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Research paper

Non-collinear libration points in CR3BP when less massive primary is an heterogeneous oblate body with N-layers

M. Javed Idrisi [1]*, Kumari Shalini [2]

[1] *Department of Mathematics, Al-Falah University, Faridabad (Haryana) −121004, India*
[2] *Department of Mathematics, Deshbandhu College, University of Delhi, Kalkaji, New Delhi -110017, India*
**Corresponding author E-mail:mjavedidrisi@gmail.com*

Abstract

In the present paper, the existence of non-collinear libration points has been shown in circular restricted three-body problem when less massive primary is a heterogeneous oblate body with N-layers. Further, the stability of non-collinear libration points is investigated in linear sense and found that the non-collinear libration points are stable for the critical value of mass parameter $\mu \le \mu_{crit} = \mu_o - 3.32792\, k_1 - 1.16808\, k_2$.

Keywords: *Celestial Mechanics; Restricted Three-Body Problem; Libration Points; Stability; Heterogeneous Oblate Spheroid.*

1. Introduction

The restricted problem of three-body describes the motion of infinitesimal mass moving in the gravitational field of two massive primaries in the same plane or out of plane called two dimensional or three dimensional problem accordingly. The primaries are revolving around their center of mass either in circular or elliptical orbits under the influence of their mutual gravitational attraction. If the orbit of the primaries around their center of mass is elliptic, problem is said to be elliptic restricted three-body problem (ER3BP or ERTBP) and if the orbit of the primaries around their center of mass is circular, problem is said to be circular restricted three-body problem or restricted three-body problem, denoted by CR3BP or CRTBP or RTBP or R3BP.

The problem possesses five equilibrium points out of which three are collinear and two non-collinear. The collinear libration points are unstable while non-collinear are stable for the mass ratio $\mu \le 0.038520896505$ [2].Some studies related to the equilibrium points in R3BP or ER3BP, taken into account the oblateness and triaxiality of the primaries, Coriolis and Centrifugal forces, variation of the masses of the primaries and the infinitesimal mass etc. are discussed by Danby [1]; Szebehely [2]; Vidyakin [3]; Sharma [4]; Subbarao and Sharma [5]; Sharma et.al. [6]; Choudhary R. K. [7]; Bhatnagar and Hallan [8]; Cid R. et. al. [9]; El-Shaboury [10]; Bhatnagar et al. [11]; Selaru D. et.al. [12]; Markellos et al. [13]; Subbarao and Sharma [14]; Khanna and Bhatnagar [15], [16]; Roberts G.E. [18]; Oberti and Vienne [19]; Sosnytskyi [20]; Perdiouet. al. [21]; Arredondo et.al. [22]; Idrisi and Taqvi [23]; Idrisi [24]; Idrisi and Amjad [25], Idrisi [26].

We got the idea of our problem from the paper 'Rotating Stratified Heterogeneous Oblate Spheriodin Newtonian Physics' by Esteban and Vazquez. [17]. in their paper they have taken three layers in a stratified non-conformal heterogeneous oblate spheroidal system. We wish to extend this study to the restricted three body problem.

2. Equations of motion

Let m_1, m_2 and m_3 be the masses of more massive primary, less massive primary and infinitesimal mass respectively. We consider the less massive primary m_2 as an heterogeneous oblate body with N-layers having different densities ρ_i and axes a_i, b_i and c_i such that $\rho_{i+1} > \rho_i$, $a_{i+1} > a_i$, $b_{i+1} > b_i$, $c_{i+1} = c_i$, $a_i = b_i > c_i$, $i = 1, 2, \ldots, N$. The primaries are revolving with angular velocity n in circular orbits about their common centre of mass O and m_3 is moving under the gravitational field of m_1 and m_2 in the same plane. The line joining m_1 and m_2 is taken as X- axis and 'O' their center of mass as origin and the line passing through O and perpendicular to OX and lying in the plane of motion of m_1 and m_2 is the Y-axis. We consider a synodic system of coordinates $O(xyz)$; initially coincident with the inertial system $O(XYZ)$, rotating with the angular velocity n about Z-axis (the z-axis is coincident with Z-axis). The distances of m_3 from m_j and O are $\boldsymbol{r_j}$ and $\boldsymbol{r}$ respectively, $j = 1, 2$. Our aim is to find the equations of motion of m_3 using the terminology of Szebehely (1967) in the synodic co-ordinate system and dimensionless variables i.e. the distance between the primaries m_1 and m_2 is unity, the unit of time t is such that the gravitational constant $G = 1$ and the sum of the masses of the primaries is unity i.e. $m_1 + m_2 = 1$.

The equations of motion of the infinitesimal mass m_3 in the synodic coordinate system and dimensionless variables are given by

$$\left.\begin{aligned} \ddot{x} - 2n\dot{y} &= \Omega_x, \\ \ddot{y} + 2n\dot{x} &= \Omega_y, \end{aligned}\right\} \tag{1}$$

where

$$\Omega = \frac{n^2}{2}\left[(1-\mu)r_1^2 + \mu r_2^2\right] + \frac{1-\mu}{r_1} + \frac{\mu}{r_2} + \frac{k_1}{2r_2^3},$$

$$\Omega_x = n^2 x - (1-\mu)\frac{(x-\mu)}{r_1^3} - \mu\frac{(x+1-\mu)}{r_2^3} - \frac{3k_1(x+1-\mu)}{2r_2^5},$$

$$\Omega_y = y\left[n^2 - \frac{(1-\mu)}{r_1^3} - \frac{\mu}{r_2^3} - \frac{3k_1}{2r_2^5}\right],$$

$n^2 = 1 + \frac{3}{2}k_2$, Is the mean motion of the primaries, (2)

$$k_1 = \frac{4\pi}{3}\sum_{i=1}^{N} a_i^2 c_i (\rho_i - \rho_{i+1})\sigma_i,$$

$$k_2 = \sum_{i=1}^{n} \sigma_i,$$

$$\sigma_i = \frac{a_i^2 - c_i^2}{5},$$

$$r_1^2 = (x-\mu)^2 + y^2, \tag{3}$$

$$r_2^2 = (x+1-\mu)^2 + y^2, \tag{4}$$

$$\mu = \frac{m_2}{m_1+m_2} < \frac{1}{2}$$
$$\Rightarrow m_1 = 1-\mu;$$
$$m_2 = \mu = \frac{4\pi}{3}\sum_{i=1}^{N} a_i^2 c_i (\rho_i - \rho_{i+1}).$$

3. Non-collinear libration points

At the libration points all the derivatives of nth order are zero, therefore the equations of motion (1) becomes

$$n^2 x - (1-\mu)\frac{(x-\mu)}{r_1^3} - \mu\frac{(x+1-\mu)}{r_2^3} - \frac{3k_1(x+1-\mu)}{2r_2^5} = 0 \tag{5}$$

and

$$y\left[n^2 - \frac{(1-\mu)}{r_1^3} - \frac{\mu}{r_2^3} - \frac{3k_1}{2r_2^5}\right] = 0. \tag{6}$$

The collinear libration points are the solution of the Equations (5) and (6) for y ≠ 0 i.e.

$$n^2 x - (1-\mu)\frac{(x-\mu)}{r_1^3} - \mu\frac{(x+1-\mu)}{r_2^3} - \frac{3k_1(x+1-\mu)}{2r_2^5} = 0 \tag{7}$$

and

$$n^2 - \frac{(1-\mu)}{r_1^3} - \frac{\mu}{r_2^3} - \frac{3k_1}{2r_2^5} = 0.$$
(8)

On substituting $k_1 = 0$ and $k_2 = 0$, the solution of Eqns. (7) and (8) is $r_1 = 1$, $r_2 = 1$ and from Eqn. (2), $n = 1$.
Now we assume that the solution of Eqns. (7) and (8) for $k_1 \neq 0$, $k_2 \neq 0$ as

$r_1 = 1 + \xi_1$, $r_2 = 1 + \xi_2$, $\xi_1, \xi_2 \ll 1$.

Substituting these values of r_1 and r_2 in the Eqns. (3) and (4), we get

$$\left.\begin{aligned} x &= \mu - \frac{1}{2} + \xi_2 - \xi_1 \\ y &= \pm\frac{\sqrt{3}}{2}\left[1 + \frac{2}{3}(\xi_2 + \xi_1)\right] \end{aligned}\right\} \tag{9}$$

Now, substituting the values of x, y from Eqns. (9) and $r_1 = 1 + \xi_1$, $r_2 = 1 + \xi_2$ in the Eqns. (7) and (8) and neglecting higher order terms, we obtain

$$\xi_1 = \left[\frac{1}{2} - \frac{\mu}{(1-\mu)}\right](k_1 + k_2),$$
$$\xi_2 = \left[1 + \frac{\mu}{(1-2\mu)}\right]k_1 - \frac{\mu}{(1-2\mu)}k_2.$$

Thus the coordinates of the non-collinear libration points $L_{4,5}$ are

$$x = \mu - \frac{1}{2} + \left[\frac{\mu}{1-\mu} + \frac{\mu}{1-2\mu}\right]k_1 - \left[1 - \frac{\mu}{1-\mu} + \frac{\mu}{1-2\mu}\right]k_2, \tag{10}$$

$$y = \pm\frac{\sqrt{3}}{2}\left[1 + \frac{2}{3}\left\{\begin{aligned} &\left(2 - \frac{\mu}{1-\mu} + \frac{\mu}{1-2\mu}\right)k_1 + \\ &\left(1 - \frac{\mu}{1-\mu} - \frac{\mu}{1-2\mu}\right)k_2 \end{aligned}\right\}\right] \tag{11}$$

4. Stability of non-collinear libration points

The equations of the motion of the infinitesimal mass are

$$\ddot{x} - 2n\dot{y} = \Omega_x,$$
$$\ddot{y} + 2n\dot{x} = \Omega_y.$$

To study the possible motion of the infinitesimal mass around the libration points let the coordinates of these points are (x_0, y_0). If we give small displacement (ζ, η) to (x_0, y_0) and considering only linear terms in ζ and η, the variation ζ and η can be written as: $\zeta = x - x_0$ and $\eta = y - y_0$ and the equations of the motion become

$$\begin{aligned} \ddot{\varsigma} - 2n\dot{\eta} &= \Omega_x(x_0+\varsigma, y_0+\eta) = \varsigma\overset{o}{\Omega}_{xx} + \eta\overset{o}{\Omega}_{xy}, \\ \ddot{\eta} + 2n\dot{\varsigma} &= \Omega_y(x_0+\varsigma, y_0+\eta) = \varsigma\overset{o}{\Omega}_{yx} + \eta\overset{o}{\Omega}_{yy}. \end{aligned} \tag{12}$$

where 'o' indicates that the partial derivatives are to be calculated at the libration points under consideration.
The characteristic equation of the Equations (12) is given by

$$\lambda^4 + (4n^2 - \overset{o}{\Omega}_{xx} - \overset{o}{\Omega}_{yy})\lambda^2 + \overset{o}{\Omega}_{xx}\overset{o}{\Omega}_{yy} - (\overset{o}{\Omega}_{xy})^2 = 0 \tag{13}$$

where

$$\overset{o}{\Omega}_{xx} = \frac{3}{4} + \left(-\frac{3\mu}{8} - \frac{3\mu}{1-2\mu} - \frac{9\mu}{4(1-\mu)} \right) k_1 + \left(\frac{33}{8} - \frac{45}{8}\mu + \frac{3\mu}{1-2\mu} - \frac{9\mu}{4(1-\mu)} \right) k_2,$$

$$\overset{o}{\Omega}_{xy} = \frac{3\sqrt{3}}{2}\left(\mu - \frac{1}{2} \right) + \frac{\sqrt{3}}{2}\left(\frac{15}{2} - \frac{45}{4}\mu + \frac{3\mu}{1-2\mu} - \frac{9\mu}{2(1-\mu)} \right) k_1 + \left(-\frac{\sqrt{3}}{8} - \frac{7\sqrt{3}}{8}\mu - \frac{2\sqrt{3}\mu}{1-2\mu} - \frac{7\sqrt{3}\mu}{4(1-\mu)} \right) k_2,$$

$$\overset{o}{\Omega}_{yy} = \frac{9}{4} + \left(6 - \frac{33}{8}\mu + \frac{3\mu}{1-2\mu} + \frac{21\mu}{4(1-\mu)} \right) k_1 + \left(\frac{3}{8} + \frac{33}{8}\mu + \frac{3\mu}{1-2\mu} + \frac{21\mu}{4(1-\mu)} \right) k_2.$$

Let $\lambda^2 = \Lambda$, therefore the characteristic Equation (13) becomes

$$\Lambda^2 + (4n^2 - \overset{o}{\Omega}_{xx} - \overset{o}{\Omega}_{yy})\Lambda + \overset{o}{\Omega}_{xx}\overset{o}{\Omega}_{yy} - (\overset{o}{\Omega}_{xy})^2 = 0 \quad (14)$$

which is a quadratic equation in Λ. If Λ_1 and Λ_2 are the roots of the Equation (14) then the roots of the characteristic Equation (13) are given by

$$\lambda_{1,2} = \pm\sqrt{\Lambda_1} \text{ and } \lambda_{3,4} = \pm\sqrt{\Lambda_2} \quad (15)$$

λ_i (i =1,…4) will be pure imaginary if Λ_1 and Λ_2 both are negative real roots and then the non-collinear libration points will be stable. Now, roots of Eqn. (14) are given by

$$\Lambda_{1,2} = \frac{1}{2}\left[\begin{array}{c} -(4n^2 - \overset{o}{\Omega}_{xx} - \overset{o}{\Omega}_{yy}) \pm \\ \sqrt{(4n^2 - \overset{o}{\Omega}_{xx} - \overset{o}{\Omega}_{yy})^2 - 4(\overset{o}{\Omega}_{xx}\overset{o}{\Omega}_{yy} - \overset{o}{\Omega}_{xy}{}^2)} \end{array} \right] \quad (16)$$

Eqn. (16) has negative real roots if

$$(4n^2 - \overset{o}{\Omega}_{xx} - \overset{o}{\Omega}_{yy})^2 - 4(\overset{o}{\Omega}_{xx}\overset{o}{\Omega}_{yy} - \overset{o}{\Omega}_{xy}{}^2) = 0 \quad i.e.$$

$$1 - 27\mu + 27\mu^2 + \left(-\frac{195}{2} + 261\mu - \frac{9\mu}{1-2\mu} + \frac{39\mu}{1-\mu} \right) k_1 + \left(-33 + \frac{105}{2}\mu - \frac{12\mu}{1-2\mu} + \frac{30\mu}{1-\mu} \right) k_2 = 0 \quad (17)$$

For $k_i = 0$ (i = 1, 2), the solution of above equation is $\mu = \mu_o$ = 0.0385208… (Szebehely, 1967). Now we consider $\mu_{crit} = \mu_o + \alpha$, $\alpha \ll 1$ be the solution of Eqn. (17), therefore

$$\alpha = \frac{p_1 k_1 + p_2 k_2}{2(14 - 27\mu_o)}, \quad (18)$$

where

$$p_1 = -\frac{195}{2} + 261\mu_o - \frac{9\mu_o}{1-2\mu_o} + \frac{39\mu_o}{1-\mu_o},$$

$$p_2 = -33 + \frac{105}{2}\mu_o - \frac{12\mu_o}{1-2\mu_o} + \frac{30\mu_o}{1-\mu_o}.$$

Thus the non-collinear libration points are stable for the critical value of mass parameter $\mu \le \mu_{crit} = \mu_o - 3.32792\, k_1 - 1.16808\, k_2$.

5. Conclusion

In the present paper the existence and stability of non-collinear libration points in restricted three-body problem considering less massive primary an oblate heterogeneous spheroid with N-layers has been discussed and this is found that there exist two non-collinear libration points which are stable for a critical value of mass parameter $\mu \le \mu_{crit} = \mu_o - 3.32792\, k_1 - 1.16808\, k_2$.

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