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Research paper

Linear stability and resonance of triangular equilibrium points in elliptic restricted three body problem with radiating primary and triaxial secondary

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Abstract

The present paper studies the linear stability of the triangular equilibrium points of the system. The system comprises of a radiating primary and a triaxial secondary in elliptic restricted three body problem. The existence of third order resonances has been shown and the linear stability has been analyzed for these resonance cases. For the resonance case, $3\lambda_2 = 1$ and $2\lambda_1 + \lambda_2 = 0$, the conditions of the linear stability are satisfied and the system is stable. But, for the resonance cases $2\lambda_1 + \lambda_2 = 1$ and $\lambda_1 - 2\lambda_2 = 2$ the system is unstable.

Keywords: *Celestial Mechanics; Elliptical Restricted Three Body Problem; Stability; Oblateness; Rigid Body; Resonance.*

1. Introduction

The elliptic restricted three body problem has been widely studied by many researchers which is a generalization of classical model. The system consists of two finite bodies (known as primaries) moving about their common center of mass, having no influence on each other. The third body is influenced by both of the primaries. The primaries in general describe elliptic path. The orbit of Jupiter around Sun is a fixed ellipse and the Trojan asteroids are influenced by the gravitational attraction of the Sun and Jupiter is an example of the above system. The study of restricted/elliptic restricted three body problem has been a subject of investigation over the years. The stability of such systems (ER3BP) moving in an elliptic orbits was subject of investigation and was investigated by many authors , Arnold [1] ; Danby [11]; Bennet [4]; Szebehely [25]; Broucke [6]; Katsiaris [19] ; Beauge[5]; Baoyin[3]; Ammar[2]; Biggs [7]; Biggs [8] and many others.

The resonance/non-resonance cases of libration points for restricted/elliptic restricted three body problem was studied and analyzed by many authors Kamel [18]; Choudhry [10]; Ferraz [12]; Henrard [15]; Kumar [20]; Henrard [17], Hadjidemetriou [13]; Hadjidemetriou [14]; Subba Rao [24]; Thakur [26]; Beauge [5]; Chandra [9]; Narayan [22] and many others.

In the present paper an attempt have been made to study and analyze the linear stability of the system. It has been shown that resonances of the third order exist under the range of linear stability. This paper is in continuation of the paper (Usha [27] and Narayan [23]). The study of linear stability in presence of resonance makes the work different from the other works. We have followed the method proposed by Kumar [20].

The present paper is organized as follows: Section 1, which is introduction; Section 2 provides the equations of motion; Section 3 gives the location of the Triangular points; Section 4 focuses on the First order stability and Normalization of Hamiltonian and Section 5 gives the Resonance Cases. The discussions and conclusions are drawn in Section 6.

2. Equations of motion

The equations of motion of the infinitesimal mass in the elliptic restricted three body problem under radiating and triaxial primaries in the barycentric, pulsating and rotating, non-dimensional coordinates are given by the differential equations derived in Usha et. al. [27] and given in the following equation (1). The notations in principle follow the book of Szebehely [25], with some minor modifications in the notation being done for adapting to the present problem, presented as:

$$x'' - 2y' = \frac{1}{1+e\cos v}\frac{\partial\Omega}{\partial x}$$
$$y'' + 2x' = \frac{1}{1+e\cos v}\frac{\partial\Omega}{\partial y} \tag{1}$$

where ' denotes differentiation with respect to *v*, and

$$\Omega = \frac{x^2+y^2}{2} + \frac{1}{n^2}\left[\frac{(1-\mu)q}{r_1} + \frac{\mu}{r_2} + \frac{\mu(2\sigma_1-\sigma_2)}{2r_2^3} - \frac{3\mu(\sigma_1-\sigma_2)}{2r_2^5}y^2\right] \tag{2}$$

where $n^2 = 1 + \frac{3}{2}(2\sigma_1 - \sigma_2)$ and $\sigma_1 = \frac{a^2-c^2}{5R^2}$; $\sigma 2 = \frac{b^2-c^2}{5R^2}$.

and

$$r_1^2=(x+\mu)^2+y^2,\ r_2^2=(x-1+\mu)^2+y^2 \tag{3}$$

Here m_1 and m_2 are the masses of the bigger and smaller primaries positioned at (x_i ,0), i= 1, 2; q=1-δ, the radiation pressure; σ_1 and σ_2 are triaxiality parameters, σ_i (i=1,2)(McCuskey [21]); and a, b, c are semi axes and R is the distance between the primaries ; r_i (i=1,2) are the distances of the infinitesimal mass from the bigger and smaller primaries respectively; while *e* is the eccentricity of the either primary around the other and *v* is the true anomaly.

3. Location of triangular equilibrium points

The equilibrium points of the system are given by the equations:

$$\frac{\partial\Omega}{\partial x}=0,\qquad \frac{\partial\Omega}{\partial y}=0 \tag{4}$$

where Ω is given by equation (2).

$$\frac{\partial\Omega}{\partial x}=x-\frac{1}{n^2}\left\{\begin{array}{l}\dfrac{(1-\mu)(x+\mu)q}{r_1^3}\\+\dfrac{\mu(x-1+\mu)}{r_2^3}\\+\dfrac{3\mu(x-1+\mu)(2\sigma_1-\sigma_2)}{2r_2^5}\\-\dfrac{15\mu(x-1+\mu)(\sigma_1-\sigma_2)y^2}{2r_2^7}\end{array}\right\}=0 \tag{5}$$

and

$$\frac{\partial\Omega}{\partial y}=y\left[1-\frac{1}{n^2}\left\{\begin{array}{l}\dfrac{(1-\mu)q}{r_1^3}+\dfrac{\mu}{r_2^3}\\+\dfrac{3\mu(4\sigma_1-3\sigma_2)}{2r_2^5}\\-\dfrac{15\mu(\sigma_1-\sigma_2)y^2}{2r_2^7}\end{array}\right\}\right]=0 \tag{6}$$

Solving the above equations, the coordinates of the triangular libration points $L_{4,5}$ are represented as:

$$x=\left[\begin{array}{l}\mu-\dfrac{1}{2}-\dfrac{(1-q)}{3}\\+\left\{\dfrac{3}{8}-(1-q)+\dfrac{\mu}{2(1-\mu)}\right\}\sigma_1\\+\left\{\dfrac{-7}{8}+\dfrac{(1-q)}{2}-\dfrac{\mu}{2(1-\mu)}\right\}\sigma_2\end{array}\right]$$

$$y=\pm\frac{\sqrt{3}}{2}\left[1+\frac{2}{3}\left\{\begin{array}{l}\dfrac{(q-1)}{3}\\+\left(\dfrac{-19}{8}-(1-q)+\dfrac{\mu}{2(1-\mu)}\right)\sigma_1\\+\left(\dfrac{15}{8}+\dfrac{(1-q)}{2}-\dfrac{\mu}{2(1-\mu)}\right)\sigma 2\end{array}\right\}\right] \tag{7}$$

4. First order stability and normalization of Hamiltonian

The Lagrangian equation of motion of the problem is written as:

$$L=\frac{1}{2}\left(\dot{x}^2+\dot{y}^2\right)+\left(x\dot{y}-y\dot{x}\right)+\frac{1}{1+e\cos v}\left\{\begin{array}{l}\dfrac{x^2+y^2}{2}\\+\dfrac{1}{n^2}\left[\begin{array}{l}\dfrac{(1-\mu)q}{r_1}+\dfrac{\mu}{r_2}\\+\dfrac{\mu(2\sigma_1-\sigma_2)}{2r_2^3}\\-\dfrac{3\mu(\sigma_1-\sigma_2)}{2r_2^5}y^2\end{array}\right]\end{array}\right\} \tag{8}$$

Now, to get the Hamiltonian function of the problem, using the formula:

$$H=-L+\frac{\partial L}{\partial\dot{x}}\dot{x}+\frac{\partial L}{\partial\dot{y}}\dot{y} \tag{9}$$

Hence, the perturbed Hamiltonian is given by:

$$H=\frac{1}{2}\left(P_x^2+P_y^2\right)+\left(P_xy-P_yx\right)+\frac{e\cos v}{1+e\cos v}(x^2+y^2)-\frac{1}{1+e\cos v}\left\{\frac{1}{n^2}\left[\begin{array}{l}\dfrac{(1-\mu)q}{r_1}+\dfrac{\mu}{r_2}\\+\dfrac{\mu(2\sigma_1-\sigma_2)}{2r_2^3}\\-\dfrac{3\mu(\sigma_1-\sigma_2)}{2r_2^5}y^2\end{array}\right]\right\} \tag{10}$$

where P_x and P_y are the generalized components of momentum.
The nature of motion near the two points will be the same as the two triangular equilibrium solutions are symmetrical to each other. Hence, in further calculations the motion near the equilibrium point L_4 will be considered.
So, shifting the origin to L_4 by the change of variables given by:

$$\begin{array}{l}x=\xi+q_1;\\ y=\eta+q_2;\\ P_x=P_\xi+P_1;\\ P_y=P_\eta+P_2.\end{array} \tag{11}$$

So, the point L_4 is given by:

$$\xi = \frac{1}{2} - \mu - \frac{\delta}{3} + \left\{ \frac{3}{8} - \delta + \frac{\mu}{2(1-\mu)} \right\} \sigma_1 + \left\{ \frac{-7}{8} + \frac{\delta}{2} - \frac{\mu}{2(1-\mu)} \right\} \sigma_2;$$

$$\eta = \frac{\sqrt{3}}{2} \left[1 + \frac{2}{3} \left\{ \begin{array}{l} \frac{-\delta}{3} + \left(\frac{-19}{8} - \delta + \frac{\mu}{2(1-\mu)} \right) \sigma_1 \\ + \left(\frac{15}{8} + \frac{\delta}{2} - \frac{\mu}{2(1-\mu)} \right) \sigma 2 \end{array} \right\} \right]; \quad (12)$$

$$P_\xi = -\eta;$$
$$P_\eta = \xi.$$

The solution of the equation (12) in the new variables are given by the equilibrium position:

$$q_1 = q_2 = P_1 = P_2 = 0.$$

Now, expanding the Hamiltonian function (10) in the powers of P_i and q_i, i=1,2, we obtain:

$$H = \sum_{k=0}^{\infty} H_k = H_0 + H_1 + H_2 + H_3 + H_4 + ... \quad (13)$$

where H_0 =constant and $H_1 = 0$.
Taking only H_2, as we are analyzing the linear stability, we have:

$$H_2 = \frac{1}{2}\left(p_1^2 + p_2^2\right) + \left(p_1 q_2 - q_1 p_2\right) + \frac{e \cos v}{1 + e \cos v}(q_1{}^2 + q_2{}^2) + \frac{1}{2(1 + e \cos v)}\left[A q_1{}^2 - B q_1 q_2 - C q_2{}^2\right] \quad (14)$$

The above equation can be written as:

$$H_2 = H_2{}^{(0)} + H_2{}^{(1)} \quad (15)$$

where,

$$H_2{}^{(0)} = \frac{1}{2}\left(p_1^2 + p_2^2\right) + \left(p_1 q_2 - q_1 p_2\right) + \frac{1}{2}\left\{A q_1{}^2 - B q_1 q_2 - C q_2{}^2\right\}$$

$$H_2{}^{(1)} = \frac{e \cos v}{1 + e \cos v} \left\{ \begin{array}{l} (1-A) q_1{}^2 \\ + B q_1 q_2 + (1+C) q_2{}^2 \end{array} \right\} \quad (16)$$

and,

$$A = \frac{1}{8} + \frac{\delta}{4} + \left(\frac{-21}{16} + \frac{3}{64}\mu \right)\sigma_1 + \left(\frac{27}{16} - \frac{99}{64}\mu \right)\sigma_2$$
$$B = \frac{\sqrt{3}}{4}\left\{ \frac{3}{4} - \frac{3\mu}{32} - \frac{2\delta}{3} + \left(\frac{7}{2} - \frac{21}{8}\mu \right)\sigma_1 + \left(\frac{-9}{2} + \frac{181}{8}\mu \right)\sigma_2 \right\}$$
$$C = \frac{5}{8} + \frac{\delta}{4} - \frac{3\mu\delta}{4} + \left(\frac{-21}{16} + \frac{183}{64}\mu \right)\sigma_1 + \left(\frac{27}{16} - \frac{183}{64}\mu \right)\sigma_2 - - \quad (17)$$

For normalizing using the canonical transformation:

$$\left[q_1, q_2, p_1, p_2\right] = \left[q_1{}', q_2{}', p_1{}', p_2{}'\right] N \quad (18)$$

where,

$$N = \begin{pmatrix} a_1 & a_1 c_1 & -a_1 c_1 & a_1(1 - \omega_1 b_1) \\ a_2 & a_2 c_2 & -a_2 c_2 & a_2(1 - \omega_2 b_2) \\ 0 & a_1 b_1 & a_1(1 - b_1) & a_1 c_1 \\ 0 & -a_2 b_2 & -a_2(1 - b_2) & -a_2 c_2 \end{pmatrix} \quad (19)$$

and,

$$a_i = \frac{1}{2}\sqrt{\frac{2 l_i}{\left|\omega_i^2 - \frac{1}{2}\right|}};\; b_i = \frac{2}{l_i};\; c_i = \frac{-B}{l_i}; \quad (20)$$

where, i=1, 2 and A, B, C are given by the equation (17). Here, $\omega_1^2 = -\left\{\lambda_{1,2}^{(0)}\right\}^2$ and $\omega_2^2 = -\left\{\lambda_{3,4}^{(0)}\right\}^2$, are the frequencies and given by the relation:

$$\omega^4 - \left\{1 - \frac{45}{8}\mu\sigma_1 + \frac{21}{8}\mu\sigma_2\right\}\omega^2 + \frac{27}{4}\mu(1-\mu)\left\{ \begin{array}{l} 1 + \frac{2}{9} \\ + \left(\frac{71}{8}\sigma_1 - \frac{181}{24}\sigma_2 \right) \end{array} \right\} = 0 \quad (21)$$

On solving the above equation, we have:

$$\omega_{1,2}^2 = \frac{1}{2}\left[\begin{array}{l} \left\{1 - \frac{45}{8}\mu\sigma_1 + \frac{21}{8}\mu\sigma_2\right\} \\ \pm \left\{ \begin{array}{l} \left(1 - \frac{45}{4}\mu\sigma_1 + \frac{21}{4}\mu\sigma_2\right) \\ -27\mu(1-\mu)\left(1 + \frac{2}{9} + \frac{71}{8}\sigma_1 - \frac{181}{24}\sigma_2\right) \end{array} \right\}^{\frac{1}{2}} \end{array} \right] \quad (22)$$

If $\omega_{1,2}^2$ are purely imaginary, we have:

$$\left\{ \begin{array}{l} \left(1 - \frac{45}{4}\mu\sigma_1 + \frac{21}{4}\mu\sigma_2\right) \\ -27\mu(1-\mu)\left(1 + \frac{2}{9} + \frac{71}{8}\sigma_1 - \frac{181}{24}\sigma_2\right) \end{array} \right\} \geq 0$$

The equality relation corresponds to resonance cases with equal frequencies, which are not considered in the study.
Now, the transformation (19) reduces the Hamiltonian as:

$$H_2{}' = \frac{1}{2}\left(p_1'{}^2 + \omega_1^2 q_1'{}^2\right) - \frac{1}{2}\left(p_2'{}^2 + \omega_2^2 q_2'{}^2\right) + \frac{e \cos v}{1 + e \cos v} \sum_{\upsilon + \mu = 2} a'_{\upsilon\mu} q_1'^{\upsilon 1} q_2'^{\upsilon 2} p_1'^{\mu 1} p_2'^{\mu 2} \quad (23)$$

Where, $\upsilon=(\upsilon_1,\upsilon_2)$, $\mu=(\mu_1,\mu_2)$ and such that

$$\upsilon+\mu=\upsilon_1+\upsilon_2+\mu_1+\mu_2=2 .$$

The coefficients $a'_{\upsilon\mu}$ are given as:

$$\begin{aligned}
a'_{2000} &= \left[(1-A)a_1^2+Ba_1^2c_1+(1+C)a_1^2c_1^2\right]\\
a'_{0200} &= \left[(1-A)a_2^2+Ba_2^2c_2+(1+C)a_2^2c_2^2\right]\\
a'_{0020} &= \left[(1+C)a_1^2b_1^2\right]\\
a'_{0002} &= \left[(1+C)a_2^2b_2^2\right]\\
a'_{1100} &= \begin{bmatrix}2(1-A)a_1a_2+Ba_1a_2c_1\\ +Ba_1a_2c_2+2(1+C)a_1a_2c_1c_2\end{bmatrix}\\
a'_{1010} &= \left[Ba_1{}^2b_1+2(1+C)a_1{}^2b_1c_1\right]\\
a'_{1001} &= \left[-Ba_1a_2b_2-2(1+C)a_1a_2b_2c_1\right]\\
a'_{0110} &= \left[Ba_1a_2b_1+2(1+C)a_1a_2b_1c_2\right]\\
a'_{0011} &= \left[-2(1+C)a_1a_2b_1b_2\right]\\
a'_{0101} &= \left[-Ba_2{}^2b_2-2(1+C)a_2{}^2b_2c_2\right]
\end{aligned} \tag{24}$$

Now, using the transformation:

$$\begin{aligned}
q'_i &= \frac{1}{\sqrt{\omega_i}}\tilde{q}_i\\
p'_i &= \sqrt{\omega_i}\,\tilde{p}_i
\end{aligned} \tag{25}$$

The Hamiltonian gets transformed as:

$$\tilde{H}_2^{(0)} = \frac{1}{2}\omega_1\left(\tilde{p}_1^2+\tilde{q}_1^2\right)-\frac{1}{2}\omega_2\left(\tilde{p}_2^2+\tilde{q}_2^2\right) \tag{26}$$

and,

$$\tilde{H}_2^{(1)} = \frac{e\cos v}{1+e\cos v}\sum_{\upsilon+\mu=2}\tilde{a}_{\upsilon\mu}\tilde{q}_1^{\upsilon1}\tilde{q}_2^{\upsilon2}\tilde{p}_1^{\mu1}\tilde{p}_2^{\mu2} \tag{27}$$

where, $\tilde{a}_{\upsilon\mu}$ is given as :

$$\begin{aligned}
&\tilde{a}_{2000}=\omega_1a'_{2000};\ \tilde{a}_{0200}=\omega_2a'_{0200};\ \tilde{a}_{0020}=\frac{1}{\omega_1}a'_{0020};\\
&\tilde{a}_{0002}=\frac{1}{\omega_2}a'_{0002};\ \tilde{a}_{1100}=\sqrt{\omega_1\omega_2}a'_{1100};\ \tilde{a}_{1010}=a'_{1010};\\
&\tilde{a}_{1001}=\sqrt{\frac{\omega_1}{\omega_2}}a'_{1001};\ \tilde{a}_{0110}=\sqrt{\frac{\omega_2}{\omega_1}}a'_{0110};\ \tilde{a}_{0011}=\frac{1}{\sqrt{\omega_1\omega_2}}a'_{0011};\\
&\tilde{a}_{0101}=a'_{0101}
\end{aligned} \tag{28}$$

Now, introducing the complex conjugate variables given by:

$$q''_j=\tilde{p}_j+i\tilde{q}_j;\ p''_j=\tilde{p}_j-i\tilde{q}_j;\ (j=1,2) \tag{29}$$

which reduces the Hamiltonian, $H''_2=2i\tilde{H}_2$, where,

$$\begin{aligned}
H''_2 &= i\omega_1q''_1p''_1+i\omega_2q''_2p''_2\\
&+2i\frac{e\cos v}{1+e\cos v}\sum_{\upsilon+\mu=2}a''_{\upsilon\mu}q_1''^{\upsilon1}q_2''^{\upsilon2}p_1''^{\mu1}p_2''^{\mu2}
\end{aligned} \tag{30}$$

The coefficients in H''_2 are such that $a''_{\upsilon\mu}=\bar{a}''_{\upsilon\mu}$

$$\begin{aligned}
a''_{2000} &= \frac{1}{4}\left\{-\tilde{a}_{2000}+\tilde{a}_{0020}-i\tilde{a}_{1010}\right\}\\
a''_{0200} &= \frac{1}{4}\left\{-\tilde{a}_{0200}+\tilde{a}_{0002}-i\tilde{a}_{0101}\right\}\\
a''_{1100} &= \frac{1}{4}\left\{-\tilde{a}_{1100}+\tilde{a}_{0011}-i\tilde{a}_{1001}-i\tilde{a}_{0110}\right\}\\
a''_{1001} &= \frac{1}{4}\left\{-\tilde{a}_{1100}+\tilde{a}_{0011}-i\tilde{a}_{1001}-i\tilde{a}_{0110}\right\}\\
a''_{1010} &= \frac{1}{2}\left\{\tilde{a}_{2000}+\tilde{a}_{0020}\right\}\\
a''_{0101} &= \frac{1}{2}\left\{\tilde{a}_{0200}+\tilde{a}_{0002}\right\}
\end{aligned} \tag{31}$$

Next, to reduce the Hamiltonian (30) to normal form in complex conjugate variables given as:

$$H_2^*\left(q_j^{**},p_j^{**}\right)=i\lambda_1q_1^{**}p_1^{**}+i\lambda_2q_2^{**}p_2^{**} \tag{32}$$

Finding a transformation of the type:

$$\left(q''_j,p''_j\right)\rightarrow\left(q_j^{**},p_j^{**}\right) \tag{33}$$

defined by the generating function:

$$q''_1p_1^{**}+p''_2q_2^{**}+S\left(q''_1,q''_2,p_1^{**},p_2^{**},v\right)$$

where

$$S=\sum_{\upsilon+\mu=2}s_{\upsilon\mu}q_1''^{\upsilon1}q_2''^{\upsilon2}p_1^{**\mu1}p_2^{**\mu2} \tag{34}$$

$s_{\upsilon\mu}$ are 2π periodic functions of *v*. The relation between the variables are given as:

$$\begin{aligned}
q_j^{**} &= q''_j+\frac{\partial s}{\partial p_j^{**}}\\
p''_j &= p_j^{**}+\frac{\partial s}{\partial q''_j}
\end{aligned} \tag{35}$$

Using the above relation, we have the equation:

$$H_2^{**}\left(q''_j+\frac{\partial s}{\partial p_j^{**}},p_j^{**},v\right)-H''_2\left(q''_j,p_j^{**}+\frac{\partial s}{\partial q''_j},v\right)=\frac{\partial s}{\partial v} \tag{36}$$

Now, expanding the Hamiltonian using Taylor's theorem, and equating the equal powers on both sides:

$$
\begin{aligned}
& i\lambda_1 q_1'' p_1^{**} + i\lambda_2 q_2'' p_2^{**} \\
& + i \sum_{\upsilon+\mu=2} \left(\mu_1\lambda_1 + \mu_2\lambda_2\right) s_{\upsilon\mu} q_1''^{\upsilon 1} q_2''^{\upsilon 2} p_1^{**\mu 1} p_2^{**\mu 2} \\
& - i\omega_1 q_1'' p_1^{**} - i\omega_2 q_2'' p_2^{**} \\
& - 2i \frac{e\cos v}{1+e\cos v} \sum_{\upsilon+\mu=2} a''_{\upsilon\mu} q_1''^{\upsilon 1} q_2''^{\upsilon 2} p_1^{**\mu 1} p_2^{**\mu 2} \\
& - i \sum_{\upsilon+\mu=2} \left(\upsilon_1\omega_1 - \upsilon_2\omega_2\right) s_{\upsilon\mu} q_1''^{\upsilon 1} q_2''^{\upsilon 2} p_1^{**\mu 1} p_2^{**\mu 2} \\
& = \sum_{\upsilon+\mu=2} \frac{ds_{\upsilon\mu}}{dv} q_1''^{\upsilon 1} q_2''^{\upsilon 2} p_1^{**\mu 1} p_2^{**\mu 2}
\end{aligned}
\tag{37}
$$

Taking the terms up to second order in e:

$$
\begin{aligned}
& i\lambda_1 q_1'' p_1^{**} + i\lambda_2 q_2'' p_2^{**} \\
& + i \sum \left(\mu_1\lambda_1 + \mu_2\lambda_2\right)\left(es^{(1)} + e^2 s^{(2)}\right) \\
& - i\omega_1 q_1'' p_1^{**} - i\omega_2 q_2'' p_2^{**} \\
& - 2i \left[e\cos v - \frac{e^2}{2}\left(1+\cos 2v\right)\right] \sum_{\upsilon+\mu=2} a''_{\upsilon\mu} q_1''^{\upsilon 1} q_2''^{\upsilon 2} p_1^{**\mu 1} p_2^{**\mu 2} \\
& - i \sum_{\upsilon+\mu=2} \left(\upsilon_1\omega_1 - \upsilon_2\omega_2\right)\left(es^{(1)} + e^2 s^{(2)}\right) \\
& = e\frac{ds^{(1)}}{dv} + e^2 \frac{ds^{(2)}}{dv}
\end{aligned}
\tag{38}
$$

where

$$s_{\upsilon\mu} = e\sum s^{(1)} + e^2 \sum s^{(2)} \tag{39}$$

and taking:

$$\lambda_i = \lambda_i^{(0)} + e\lambda_i^{(1)} + e^2\lambda_i^{(2)} + \ldots\ldots\ldots\ldots\ldots, i = 1, 2 \tag{40}$$

Now, equating the equal powers of e on both sides of equation (38), we have:

$$
\begin{aligned}
& \lambda_1^{(0)} = \omega_1; \lambda_2^{(0)} = \omega_2; i\lambda_1^{(1)} + i\lambda_1^{(2)} \\
& + i\sum\left(\mu_1\lambda_1^{(0)} + \mu_2\lambda_2^{(0)}\right) s^{(1)} - 2ia''_{\upsilon\mu}\cos v \\
& - i\sum\left(\upsilon_1\mu_1 - \upsilon_2\mu_2\right) = \frac{ds^{(1)}}{dv}
\end{aligned}
\tag{41}
$$

From which we get the relation:

$$\frac{ds_{\upsilon\mu}}{dv} + i\left[\left(\upsilon_1 - \mu_1\right)\omega_1 - \left(\upsilon_2 - \mu_2\right)\omega_2\right] = -2ia''_{\upsilon\mu}\cos v \tag{42}$$

Solving, the above equation, we have:

$$s_{\upsilon\mu}^{(1)} = \frac{2ia''_{\upsilon\mu}\left[\sin v + i\left\{\left(\upsilon_1 - \mu_1\right)\omega_1 - \left(\upsilon_2 - \mu_2\right)\omega_2\right\}\cos v\right]}{\left[\left(\upsilon_1 - \mu_1\right)\omega_1 - \left(\upsilon_2 - \mu_2\right)\omega_2\right]^2 - 1} \tag{43}$$

By virtue of periodicity of $s_{1010}^{(1)}$ and $s_{0101}^{(1)}$, we have $\lambda_1^{(1)} = \lambda_2^{(1)} = 0$. So, equation (43) completely determines the complex-valued generating function S correct upto O[e].

Finally, reducing the Hamiltonian $\tilde{H}_2 = \tilde{H}_2^{(0)} + \tilde{H}_2^{(1)}$ to the normal form given by:

$$H_2 = \frac{1}{2}\lambda_1\left(q_1^{*2} + p_1^{*2}\right) + \frac{1}{2}\lambda_2\left(q_2^{*2} + p_2^{*2}\right) \tag{44}$$

With the real-valued transformation $\left(\tilde{q}_j, \tilde{p}_j\right) \to \left(q_j^*, p_j^*\right)$ by means of generating function $\tilde{q}_1 p_1^* + \tilde{q}_2 p_2^* + K\left(\tilde{q}_j, p_j^*, v\right)$, up to O[$e$], using the transformation formula given by:

$$q_j^* = \tilde{q}_j + \frac{\partial K}{\partial p_j^*}, \quad \tilde{p}_j = p_j^* + \frac{\partial K}{\partial q_j^*} \tag{45}$$

which can be written using Implicit function theorem as:

$$\tilde{q}_j = q_j^* - \frac{\partial K}{\partial p_j^*}, \quad \tilde{p}_j = p_j^* + \frac{\partial K}{\partial q_j^*} \tag{46}$$

Now, using equation (33), up to first order terms in e, we have the relation:

$$q_j'' = q_j^{**} - \frac{\partial S^{**}}{\partial p_j^{**}}, \quad p_j'' = p_j^{**} + \frac{\partial S^{**}}{\partial q_j^{**}} \tag{47}$$

where, $S^{**} = S^{(1)}\left(q^{**}, p^{**}, v\right)$. Now, using the relations:

$$
\begin{aligned}
& q_j'' = \tilde{p}_j + i\tilde{q}_j, \; p_j'' = \tilde{p}_j - i\tilde{q}_j \\
& q_j^{**} = p_j^* + iq_j^*, \; p_j^{**} = p_j^* - iq_j^*
\end{aligned}
$$

Let $S^{(1)} = W$, so from equation(46) we have:

$$\tilde{q}_j = q_j^* - \frac{1}{2i}\frac{\partial W}{\partial p_j^*}, \; \tilde{p}_j = p_j^* + \frac{1}{2i}\frac{\partial W}{\partial q_j^*} \tag{48}$$

From, equations (46) and (48), we have:

$$K = \frac{1}{2i}W \tag{49}$$

where, the function $K = \sum k_{\upsilon\mu} q_1^{*\upsilon 1} q_2^{*\upsilon 2} p_1^{*\mu 1} p_2^{*\mu 2}$ is a real valued function. The coefficients are given by using the relations (25) and (43) as:

$$k_{2000} = \frac{1}{2i}\left(-s_{2000}^{(1)} - s_{0020}^{(1)} + s_{1010}^{(1)}\right);$$

$$k_{0200} = \frac{1}{2i}\left(-s_{0200}^{(1)} - s_{0002}^{(1)} + s_{0101}^{(1)}\right);$$

$$k_{0020} = \frac{1}{2i}\left(s_{2000}^{(1)} - s_{0020}^{(1)} + s_{1010}^{(1)}\right);$$

$$k_{0002} = \frac{1}{2i}\left(s_{0200}^{(1)} - s_{0002}^{(1)} + s_{0101}^{(1)}\right);$$

$$k_{1100} = \frac{1}{2i}\left(s^{(1)}_{1100} + s^{(1)}_{1001} + s^{(1)}_{0110} - s^{(1)}_{0011}\right);$$

$$k_{1010} = \left(s^{(1)}_{2000} - s^{(1)}_{0020}\right);$$

$$k_{1001} = \frac{1}{2}\left(s^{(1)}_{1100} + s^{(1)}_{1001} - s^{(1)}_{0110} - s^{(1)}_{0011}\right);$$

$$k_{1001} = \frac{1}{2}\left(s^{(1)}_{1100} + s^{(1)}_{1001} - s^{(1)}_{0110} - s^{(1)}_{0011}\right);$$

$$k_{0101} = \left(s^{(1)}_{0200} - s^{(1)}_{0002}\right);$$

$$k_{0110} = \frac{1}{2}\left(s^{(1)}_{1100} - s^{(1)}_{1001} + s^{(1)}_{0110} - s^{(1)}_{0011}\right);$$

$$k_{0011} = \frac{1}{2i}\left(s^{(1)}_{1100} + s^{(1)}_{1001} + s^{(1)}_{0110} + s^{(1)}_{0011}\right) \tag{50}$$

Thus, the Hamiltonian H_2 has been transformed to normal form given by equation (44), correct up to first order in eccentricity *e*. This is obtained using equations (19), (25) and (46) and the corresponding coefficients of the generating function K are given by (50).

5. Resonance cases

For the study of stability, we have to examine the presence of resonances. For this, taking third order terms and applying the KAM theorem. So, considering the value of λ_1 and λ_2 up to second order of *e*. As the value f $\lambda^{(1)}_1 = \lambda^{(1)}_2 = 0$, so the quantities $\lambda^{(2)}_1$ and $\lambda^{(2)}_2$ are found by the periodicity conditions of the functions $s^{(2)}_{1010}$ and $s^{(2)}_{0101}$. Using (38) and equating the terms of e^2, we have:

$$\frac{ds^{(2)}_{1010}}{dv} = -2i\cos v\begin{pmatrix} 4a''_{0020}s^{(1)}_{2000} \\ +a''_{1010}s^{(1)}_{1010} \\ +a''_{1001}s^{(1)}_{1001} \\ +a''_{0110}s^{(1)}_{1100} \end{pmatrix} +2i\cos^2 v a''_{1010} + i\lambda^{(2)}_1$$

$$\frac{ds^{(2)}_{0101}}{dv} = -2i\cos v\begin{pmatrix} 4a''_{0002}s^{(1)}_{0200} \\ +a''_{0110}s^{(1)}_{1001} \\ +a''_{0101}s^{(1)}_{0101} \\ +a''_{0011}s^{(1)}_{1100} \end{pmatrix} +2i\cos^2 v a''_{0101} + i\lambda^{(2)}_2 \tag{51}$$

Now, using equation (43) along with the condition of periodicity of $s^{(2)}_{1010}$ and $s^{(2)}_{0101}$, the values of $\lambda^{(2)}_1$ and $\lambda^{(2)}_2$ using equations (14) and (21) are given as:

$$\lambda^{(2)}_1 = -\frac{\omega_1\omega_2^2\left(6\omega_1^2 - 7\right)}{4\left(4\omega_1^2 - 1\right)\left(2\omega_1^2 - 1\right)};$$

$$\lambda^{(2)}_2 = \frac{\omega_1^2\omega_2\left(6\omega_2^2 - 7\right)}{4\left(4\omega_2^2 - 1\right)\left(2\omega_2^2 - 1\right)} \tag{52}$$

Now, let the value of μ giving the resonance be:

$$k_1\lambda_1 + k_2\lambda_2 = N \tag{53}$$

For small values of *e*. The value of μ correct up to order of e^2 is given as:

$$\mu = \mu^{(0)} + e^2\mu^{(2)}$$

where $\mu^{(0)}$ is the value of μ for $e = 0$ and $\mu^{(2)}$ is the value of μ for $e \neq 0$ taken up to order of e^2, given by the equation, following Usha et. al[27] is :

$$\mu(e) = \left[-\left\{\begin{matrix} 27 + \frac{4311}{16}\sigma_1 \\ -\frac{5469}{16}\sigma_2 \\ +\left(1-e^2\right)^{\frac{1}{2}}\left(16\delta + 45\sigma_1 - 69\sigma_2\right) \end{matrix}\right\} \pm \left\{\begin{matrix} -1971 + 300\delta + \frac{228457}{8}\sigma_1 - \frac{642583}{8}\sigma_2 \\ +e^2\left(1728 - 192\delta - 27800\sigma_1 + 37304\sigma_2\right) \\ +\left(1-e^2\right)^{\frac{1}{2}}\left(2592 - 1632\delta - 4377\sigma_1 + 59330\sigma_2\right) \end{matrix}\right\}^{\frac{1}{2}}\right] \div 2\left[\begin{matrix} 27 - 3\delta - \frac{3379}{8}\sigma_1 + \frac{4567}{8}\sigma_2 \\ +8\left(1-e^2\right)\left(\sigma_1 - \sigma_2\right) \end{matrix}\right] \tag{54}$$

Taking, $\lambda_i = \lambda_i\left(\mu^{(0)} + e^2\mu^{(2)}\right)$, i=1,2 and expanding by Taylor's theorem,

$$\lambda_i = \lambda^{(0)}_i + e^2\lambda^{(2)}_i + e^2\mu^{(2)}\left(\frac{d\lambda_i}{d\mu}\right)_0, \text{ i=1, 2}$$

where, $\lambda^{(2)}_1$, $\lambda^{(2)}_2$ are given by equation by (52). Now, substituting the above obtained values in equation (53) and equating the coefficient of e^2 to zero, we have:

$$\mu^{(2)} = \frac{k_1\lambda^{(2)}_1 + k_2\lambda^{(2)}_2}{k_2\frac{d\omega_2}{d\mu} - k_1\frac{d\omega_1}{d\mu}} \tag{55}$$

The value of $\mu^{(2)}$ is calculated on putting the value $\mu = \mu^{(0)}$ calculated from equation (54) and substituting different values of k_1, k_2, N in the above equation, the values of third order resonances are calculated.

Let us consider, the third order resonance occur for:

$$3\lambda_2 = -1;\ 2\lambda_1 + \lambda_2 = 0;\ 2\lambda_1 + \lambda_2 = 1;\ \lambda_1 - 2\lambda_2 = 2. \quad (56)$$

Calculating the values of $\mu^{(0)}$ and $\mu^{(2)}$ at $e = 0$ for different resonance cases and plotting the graphs we can analyze the stability of the system. The graphs have been plotted using MATLAB and the calculations have been verified using the software Mathematica.

6. Conclusion

The resonance cases and the linear stability of the elliptic restricted three body problem with radiating primary and triaxial secondary has been analyzed. It is observed from the graphs that the linear stability is satisfied for the resonance cases $3\lambda_2 = -1$ and $2\lambda_1 + \lambda_2 = 0$ (Figures1 & 2). On the other hand, the condition of linear stability does not hold for the resonance cases $\lambda_1 - 2\lambda_2 = 2$ and $2\lambda_1 + \lambda_2 = 1$ which are clear from the graphs (Figures 3 & 4). These results are in confirmation with Kumar [20].

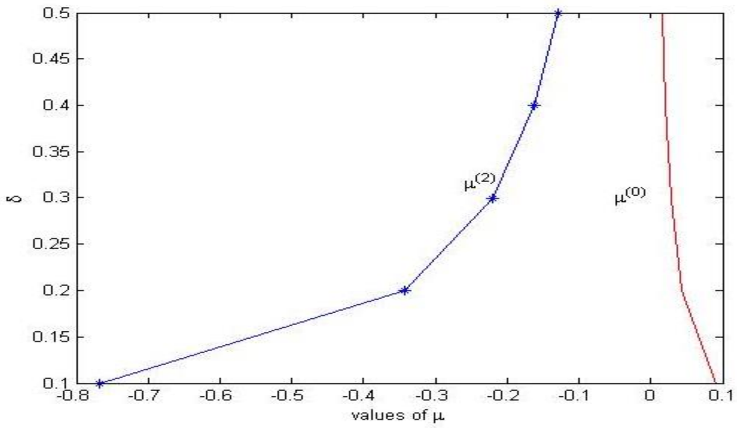

Fig. 1: Analysis of Stability For $3\lambda_2 = -1$.

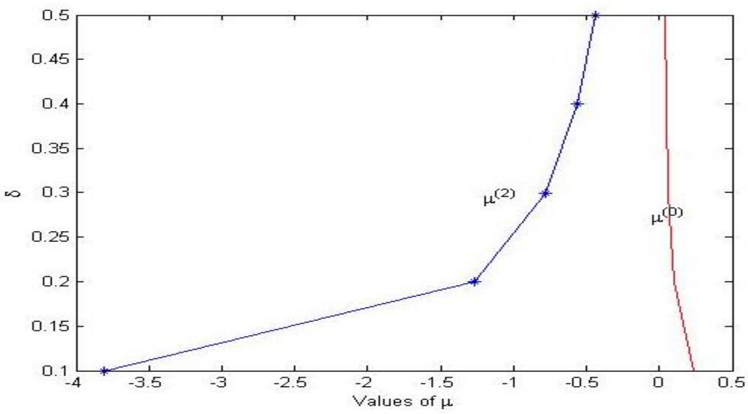

Fig. 2: Analysis of Stability for $2\lambda_1 + \lambda_2 = 0$.

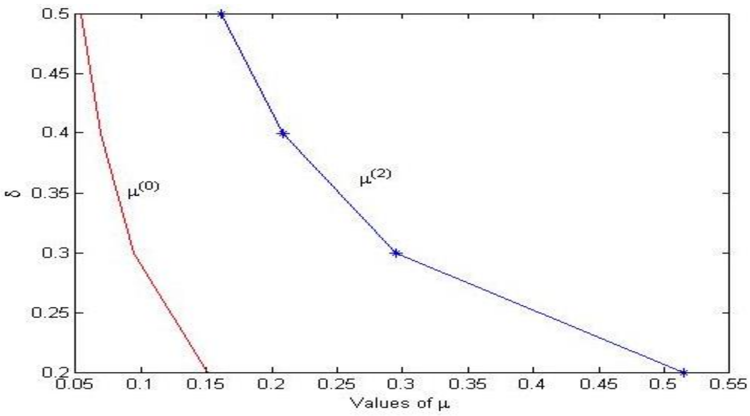

Fig. 3: Analysis of Stability for $2\lambda_1 + \lambda_2 = 1$.

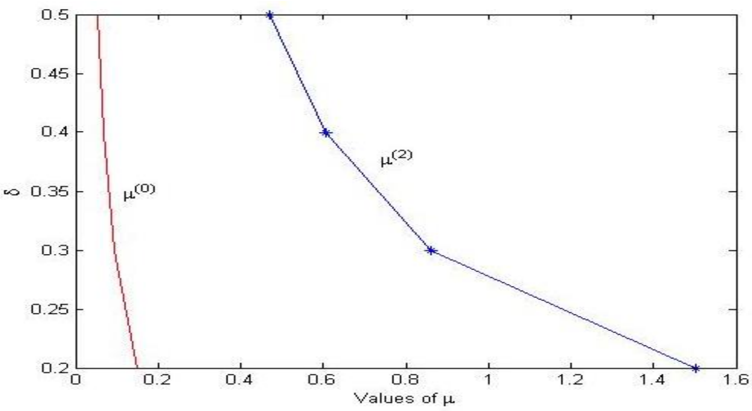

Fig. 4: Analysis of Stability for $\lambda_1 - 2\lambda_2 = 2$.

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