International Journal of Advanced Astronomy, 4 (1) (2016) 43-48

International Journal of Advanced Astronomy

Website: www.sciencepubco.com/index.php/IJAA
doi: 10.14419/ijaa.v4i1.5711
Research paper

Existence and stability of triangular points in the relativistic R3BP when the bigger primary is a triaxial rigid body and a source of radiation

Jagadish Singh [1]*, Nakone Bello [2]

[1] *Department of Mathematics, Faculty of Science, Ahmadu Bello University, Zaria. Nigeria*
[2] *Department of Mathematics, Faculty of Science Usmanu Danfodiyo University, Sokoto, Nigeria*
**Corresponding author E-mail:jgds2004@yahoo.com*

Abstract

We study the effect of triaxiality and radiation of the bigger primary on the positions and stability of the triangular points in the relativistic R3BP. It is found that the locations of the triangular points are affected by the relativistic terms apart the radiation force and the triaxiality of the bigger primary. It is also seen that for these points, the range of stability region increases or decreases according as without which depends upon the relativistic terms, the radiation and triaxiality coefficient is greater than or less than zero. A practical application of this model could be the study of the motion of a dust grain particle near the Sun-Earth system. A practical application of this model could be the study of the motion of a dust grain particle near the Sun-Earth system.

Keywords: *Celestial Mechanics; Radiation; Triaxiality; Relativity; R3BP.*

1. Introduction

The circular restricted three-body problem (CR3BP) describes the dynamics of a body having infinitesimal mass and moving in the gravitational field of two massive bodies called primaries, which revolve around their common center of mass in circular orbits on account of their mutual attraction. It is originally formulated due to the approximately circular motion of the planets around the Sun, and the small masses of asteroids and the satellites of planets compared to the planets' masses.

The infinitesimal mass can be at rest in a rotating coordinate frame, at five equilibrium points, where the gravitational and centrifugal forces just balance each other. Three of them are the collinear points L_1, L_2, L_3 lying on the line connecting the primaries, while the other two are the triangular points L_4, L_5, forming equilateral triangles with the primaries. The latters are linearly stable for the mass ratio μ of the primaries less than $\mu < 0.03852...$ (Szebehely [1]). Their stability occurs although the potential energy has a maximum rather than a minimum at L_4 and L_5. The stability is actually achieved through the influence of the Coriolis force, because the coordinate system is rotated (Wintner [2]; Contopoulos [3]). The bodies in the R3BP are strictly spherical in shape, but in nature, celestial bodies are not perfect spheres. They are either oblate or triaxial. The lack of sphericity or the oblateness of the planets causes large perturbations from a two-body orbit. The motions of artificial Earth satellites are examples of this. This motivates many investigators (SubbaRao and Sharma [4]; AbdulRaheem and Singh [5]; Sharma [6]; Idrisi et al. [7]). The effect of triaxiality and radiation of the primaries on the existence and stability of libration points in the CR3BP was analyzed by e.g El-Shaboury [8], Sharma et al. [9], [10], Khanna and Bhatnagar [11], Singh [12] to study CR3BP with oblateness or triaxiality of the bodies.

In general relativity, even writing down the equations of motion in the simplest case N=2 is difficult. Unlike in Newton's theory, it is impossible to express the acceleration by means of the positions and velocities, in a way which would be valid within the "Exact" theory. Therefore, the approximation method is needed.

Historically, the equations of motion of the problem of N bodies considered as point masses were first obtained by generalizing the geodesic principle. By the use of this method, De sitter [13] first derived the relativistic equation of N-body problem. Some arithmetic errors occurred in these equations are reproduced in the encyclopedic paper of Kottler [14] and treatises by Chazy [15], [16], but were corrected by Eddington and Clark [17]. Brumberg [18], [19] studied the problem in more details and collected most of the important results of relativistic celestial mechanics. He has not only obtained the equations of motion for the general problem of the three bodies, but also deduced the equations of motion for the restricted problem of three bodies.

Bhatnagar and Hallan [20] studied the existence and linear stability of the triangular points $L_{4,5}$ in the relativistic R3BP, they concluded that $L_{4,5}$ are always unstable in the whole range $0 \le \mu \le \frac{1}{2}$ in contrast to the classical R3BP where they are stable for $0 < \mu < \mu_0$, where μ is the mass ratio and $\mu_0 = 0.03852...$ is the Routh's value. Douskos and Perdios [21] examined the stability of the triangular points in the relativistic R3BP and contrary to the results of Bhatnagar and Hallan [20], they obtained a region of linear stability in the parameter space $0 \le \mu < \mu_0 - \frac{17\sqrt{69}}{486c^2}$ where $\mu_0 = 0.03852...$ is Routh's value.

In recent times, many perturbing forces, i.e. oblateness and radiation forces of the primaries, Coriolis and centrifugal forces, have been included in the study of the relativistic R3BP.

The locations of libration points in the relativistic R3BP, when one or more additional effects are included in the potential due to radiation pressure and the oblateness of the primaries, were studied by Abd El-Salam and Abd El-Bar [22] and Katour et al. [23]. The locations of triangular points and their linear stability when the bigger primary is radiating in the relativistic R3BP were examined by Singh and Bello [24]. The locations of triangular points and their linear stability in the presence of small perturbation given to the centrifugal force were also investigated by Singh and Bello [25].

In all the studies previously mentioned in the relativistic R3BP, no work is performed in the direction of linear stability of the triangular point in the presence of both radiation and triaxiality. Hence, the idea of the radiation pressure force together with triaxiality of bigger primary raises a curiosity in our mind to study the "stability of triangular points in the relativistic R3BP".

This paper is organized as follows: In Sect. 2, the equations governing the motion are presented; Sect. 3 describes the positions of triangular points, while their linear stability is analyzed in Sect.4; a discussion of these results is given in Sect. 5, finally sect. 6 summarizes the conclusions and findings of our paper.

2. Equations of motion

The pertinent equations of motion of an infinitesimal mass in the relativistic R3BP in a barycentric synodic coordinate system (ξ,η) and dimensionless variables can be written as Brumberg [18] and Bhatnagar and Hallan [20]:

$$\ddot{\xi}-2n\dot{\eta}=\frac{\partial W}{\partial \xi}-\frac{d}{dt}\left(\frac{\partial W}{\partial \dot{\xi}}\right)$$

$$\ddot{\eta}+2n\dot{\xi}=\frac{\partial W}{\partial \eta}-\frac{d}{dt}\left(\frac{\partial W}{\partial \dot{\eta}}\right) \tag{1}$$

With

$$\begin{aligned} W=&\frac{1}{2}(\xi^2+\eta^2)+\frac{1-\mu}{\rho_1}+\frac{\mu}{\rho_2}+\frac{1}{c^2}\Bigg[-\frac{3}{2}\left(1-\frac{1}{3}\mu(1-\mu)\right)(\xi^2+\eta^2)+\\ &\frac{1}{8}\left\{\dot{\xi}^2+\dot{\eta}^2+2(\xi\dot{\eta}-\eta\dot{\xi})+(\xi^2+\eta^2)\right\}^2\\ &+\frac{3}{2}\left(\frac{1-\mu}{\rho_1}+\frac{\mu}{\rho_2}\right)\left(\dot{\xi}^2+\dot{\eta}^2+2(\xi\dot{\eta}-\eta\dot{\xi})+(\xi^2+\eta^2)\right)-\\ &\frac{1}{2}\left(\frac{(1-\mu)^2}{\rho_1^2}+\frac{\mu^2}{\rho_2^2}\right)+\mu(1-\mu)\Bigg\{\left(4\dot{\eta}+\frac{7}{2}\xi\right)\left(\frac{1}{\rho_1}-\frac{1}{\rho_2}\right)\\ &-\frac{\eta^2}{2}\left(\frac{\mu}{\rho_1^3}+\frac{1-\mu}{\rho_2^3}\right)+\left(\frac{-1}{\rho_1\rho_2}+\frac{3\mu-2}{2\rho_1}+\frac{1-3\mu}{2\rho_2}\right)\Bigg\}\Bigg]\end{aligned} \tag{2}$$

$$n=1-\frac{3}{2c^2}\left(1-\frac{1}{3}\mu(1-\mu)\right) \tag{3}$$

$$\rho_1^2=(\xi+\mu)^2+\eta^2$$

$$\rho_2^2=(\xi+\mu-1)^2+\eta^2 \tag{4}$$

Where $0<\mu\le\frac{1}{2}$ is the ratio of the mass of the smaller primary to the total mass of the primaries, ρ_1 and ρ_2 are distances of the infinitesimal mass from the bigger and smaller primary, respectively; n is the mean motion of the primaries; c is the velocity of light.

We now introduce the triaxiality factors of the bigger primary with the help of the parameter $\sigma_i \ll 1 (i=1,2)$, where $\sigma_1=\frac{a^2-c^2}{5R^2}$, $\sigma_2=\frac{b^2-c^2}{5R^2}$. (McCuskey [26]). Here σ_1,σ_2 characterize the triaxiality of the bigger primary with a,b,c as lengths of its semi-axes and R is the dimensional distance between the primaries. The radiation factor q_1 is given by $F_{p1}=F_{g1}(1-q_1)$ such that $0\le(1-q_1)\ll 1$ Radzievskii [27] For simplicity, putting $q_1=1-(1-q_1)=1-\delta$ where $0\le\delta=1-q_1\ll 1$ and neglecting second and higher powers of $\sigma_i(i=1,2)$ and δ, and also their products, we take the equations of motion as:

$$\ddot{\xi}-2n\dot{\eta}=\frac{\partial W}{\partial \xi}-\frac{d}{dt}\left(\frac{\partial W}{\partial \dot{\xi}}\right)$$

$$\ddot{\eta}+2n\dot{\xi}=\frac{\partial W}{\partial \eta}-\frac{d}{dt}\left(\frac{\partial W}{\partial \dot{\eta}}\right) \tag{5}$$

Where W is the potential-like function of the relativistic R3BP. As Katour et al. [23], we do not include the parameters $\sigma_i(i=1,2)$ and δ in the relativistic part of W since the magnitude of these terms is so small due to c^{-2}.

$$\begin{aligned} W=&\frac{1}{2}\left(1+\frac{3}{2}(2\sigma_1-\sigma_2)\right)(\xi^2+\eta^2)+\frac{(1-\delta)(1-\mu)}{\rho_1}+\frac{\mu}{\rho_2}\\ &+\frac{1-\mu}{2\rho_1^3}(2\sigma_1-\sigma_2)+\frac{3(1-\mu)\eta^2}{2\rho_1^5}(\sigma_2-\sigma_1)\\ &+\frac{1}{c^2}\Bigg[-\frac{3}{2}\left\{1-\frac{1}{3}\mu(1-\mu)\right\}(\xi^2+\eta^2)\\ &+\frac{1}{8}\left\{\dot{\xi}^2+\dot{\eta}^2+2\left(\xi\dot{\eta}-\eta\dot{\xi}\right)+(\xi^2+\eta^2)\right\}^2\\ &+\frac{3}{2}\left(\frac{1-\mu}{\rho_1}+\frac{\mu}{\rho_2}\right)\left\{\begin{matrix}\dot{\xi}^2+\dot{\eta}^2++2\left(\xi\dot{\eta}-\eta\dot{\xi}\right)\\ +(\xi^2+\eta^2)\end{matrix}\right\}-\frac{1}{2}\left(\frac{(1-\mu)^2}{\rho_1^2}+\frac{\mu^2}{\rho_2^2}\right)\\ &+\mu(1-\mu)\left\{\left(4\dot{\eta}+\frac{7}{2}\xi\right)\left(\frac{1}{\rho_1}-\frac{1}{\rho_2}\right)\begin{matrix}-\frac{\eta^2}{2}\left(\frac{\mu}{\rho_1^3}+\frac{1-\mu}{\rho_2^3}\right)\\ +\left(\frac{-1}{\rho_1\rho_2}+\frac{3\mu-2}{2\rho_1}+\frac{1-3\mu}{2\rho_2}\right)\end{matrix}\right\}\Bigg],\end{aligned} \tag{6}$$

And n is the perturbed mean motion of the primaries and is given by

$$n=1+\frac{3}{4}(2\sigma_1-\sigma_2)-\frac{3}{2c^2}\left(1-\frac{1}{3}\mu(1-\mu)\right) \tag{7}$$

3. Location of triangular points

The libration points are obtained from equation (5) after putting

$$\dot{\xi}=\dot{\eta}=\ddot{\xi}=\ddot{\eta}=0.$$

These points are the solutions of the equations

$$\frac{\partial W}{\partial \xi}=0=\frac{\partial W}{\partial \eta} \text{ With } \dot{\xi}=\dot{\eta}=0.$$

That is

$$\xi-\frac{(1-\delta)(1-\mu)(\xi+\mu)}{\rho_1^3}-\frac{\mu(\xi-1+\mu)}{\rho_2^3}$$
$$+\left(3\sigma_1-\frac{3}{2}\sigma_2\right)\xi-\frac{3(1-\mu)(\xi+\mu)(2\sigma_1-\sigma_2)}{2\rho_1^5}$$
$$-\frac{15(1-\mu)(\xi+\mu)(\sigma_2-\sigma_1)\eta^2}{2\rho_1^7}+\frac{1}{c^2}\left[\begin{array}{l}-3\xi\left\{1-\frac{\mu(1-\mu)}{3}\right\}\\+\frac{1}{2}\xi(\xi^2+\eta^2)\end{array}\right.$$
$$-\frac{3}{2}(\xi^2+\eta^2)\left\{\frac{(1-\mu)(\xi+\mu)}{\rho_1^3}+\frac{\mu(\xi-1+\mu)}{\rho_2^3}\right\}$$
$$+3\left(\frac{1-\mu}{\rho_1}+\frac{\mu}{\rho_2}\right)\xi+\frac{(1-\mu)^2(\xi+\mu)}{\rho_1^4}$$
$$+\frac{\mu^2(\xi-1+\mu)}{\rho_2^4}+\mu(1-\mu)\left\{\frac{7}{2}\left(\frac{1}{\rho_1}-\frac{1}{\rho_2}\right)\right.$$
$$+\frac{7}{2}\xi\left(-\frac{(\xi+\mu)}{\rho_1^3}+\frac{(\xi-1+\mu)}{\rho_2^3}\right)+\frac{3}{2}\eta^2\left(\frac{\mu(\xi+\mu)}{\rho_1^5}+\frac{(1-\mu)(\xi-1+\mu)}{\rho_2^5}\right)$$
$$+\frac{(\xi+\mu)}{\rho_1^3\rho_2}+\frac{(\xi-1+\mu)}{\rho_1\rho_2^3}$$
$$\left.\left.-\frac{(3\mu-2)(\xi+\mu)}{2\rho_1^3}-\frac{(1-3\mu)(\xi-1+\mu)}{2\rho_2^3}\right\}\right]=0 \tag{8}$$

And

$$\eta F=0,$$

With

$$F=\left(1-\frac{(1-\delta)(1-\mu)}{\rho_1^3}-\frac{\mu}{\rho_2^3}\right)+\left(3\sigma_1-\frac{3}{2}\sigma_2\right)+\frac{3(1-\mu)}{\rho_1^5}\left(\frac{3}{2}\sigma_2-2\sigma_1\right)$$
$$-\frac{15(1-\mu)(\sigma_2-\sigma_1)\eta^2}{2\rho_1^7}$$
$$+\frac{1}{c^2}\left[-3\left(1-\frac{\mu(1-\mu)}{3}\right)+\frac{1}{2}(\xi^2+\eta^2)+3\left(\frac{1-\mu}{\rho_1^3}+\frac{\mu}{\rho_2^3}\right)\right.$$
$$-\frac{3}{2}(\xi^2+\eta^2)\left(\frac{1-\mu}{\rho_1^3}+\frac{\mu}{\rho_2^3}\right)+\left(\frac{(1-\mu)^2}{\rho_1^4}+\frac{\mu^2}{\rho_2^4}\right)$$
$$+\mu(1-\mu)\left\{\frac{7}{2}\xi\left(-\frac{1}{\rho_1^3}+\frac{1}{\rho_2^3}\right)-\left(\frac{\mu}{\rho_1^3}+\frac{1-\mu}{\rho_2^3}\right)+\frac{3}{2}\eta^2\left(\frac{\mu}{\rho_1^5}+\frac{1-\mu}{\rho_2^5}\right)\right.$$
$$\left.\left.+\frac{1}{\rho_1^3\rho_2}+\frac{1}{\rho_1\rho_2^3}-\frac{(3\mu-2)}{2\rho_1^3}-\frac{(1-3\mu)}{2\rho_2^3}\right\}\right]$$

Following as Singh and Bello [24], from the system (8) with $\eta\neq 0$, we have obtained the coordinates of the triangular points $(\xi,\pm\eta)$ as

$$\xi=\frac{1-2\mu}{2}\left(1+\frac{5}{4c^2}\right)+\left(\frac{1}{8}-\frac{1}{2\mu}\right)\sigma_1-\left(\frac{1}{2\mu}+\frac{3}{8}\right)\sigma_2-\frac{1}{3}\delta \tag{9}$$

$$\eta=\pm\frac{\sqrt{3}}{2}\left[1+\frac{1}{12c^2}\left(-5+6\mu-6\mu^2\right)+\frac{2}{3}\left\{\begin{array}{l}\left(-\frac{23}{8}+\frac{1}{2\mu}\right)\sigma_1\\+\left(\frac{19}{8}-\frac{1}{2\mu}\right)\sigma_2\end{array}\right\}-\frac{2}{9}\delta\right].$$

These points are denoted by L_4 and L_5 respectively.

4. Stability of triangular points

Since the nature of linear stability about the point L_5 will be similar to that about L_4, it will be sufficient to consider here the stability only near L_4.

Let (a,b) be the coordinates of the triangular point L_4

We set $\xi=a+\alpha, \eta=b+\beta, (\alpha,\beta<<1)$ in the equations (5) of motion. First, we compute the terms of their R.H.S, neglecting second and higher order terms, we get

$$\left(\frac{\partial W}{\partial \xi}\right)_{\xi=a+\alpha,\,\eta=b+\beta}=A\alpha+B\beta+C\dot{\alpha}+D\dot{\beta}$$

Where,

$$A=\frac{3}{4}\left\{1+\frac{1}{2c^2}\left(2-19\mu+19\mu^2\right)\right\}+\frac{3\left(15\mu^2+19\mu-8\right)}{16\mu}\sigma_1$$
$$-\frac{3\left(31\mu^2+\mu-8\right)}{16\mu}\sigma_2+\frac{1}{2}(3\mu-1)\delta,$$

$$B=\frac{3\sqrt{3}}{4}(1-2\mu)\left(1-\frac{2}{3c^2}\right)-\frac{\sqrt{3}\left(89\mu^2-47\mu+8\right)}{16\mu}\sigma_1$$
$$+\frac{\sqrt{3}\left(37\mu^2-9\mu+8\right)}{16\mu}\sigma_2-\frac{\sqrt{3}}{6}(1+\mu)\delta,$$

$$C=\frac{\sqrt{3}}{2c^2}(1-2\mu),$$

$$D=\frac{6-5\mu+5\mu^2}{2c^2}.$$

Similarly, we obtain

$$\left(\frac{\partial W}{\partial \eta}\right)_{\xi=a+\alpha,\,\eta=b+\beta}=A_1\alpha+B_1\beta+C_1\dot{\alpha}+D_1\dot{\beta}$$

Where,

$$A_1=\frac{3\sqrt{3}}{4}(1-2\mu)\left(1-\frac{2}{3c^2}\right)-\frac{\sqrt{3}\left(89\mu^2-47\mu+8\right)}{16\mu}\sigma_1$$
$$+\frac{\sqrt{3}\left(37\mu^2-9\mu+8\right)}{16\mu}\sigma_2-\frac{\sqrt{3}}{6}(1+\mu)\delta,$$

$$B_1 = \frac{9}{4}\left\{1+\frac{7}{6c^2}\left(-2+3\mu-3\mu^2\right)\right\} - \frac{3\left(15\mu^2-29\mu-8\right)}{16\mu}\sigma_1 + \frac{3\left(15\mu^2-7\mu-8\right)}{16\mu}\sigma_2 + \frac{1}{2}(1-3\mu)\delta,$$

$$C_1 = \frac{1}{2c^2}\left(-4+\mu-\mu^2\right),$$

$$D_1 = -\frac{\sqrt{3}(1-2\mu)}{2c^2}.$$

$$\frac{d}{dt}\left(\frac{\partial W}{\partial \dot{\xi}}\right)_{\xi=a+\alpha,\,\eta=b+\beta} = A_2\dot{\alpha} + B_2\dot{\beta} + C_2\ddot{\alpha} + D_2\ddot{\beta}$$

Where,

$$A_2 = \frac{\sqrt{3}}{2c^2}(1-2\mu),$$

$$B_2 = \frac{1}{2c^2}\left(-4+\mu-\mu^2\right),$$

$$C_2 = \frac{1}{4c^2}\left(17-2\mu+2\mu^2\right),$$

.

$$D_2 = -\frac{\sqrt{3}}{4c^2}(1-2\mu).$$

$$\frac{d}{dt}\left(\frac{\partial W}{\partial \dot{\eta}}\right)_{\xi=a+\alpha,\,\eta=b+\beta} = A_3\dot{\alpha} + B_3\dot{\beta} + C_3\ddot{\alpha} + D_3\ddot{\beta}$$

Where

$$A_3 = \frac{1}{2c^2}\left(6-5\mu+5\mu^2\right),$$

$$B_3 = -\frac{\sqrt{3}}{2c^2}(1-2\mu),$$

$$C_3 = -\frac{\sqrt{3}}{4c^2}(1-2\mu),$$

$$D_3 = \frac{3\left(5-2\mu+2\mu^2\right)}{4c^2}.$$

The characteristic equation of the variational equations of motion corresponding to (5) can be expressed as

$$\lambda^4 + \left\{1-\frac{9}{c^2}+3\sigma_1+\frac{3}{2}(2\mu-3)\sigma_2\right\}\lambda^2 + \left\{\frac{27\mu(1-\mu)}{4} + \frac{\left(108\mu^4-216\mu^3+693\mu^2-585\mu\right)}{8c^2} + \frac{3}{2}\mu(1-\mu)\delta + \frac{9\left(-89\mu^2+99\mu-10\right)}{16}\sigma_1 + \frac{9\left(37\mu^2-47\mu+10\right)}{16}\sigma_2\right\} = 0 \quad (10)$$

For $\frac{1}{c^2} \to 0$ and when the bigger primary is non-luminous and non triaxial (i.e. $\sigma_1 = \sigma_2 = \delta = 0$), this reduces to its well-known classical restricted problem form (see e.g. Szebehely, [1]):

$$\lambda^4 + \lambda^2 + \frac{27}{4}\mu(1-\mu) = 0.$$

The discriminant of (10) is

$$\Delta = \frac{-54}{c^2}\mu^4 + \frac{108}{c^2}\mu^3 + \left(27+\frac{801}{4}\sigma_1-\frac{333}{4}\sigma_2+6\delta-\frac{693}{2c^2}\right)\mu^2 + \left(-27-\frac{891}{4}\sigma_1+\frac{447}{4}\sigma_2-6\delta+\frac{585}{2c^2}\right)\mu + 1 - \frac{18}{c^2} + \frac{57}{2}\sigma_1 - \frac{63}{2}\sigma_2 \quad (11)$$

Its roots are

$$\lambda^2 = \frac{-b \pm \sqrt{\Delta}}{2} \quad (12)$$

Where

$$b = \left(1-\frac{9}{c^2}\right) + 3\sigma_1 + \frac{3}{2}(2\mu-3)\sigma_2$$

From (11), we have

$$\frac{d\Delta}{d\mu} = \frac{-216}{c^2}\mu^3 + \frac{324}{c^2}\mu^2 + 2\left(27+\frac{801}{4}\sigma_1-\frac{333}{4}\sigma_2+6\delta-\frac{693}{2c^2}\right)\mu \left(-27-\frac{891}{4}\sigma_1+\frac{447}{4}\sigma_2-6\delta+\frac{585}{2c^2}\right) < 0\,\forall\mu\in\left(0,\frac{1}{2}\right]. \quad (13)$$

From (13), it can be easily seen that Δ is monotone decreasing in $\left(0,\frac{1}{2}\right]$.

But

$$(\Delta)_{\mu=0} = 1+\frac{57}{2}\sigma_1-\frac{63}{2}\sigma_2-\frac{18}{c^2} > 0$$

$$(\Delta)_{\mu=\frac{1}{2}} = -\frac{23}{4}-\frac{525}{16}\sigma_1+\frac{57}{16}\sigma_2-\frac{3}{2}\delta+\frac{207}{4c^2} < 0 \quad (14)$$

Since $(\Delta)_{\mu=0}$ and $(\Delta)_{\mu=\frac{1}{2}}$ are of opposite signs, and Δ is monotone decreasing and continuous, there is one value of μ, e.g. μ_c in the interval $\left(0,\frac{1}{2}\right]$ for which Δ vanishes.

Solving the equation $\Delta = 0$, using (11), we obtain critical value of the mass parameter as

$$\mu_c = \frac{1}{2} - \frac{1}{18}\sqrt{69} - \frac{17\sqrt{69}}{486c^2} + \frac{1}{2}\left(\frac{5}{6} + \frac{59}{9\sqrt{69}}\right)\sigma_1 - \frac{1}{2}\left(\frac{19}{18} + \frac{85}{9\sqrt{69}}\right)\sigma_2 - \frac{2}{27\sqrt{69}}\delta \quad (15)$$

$$\mu_c = \mu_0 - \frac{17\sqrt{69}}{486c^2} + \frac{1}{2}\left(\frac{5}{6} + \frac{59}{9\sqrt{69}}\right)\sigma_1 - \frac{1}{2}\left(\frac{19}{18} + \frac{85}{9\sqrt{69}}\right)\sigma_2 - \frac{2}{27\sqrt{69}}\delta$$

Where $\mu_0 = 0.03852...$ is the Routh's value?

We consider the following three regions of the values of μ separately.

When $0 \le \mu < \mu_c$, $\Delta > 0$ the values of λ^2 given by (12) are negative and therefore all the four characteristic roots are distinct pure imaginary numbers. Hence, the triangular points are stable.

When $\mu_c < \mu \le \frac{1}{2}, \Delta < 0$ $\mu_c < \mu \le \frac{1}{2}, \Delta < 0$, the real parts of the characteristic roots are positive. Therefore, the triangular points are unstable.

When $\mu = \mu_c, \Delta = 0$, the values of λ^2 given by (12) are the same. This induces instability of the triangular points due to the presence of secular terms in the solution of linearized equations of motion in the vicinity of these points.

Hence, the stability region is

$$0 < \mu < \mu_0 - \frac{17\sqrt{69}}{486c^2} + \frac{1}{2}\left(\frac{5}{6} + \frac{59}{9\sqrt{69}}\right)\sigma_1 - \frac{1}{2}\left(\frac{19}{18} + \frac{85}{9\sqrt{69}}\right)\sigma_2 - \frac{2}{27\sqrt{69}}\delta \quad (16)$$

5. Discussion

Equations (5) - (6) describe the motion of a third body under the influence of relativistic terms and triaxiality and radiation of the bigger primary. Equations (9) and (15) give respectively the positions of triangular points and critical mass parameter. Equation (16) describes the region of stability. It can be seen both positions, and critical mass depend upon relativistic terms, triaxiality and radiation factors. It may be noted here that in this problem, the triangular points no longer form equilateral triangles with the primaries as they do in the classical case. Rather, they form scalene triangles with the primaries. It can also be seen from (16) that the relativistic, radiation and triaxiality terms all reduce the size of stability region.

In the absence of radiation and triaxiality $(\text{i.e. } \delta = \sigma_1 = \sigma_2 = 0)$, the positions of triangular points obtained in this study correspond to those of Bhatnagar and Hallan [20], Douskos and Perdios [21].

In the absence of triaxiality $(\text{i.e. } \sigma_1 = \sigma_2 = 0)$ the results of the present study are in accordance with those of Singh and Bello [24] when the coupling terms $\frac{\delta}{c^2}$ neglected in their study.

In the absence of radiation and triaxiality $(\text{i.e. } \delta = \sigma_1 = \sigma_2 = 0)$, the stability results obtained are in agreement with those of Douskos and Perdios [21] and disagree with those of Bhatnagar and Hallan [20].

In the absence of relativistic terms, the results of the present study coincide with those of Sharma et al. [10] and with those of Singh [12] when the perturbations are absent and the bigger primary is triaxial and luminous only.

6. Conclusion

By considering the bigger primary as radiating and triaxial rigid body in the relativistic R3BP, we have determined the positions of triangular points and have examined their linear stability. It is found that their positions and stability region are affected by relativistic terms, radiation and triaxiality of the bigger primary. It is also noticed that the expression for A, D, A_2, C_2 in Bhatnagar and Hallan [20] differ from the present study when the radiation pressure and triaxiality are absent. Consequently, the characteristic equations are also different. This led them (Bhatnagar and Hallan [20]) to conclude that triangular points are unstable, contrary to Douskos and Perdios [21] and our results.

References

[1] Szebehely V (1967), Theory of orbits. The restricted problem of three- bodies. Academic Press, New York.

[2] Wintner A(1941), The Analytical Foundations of celestial mechanics, Princeton University Press, Princeton

[3] Contopolous G (2002), rder and Chaos in Dynamical Astronomy. Springer, Berlin.

[4] SubbaRao PV&Sharma RK (1975), a note on the stability of the triangular points of equilibrium in the restricted three-body problem, Astronomy & Astrophysics 43, 381-383.

[5] AbdulRaheem A & Singh J (2006), Combined effects of perturbations,radiation and oblateness on the stability of equilibrium points in the restricted three-body problem, Astronomical Journal,131,1880-1885 http://dx.doi.org/10.1086/499300.

[6] Sharma RK (1987), The linear stability of libration points of the photogravitational restricted three-body problem when the smaller primary is an oblate spheroid, Astrophysics & Space Science, 135, 271-281 http://dx.doi.org/10.1007/BF00641562.

[7] Idrisi M J, Imran M& Taqvi Z A(2013), Existence and stability of libration points in the restricted three-body problem when one of the primaries is an oblate spheroid, International Journal of Applied Mathematics and Mechanics, 9(17), 1-12 .

[8] El-Shaboury SM (1991), The libration points of a triaxial satellite in the photogravitational restricted problem of three bodies, Indian Journal of Pure and Applied Mathematics, 22(8), 703-712 .

[9] Sharma RK, Taqvi ZA& Bhatnagar KB(2001), Existence and stability of the libration points in the restricted three-body problem the when the primaries are triaxial rigid bodies, Indian Journal of Pure and Applied Mathematics, 32(7), 944-981 .

[10] Sharma RK ,Taqvi ZA&Bhatnagar K B (2001), Existence and stability of the libration points in the restricted three-body problem when the bigger primary is a triaxial rigid body and source of radiation, Indian Journal of Pure and Applied Mathematics, 32(2) 255-266

[11] Khanna M & Bhatnagar KB (1999), Existence and stability of libration points in the restricted three-body problem when the smaller primary is a triaxial rigid body and the bigger one an oblate spheroid, Indian Journal of Pure and Applide Mathematics, 30 (7), 721-733

[12] Singh J (2013), the equilibrium points in the perturbed R3BP with triaxial and luminous primaries. Astrophysics &Space Science, 346, 41-50 http://dx.doi.org/10.1007/s10509-013-1420-7.

[13] De Sitter W (1916), On Einstein's theory of gravitation and its astronomical consequences, Monthly Notes of Royal Astronomical Society, 76, 699-728. http://dx.doi.org/10.1093/mnras/76.9.699.

[14] Kottler F (1922), Gravitation Und Relativitats theorie. Encykl. Math. Wiss 6 (2), No 22a.

[15] Chazy J (1928), La theorie de la relativite et la mecanique celeste vol.1, Paris Gauthier- villars .

[16] Chazy J (1930), La theorie de la relativite et la mecanique celeste vol.1, Paris Gauthier- villars.

[17] Eddington AS & Clark GL (1938), The problem of n bodies in general relativity theory, Proceeding of Royal Society of America,166, 465 http://dx.doi.org/10.1098/rspa.1938.0104.

[18] Brumberg V A (1972), Relativistic Celestial Mechanics. Nauka, Moscow

[19] Brumberg VA (1991), Essential Relativistic Celestial Mechanics. New York Adam Hilger.

[20] Bhatnagar K B & Hallan PP (1998), Existence and stability of $L_{4,5}$ in the relativistic restricted three-body problem, Celestial Mechanics and Dynamica Astronomy. 69(3), 271-281. http://dx.doi.org/10.1023/A:1008271021060.

[21] Douskos C N & Perdios E A (2002), on the stability of equilibrium points in the relativistic three-body problem, Celestial Mechanics and Dynamical Astroonomy, 82, 317-321. http://dx.doi.org/10.1023/A:1015296327786.

[22] Abd El-Salam F a & Abd El-Bar S E (2014), On the triangular equilibrium points in the photogravitional relativistic restricted

three-body problem, Astrophysics & Space Science, 349, 125-135. http://dx.doi.org/10.1007/s10509-013-1629-5.

[23] Katour D A, Abd El-Salam F A& Shaker MO (2014), Relativistic restricted three-body problem with oblateness and photo-gravitational corrections to triangular equilibrium points, Astrophysics & Space Science, 351(1), 143-149. http://dx.doi.org/10.1007/s10509-014-1826-x.

[24] Singh J & Bello N (2014), Effect of radiation pressure on the stability of $L_{4,\ 5}$ in relativistic R3BP, Astrophysics & Space Science, 351(2), 483-490. http://dx.doi.org/10.1007/s10509-014-1858-2.

[25] Singh J & Bello N (2014), Motion around L_4 in the perturbed relativistic R3BP, Astrophysics & Space Science, 351(2), 491-497. http://dx.doi.org/10.1007/s10509-014-1870-6.

[26] McCuskey S W (1963), Introduction to celestial mechanics, Addison-Wesley.

[27] Radzievskii VV (1950), The restricted problem of three odies taking account of light pressure,Astronomical Journal,250-256.