28614×10^{7}	-2.72591×10^{8}	-6.34337×10^{-7}	±8535.89	±16510.3i
33	0.0121496	−0.0000164059	7.96871×10^{7}	-3.00979×10^{8}	-5.97975×10^{-7}	±8926.76	±17348.7i
34	0.0121496	−0.0000154957	8.69041×10^{7}	-3.31451×10^{8}	-5.64801×10^{-7}	±9322.23	±18205.8i
35	0.0121496	−0.0000146631	9.45188×10^{7}	-3.64114×10^{8}	-5.34452×10^{-7}	±9722.08	±19081.8i
36	0.0121496	−0.0000138994	1.02544×10^{8}	-3.99065×10^{8}	-5.06616×10^{-7}	±10126.4	±19976.6i
37	0.0121496	−0.0000131974	1.10983×10^{8}	-4.36427×10^{8}	-4.81029×10^{-7}	±10534.8	±20890.8i
38	0.0121496	−0.0000125506	1.19846×10^{8}	-4.76311×10^{8}	-4.57454×10^{-7}	±10947.4	±21824.5i
39	0.0121496	−0.0000119535	1.29138×10^{8}	-5.18848×10^{8}	-4.35691×10^{-7}	±11363.9	±22778.2i
40	0.0121496	−0.0000114013	1.38859×10^{8}	-5.64185×10^{8}	-4.15564×10^{-7}	±11783.9	±23752.6i
41	0.0121496	−0.0000108895	1.49026×10^{8}	-6.12433×10^{8}	-3.96909×10^{-7}	±12207.6	±24747.4i
42	0.0121496	−0.0000104144	1.59639×10^{8}	-6.63751×10^{8}	-3.79592×10^{-7}	±12634.9	±25763.4i
43	0.0121496	−0.0000099726	1.70709×10^{8}	-7.18281×10^{8}	-3.63489×10^{-7}	±13065.5	±26800.8i
44	0.0121496	−0.0000095612	1.82233×10^{8}	-7.76204×10^{8}	-3.48494×10^{-7}	±13499.4	±27860.4i

From the nature of the roots, it is observed that the non-collinear libration points are unstable. Similarly, L_5 is also unstable.

5. Conclusion

In the present paper existence of collinear and non-collinear libration points for Earth-Moon system has been analyzed and this is found that all the libration points exist only in the interval $0^\circ < \varphi < 45^\circ$. There exist three collinear libration points and non-collinear libration points have been calculated by Newton-Raphson method taking initial values $x_0 = \mu - ½$ and $y_0 = \pm\sqrt{3}/2$ (Table 2). The non-collinear libration points are forming a right triangle with the primaries, right angled at the bigger primary. Further, all the libration points are unstable in $0^\circ < \varphi < 45^\circ$.

References

[1] Abouelmagd, E. I., Asiri, H. M., Sharaf, M. A., 2013. "The effect of oblateness in the perturbed restricted three-body problem", Meccanica, vol. 48 (10), p. 2479-2490.http://dx.doi.org/10.1007/s11012-013-9762-3.

[2] Byrd, P. F. & Friedman, M. D., 1954. "Handbook of Elliptic Integrals for Engineers and Physicists", Springer-Verlag, Germany, p. 4-5.http://dx.doi.org/10.1007/978-3-642-52803-3.

[3] Idrisi, M. Javed, Taqvi, Z. A., 2013. "Restricted three-body problem when one of the primaries is an ellipsoid", Astrophysics and Space Science, vol. 348 (1), p. 41-56.http://dx.doi.org/10.1007/s10509-013-1534-y.

[4] Idrisi, M. Javed, Taqvi, Z. A., 2014. "Existence and stability of the non-collinear libration points in the restricted three body problem when both the primaries are ellipsoid", Astrophysics and Space Science, vol. 350 (1), p. 133-141.http://dx.doi.org/10.1007/s10509-013-1718-5.

[5] Idrisi, M. Javed, 2014. "Existence and stability of libration points in CR3BP when the smaller primary is an oblate spheroid", Astrophysics and Space Science, http://dx.doi.org/10.1007/s10509-014-2031-7.

[6] Khanna, M. & Bhatnagar, K. B., 1999. "Existence and stability of libration points in the restricted three body problem when the smaller primary is a triaxial rigid body and the bigger one an oblate spheroid", Indian Journal of Pure and Applied Mathematics, vol. 30 (7), p. 721-733.

[7] Kumari, Reena, Kushvah, Badam Singh, 2014. "Stability regions of equilibrium points in restricted four-body problem with oblateness effects", Astrophysics and Space Science, vol. 349 (2), p. 693-704.http://dx.doi.org/10.1007/s10509-013-1689-6.

[8] Narayan, A., Kumar, C. Ramesh, 2011, "Effects of photogravitational and oblateness on the triangular lagrangian points in the elliptical restricted three body problem", International Journal of Pure and Applied Mathematics, vol. 68(2), p. 201-224.

[9] Raheem, Abdul Razzaq Abdul, Singh, Jagdish, 2006, "Combined effects of perturbations, radiation and oblateness on the stability of equilibrium points in the restricted three-body problem", The Astronomical Journal, vol. 131, p. 1880-1885.http://dx.doi.org/10.1086/499300.

[10] Singh, Jagdish, Taura, J. J., 2013. "Motion in the generalized restricted three-body problem", Astrophysics and Space Science, vol. 343 (1), p. 95-106.http://dx.doi.org/10.1007/s10509-012-1225-0.

[11] Szebehely, Victor, 1967. "Theory of Orbits, The restricted problem of three bodies", Academic Press, New York and London.

Appendix I

$$\Omega_{xx} = n^2 - \frac{\mu}{r_2^3} + \frac{3\mu}{r_2^5}(x+1-\mu)^2$$

$$-3(1-\mu)\left[\frac{E(\varphi,k)-F(\varphi,k)}{p_6\,p_8}\right.$$

$$-\left\{1-k^2\sin^2\varphi\,\frac{(x-\mu)^2}{p_6}\right.$$

$$\left.+\left(\frac{1}{p_6}+\frac{1-k^2\sin^2\varphi}{p_9}\right)y^2\right\}$$

$$\times\frac{(\lambda+p_3)}{2(\lambda+a^2)(2\lambda+p_1-r_1^2)\sqrt{\lambda+c^2}\sqrt{1-k^2\sin^2\varphi}}$$

$$\left.-\frac{(\lambda^2+2c^2\lambda+p_{11})(\lambda+p_3)y^2}{2p_7(2\lambda+p_1-r_1^2)\sqrt{\lambda+c^2}\,(\lambda^2+p_5\lambda+p_{10})^{\frac{3}{2}}}\right]$$

$$-3(1-\mu)(x-\mu)$$

$$\left[A_1-\frac{(\lambda+p_3)A_2}{2(\lambda+a^2)(2\lambda+p_1-r_1^2)\sqrt{\lambda+c^2}\sqrt{1-k^2\sin^2\varphi}}\right.$$

$$\left.-\left\{1-k^2\sin^2\varphi\,\frac{(x-\mu)^2}{p_6}+\left(\frac{1}{p_6}+\frac{1-k^2\sin^2\varphi}{p_9}\right)y^2\right\}A_3-A_4\right],$$

$$\Omega_{xy} = \frac{3\mu}{r_2^5}(x+1-\mu)\,y - 3(1-\mu)(x-\mu)$$

$$\left[A_5-\frac{(\lambda+p_3)A_6}{2(\lambda+a^2)(2\lambda+p_1-r_1^2)\sqrt{\lambda+c^2}\sqrt{1-k^2\sin^2\varphi}}\right.$$

$$-\left\{1-k^2\sin^2\varphi\,\frac{(x-\mu)^2}{p_6}+\left(\frac{1}{p_6}+\frac{1-k^2\sin^2\varphi}{p_9}\right)y^2\right\}A_7$$

$$-A_8\,y^2-$$

$$\left.\frac{(\lambda^2+2c^2\lambda+p_{11})(\lambda+p_3)\,y}{p_7(2\lambda+p_1-r_1^2)\sqrt{\lambda+c^2}\,(\lambda^2+p_5\lambda+p_{10})^{\frac{3}{2}}}\right],$$

$$\Omega_{yy} = n^2-\frac{\mu}{r_2^3}+\frac{3\mu}{r_2^5}y^2-3(1-\mu)\left[\frac{1}{p_8}\left(\frac{E(\varphi,k)}{p_9}+\frac{F(\varphi,k)}{p_6}\right)\right.$$

$$-\left\{1-k^2\sin^2\varphi\,\frac{(x-\mu)^2}{p_6}+\right.$$

$$\left.\left(\frac{1}{p_6}+\frac{1-k^2\sin^2\varphi}{p_9}\right)y^2\right\}$$

$$\times\frac{(\lambda+p_4)}{2(\lambda+a^2)(2\lambda+p_1-r_1^2)\sqrt{\lambda+c^2}\sqrt{1-k^2\sin^2\varphi}}-$$

$$\left.\frac{(\lambda^2+2c^2\lambda+p_{11})(\lambda+p_4)y^2}{2p_7(2\lambda+p_1-r_1^2)\sqrt{\lambda+c^2}\,(\lambda^2+p_5\lambda+p_{10})^{\frac{3}{2}}}\right]-$$

$$3(1-\mu)y\left[A_9-A_6\,\frac{(\lambda+p_4)}{2(\lambda+a^2)(2\lambda+p_1-r_1^2)\sqrt{\lambda+c^2}\sqrt{1-k^2\sin^2\varphi}}-\right.$$

$$\left\{1-k^2\sin^2\varphi\,\frac{(x-\mu)^2}{p_6}+\left(\frac{1}{p_6}+\frac{1-k^2\sin^2\varphi}{p_9}\right)y^2\right\}A_{10}-$$

$$\left.\frac{(\lambda^2+2c^2\lambda+p_{11})(\lambda+p_4)y}{p_7(2\lambda+p_1-r_1^2)\sqrt{\lambda+c^2}\,(\lambda^2+p_5\lambda+p_{10})^{\frac{3}{2}}}-A_{11}\,y^2+A_{12}\right],$$

where

$$A_1=\frac{(x-\mu)(\lambda+p_3)k^2\sin^2\varphi}{p_6(\lambda+a^2)(2\lambda+p_1-r_1^2)\sqrt{\lambda+c^2}\sqrt{1-k^2\sin^2\varphi}},$$

$$A_2=\frac{p_8(x-\mu)(\lambda+p_3)k^2\sin 2\varphi}{(\lambda+a^2)(2\lambda+p_1-r_1^2)\sqrt{\lambda+c^2}}\left(\frac{(x-\mu)^2}{p_6}+\frac{y^2}{p_9}\right)-\frac{2k^2(x-\mu)\sin^2\varphi}{p_6},$$

$$A_3=\frac{(x-\mu)(\lambda+p_3)}{(\lambda+a^2)(2\lambda+p_1-r_1^2)^2\sqrt{\lambda+c^2}\sqrt{1-k^2\sin^2\varphi}}$$

$$\left[1-\frac{\lambda+p_3}{\lambda+a^2}-\left(\frac{2(\lambda+p_3)}{2\lambda+p_1-r_1^2}-1\right)-\frac{\lambda+p_3}{2(\lambda+c^2)}\right.$$

$$\left.-\frac{p_8(\lambda+p_3)k^2\sin 2\varphi}{4(\lambda+a^2)(1-k^2\sin^2\varphi)\sqrt{\lambda+c^2}}\right],$$

$$A_4=\frac{y^2(x-\mu)(\lambda+p_3)}{2p_7\sqrt{\lambda+c^2}\,(2\lambda+p_1-r_1^2)^2(\lambda^2+p_5\lambda+p_{10})^{\frac{3}{2}}}$$

$$\left[4(\lambda+c^2)(\lambda+p_3)+2(\lambda^2+2c^2\lambda+p_{11})-\right.$$

$$\left.(\lambda^2+2c^2\lambda+p_{11})(\lambda+p_3)\left\{\frac{4}{(2\lambda+p_1-r_1^2)}-\frac{2}{\lambda+p_3}+\frac{1}{\lambda+c^2}+\frac{3(2\lambda+p_5)}{\lambda^2+p_5\lambda+p_{10}}\right\}\right],$$

$$A_5=\frac{(\lambda+p_4)\,y\,k^2\sin^2\varphi}{p_6(\lambda+a^2)(2\lambda+p_1-r_1^2)\sqrt{\lambda+c^2}\sqrt{1-k^2\sin^2\varphi}},$$

$$A_6 = \frac{p_8\ (\lambda + p_4)\ y\ \ k^2 \sin 2\varphi}{(\lambda + a^2)(2\lambda + p_1 - r_1^2)\sqrt{\lambda + c^2}} \left(\frac{(x-\mu)^2}{p_6} + \frac{y^2}{p_9} \right)$$

$$+2y\left(\frac{1}{p_6} + \frac{1-k^2 \sin^2\varphi}{p_9} \right),$$

$$A_7 = \frac{(\lambda + p_4)\ y}{(\lambda + a^2)(2\lambda + p_1 - r_1^2)^2\sqrt{\lambda + c^2}\sqrt{1-k^2\sin^2\varphi}}$$

$$\left[1 - \frac{\lambda + p_3}{\lambda + a^2} - \left(\frac{2}{2\lambda + p_1 - r_1^2} - \frac{1}{\lambda + p_4} \right)(\lambda + p_3) \right.$$

$$\left. - \frac{\lambda + p_3}{2(\lambda + c^2)} - \frac{p_8\ (\lambda + p_3)k^2 \sin 2\varphi}{4(\lambda + a^2)\sqrt{\lambda + c^2}(1-k^2\sin^2\varphi)} \right],$$

$$A_8 = \frac{y}{2p_7\sqrt{\lambda + c^2}(2\lambda + p_1 - r_1^2)^2(\lambda^2 + p_5\lambda + p_{10})^{\frac{3}{2}}}$$

$$\Big[2(2\lambda + p_1 - r_1^2)(\lambda + p_3)(\lambda^2 + 2c^2\lambda + p_{11}) +$$

$$2(\lambda + p_4)y^2\left\{ 2(\lambda + c^2)(\lambda + p_3) + (\lambda^2 + 2c^2\lambda + p_{11}) \right\}$$

$$-(\lambda^2 + 2c^2\lambda + p_{11})(\lambda + p_3)(\lambda + p_4)y^2 \times$$

$$\left. \left\{ \frac{4}{(2\lambda + p_1 - r_1^2)} - \frac{2}{\lambda + p_4} + \frac{1}{\lambda + c^2} + \frac{3(2\lambda + p_5)}{\lambda^2 + p_5\lambda + p_{10}} \right\} \right],$$

$$A_9 = \frac{-y\ (\lambda + p_4)}{(\lambda + a^2)(2\lambda + p_1 - r_1^2)\sqrt{\lambda + c^2}\sqrt{1-k^2\sin^2\varphi}}$$

$$\left[\frac{1}{p_6} + \frac{1-k^2\sin^2\varphi}{p_9} \right],$$

$$A_{10} = \frac{(\lambda + p_4)\ y}{(\lambda + a^2)(2\lambda + p_1 - r_1^2)^2\sqrt{\lambda + c^2}\sqrt{1-k^2\sin^2\varphi}}$$

$$\left[1 - \frac{\lambda + p_4}{\lambda + a^2} - \left(\frac{\lambda + p_4}{2\lambda + p_1 - r_1^2} - 1 \right) \right.$$

$$\left. - \frac{\lambda + p_4}{2(\lambda + c^2)} - \frac{p_8\ (\lambda + p_4)k^2 \sin 2\varphi}{4(\lambda + a^2)(1-k^2\sin^2\varphi)\sqrt{\lambda + c^2}} \right],$$

$$A_{11} = \frac{y\,(\lambda + p_4)}{p_7\,(2\lambda + p_1 - r_1^2)^2\sqrt{\lambda + c^2}(\lambda^2 + p_5\lambda + p_{10})^{\frac{3}{2}}} \times$$

$$\left[\begin{array}{l} (2\lambda + p_1 - (x-\mu)^2)(\lambda^2 + 2c^2\lambda + p_{11}) \\ +2y^2(\lambda + c^2)(\lambda + p_4) - (\lambda^2 + 2c^2\lambda + p_{11})(\lambda + p_4)y^2 \\ \left\{ \frac{2}{\left(2\lambda + p_1 - r_1^2\right)} - \frac{1}{(\lambda + p_4)} + \frac{1}{2\left(\lambda + c^2\right)} + \frac{3(2\lambda + p_5)}{2(\lambda^2 + p_5\lambda + p_{10})} \right\} \end{array} \right],$$

$$A_{12} = \frac{y\,(\lambda + p_4)\left[\lambda^2 + p_5\lambda + p_{10} - (\lambda + c^2)(2\lambda + p_5) \right]}{p_7\,(2\lambda + p_1 - r_1^2)\sqrt{\lambda + c^2}(\lambda^2 + p_5\lambda + p_{10})^{\frac{3}{2}}}.$$