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Research paper

Comparative study of the dynamics of Mimas-Tethys system at different resonances

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Abstract

Here we have done the comparative study on the dynamics of Mimas-Tethys system at $i_1i_3, i_1i_3e_3, i_1^2e_3, i_3^2e_3$ and $i_3^2, i_1i_3e_3, i_1^2e_3, i_3^2e_3$ resonances along with secular resonance of all inner satellites and Saturn's oblateness. We have drawn Poincare surface of sections and Time-series graph to compare their effect.

Keywords: *Secular Resonance; Three-Body Problem; Disturbing Function; Oblateness and Poincare Surface of Section.*

1. Introduction

Allan [1] and Sinclair [2] investigated the dynamics of the Mimas-Tethys system under the hypothesis of circular orbits. Allan found (backward in time) the values of the satellites inclinations before capture in the present resonance to be $i_1 = 0.42^o$ and $i_3 = 1.05^o$. Sinclair found the probability of capture is 0.04.
Champenois and Vienne [3],[4] numerically investigated the role of 200 year period and found Mimas's inclination before capture may have been higher (up to 0.7 degree) or lower (down to 0.03 degree) than the previously considered 0.42 degree. Furthermore, Tethys's eccentricity on capture may have been quite higher (0.0008 versus 0). This value of eccentricity was found by a method which takes chaos into account. They also found that probability of capture in i_1i_3 resonance is 0.76 at eccentricity of Tethys is 0.0008.
Jha and Agrawal [5] have done the comparative study of the dynamics of Mimas-Tethys at i_1^2, i_1i_3 and i_3^2 resonances along with three third-order resonances without considering the secular term of all inner satellites and oblateness of Saturn. Jha and Agrawal [6] have studied the effect of secular resonance of Enceladus on Mimas-Tethys system. Jha and Jha [7-9] have studied the secular resonance effect of Dione, Rhea and Titan on this system.
Here we are extending the work of Jha and Agrawal [6] by considering the effect of secular term of all inner satellites along with Saturn's oblateness.
The physical model was taken to be Mimas and Tethys on eccentric orbits inclined on the equatorial plane of Saturn. Saturn's gravitational momenta are essential as they provide the main contribution to the orbital precession rates and we had to take into account the lowest degree oblateness terms $J_2, J_4, J_2^2, J_6, J_2J_4$ and J_2^3 (See table 2) also the actions of the Japet in the equations whereas the Sun and the small satellites of Saturn are not taken into account because of their weak effects in the generations of the purturbative frequency $\dot\sigma$.
Here the notations $a_1, n_1, e_1, i_1, \gamma_1, \varpi_1, \Omega_1$ and λ_1 are orbital semi-major axis, mean motion, eccentricity, inclination, sine of semi inclination, longitudes of periapse, longitude of ascending node and mean longitude of Mimas respectively. Corresponding notions with subscript 2, 3, 4, 5 and 6 refer to the Enceladus, Tethys, Dione, Rhea and Titan's orbital elements. Small m_1, m_2, m_3, m_4, m_5 and m_6 stand for Mimas, Enceladus, Tethys, Dione, Rhea and Titan's masses with unit of Saturn.

2. Equation of motion when the system is locked in $i_3^2, i_1^2e_3, i_1i_3e_3$ and $i_3^2e_3$ resonances

(Equation of motion is taken from Jha and Agrawal [5].

$$\ddot\chi = A_\chi \mathrm{Sin}[\chi] + A_{\psi+\varphi}\mathrm{Sin}[\psi+\varphi] + A_\psi \mathrm{Sin}[\psi] + A_{\psi-\varphi}\mathrm{Sin}[\psi-\varphi]$$

$$= A_\chi[\mathrm{Sin}(\chi) + A_{\chi 1}\mathrm{Sin}(\frac{3}{2}\chi + \frac{3}{2}(\Omega_1 - \Omega_3) + \sigma)$$

$$+ A_{\chi 2}\mathrm{Sin}(\frac{1}{2}\chi + \frac{1}{2}(\Omega_1 - \Omega_3) + \sigma)$$

$$+ A_{\chi 3}\mathrm{Sin}[-\frac{1}{2}\chi - \frac{1}{2}(\Omega_1 - \Omega_3) + \sigma] \tag{1}$$

$$+ \left(\frac{d^2\varphi}{dt^2}\right)_o + \left(\frac{d^2\sigma}{dt^2}\right)_o + \left(\frac{d^2\varphi}{dt^2}\right)_s + \left(\frac{d^2\sigma}{dt^2}\right)_s$$

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Where,

$$\chi = 2\lambda_1 - 4\lambda_3 + 2\Omega_3 ,$$

$$\psi + \varphi = 3\lambda_1 - 6\lambda_3 + 2\Omega_1 + \varpi_3 = \frac{3}{2}(\chi + \Omega_1 - \Omega_3) + \sigma$$

$$\psi = \lambda_1 - 2\lambda_3 + \Omega_1 - \Omega_3 + \varpi_3 = \frac{1}{2}(\chi + \Omega_1 - \Omega_3) + \sigma$$

And

$$\psi - \varphi = -\lambda_1 + 2\lambda_3 - 2\Omega_3 + \varpi_3 = -\frac{1}{2}(\chi + \Omega_1 - \Omega_3) + \sigma$$

With

$$\sigma = ft + \sigma_0 ,$$

$$A_\chi = (6n_1^2 m_3 \alpha_{13} + 24 n_3^2 m_1)(-f_0(\alpha_{13}))\gamma_3^2$$

$$A_{\chi 1} = -3e_3 \frac{f_1(\alpha_{13})}{f_0(\alpha_{13})}\frac{\gamma_1^2}{\gamma_3^2},\; A_{\chi 2} = -e_3 \frac{f_2(\alpha_{13})}{f_0(\alpha_{13})}\frac{\gamma_1}{\gamma_3},\; A_{\chi 3} = e_3 \frac{f_3(\alpha_{13})}{f_0(\alpha_{13})}$$

The value of $f_0(\alpha_{13})$, $f_1(\alpha_{13}), f_2(\alpha_{13})$ and $f_3(\alpha_{13})$ depend on the Laplace's Coefficients $b_s^{(k)}(\alpha_{ij})$ and given in table 1.

3. Equation of motion when the system is locked in $i_1 i_3, i_1^2 e_3, i_1 i_3 e_3$ and $i_3^2 e_3$ resonances

$$\frac{d^2\varphi}{dt^2} = A_0 \begin{pmatrix} \sin\varphi + A_1 \sin(\frac{3}{2}\varphi + f\,t + \sigma_0) \\ + A_2 \sin(\frac{1}{2}\varphi + f\,t + \sigma_0) \\ + A_3 \sin(-\frac{1}{2}\varphi + f\,t + \sigma_0) \end{pmatrix} + \left(\frac{d^2\varphi}{dt^2}\right)_o + \left(\frac{d^2\sigma}{dt^2}\right)_o + \left(\frac{d^2\varphi}{dt^2}\right)_s + \left(\frac{d^2\sigma}{dt^2}\right)_s \tag{2}$$

Where

$$\varphi = 2\lambda_1 - 4\lambda_3 + \Omega_1 + \Omega_3 ,$$

$$\psi + \varphi = 3\lambda_1 - 6\lambda_3 + 2\Omega_1 + \varpi_3 = \frac{3}{2}\varphi + \sigma$$

$$\psi = \lambda_1 - 2\lambda_3 + \Omega_1 - \Omega_3 + \varpi_3 = \frac{1}{2}\varphi + \sigma$$

And

$$\psi - \varphi = -\lambda_1 + 2\lambda_3 - 2\Omega_3 + \varpi_3 = -\frac{1}{2}\varphi + \sigma$$

With

$\sigma = ft + \sigma_0$ where $\sigma_0 = -0.031391995$

And

$$A_0 = (12 n_1^2 m_3 \alpha_{13} + 48 n_3^2 m_1) f_0(\alpha_{13})\gamma_1\gamma_3$$

$$A_1 = \frac{3}{2} e_3 \frac{f_1(\alpha_{13})}{f_0(\alpha_{13})}\frac{\gamma_1}{\gamma_3},\; A_2 = \frac{1}{2} e_3 \frac{f_2(\alpha_{13})}{f_0(\alpha_{13})}$$

and $A_3 = -\dfrac{1}{2}\dfrac{f_3(\alpha_{13})}{f_0(\alpha_{13})}\dfrac{\gamma_3}{\gamma_1}$

The value of $f_0(\alpha_{13})$, $f_1(\alpha_{13}), f_2(\alpha_{13})$ and $f_3(\alpha_{13})$ depend on the Laplace's Coefficients $b_s^{(k)}(\alpha_{ij})$ and given in table 1.

With

$$\left(\frac{d^2\lambda_1}{dt^2}\right)_S = -2\left[\Sigma_{i=2}^{5} m_i \alpha_{1i}^2 \frac{d^2 A(\alpha_{1i})}{d\alpha_{1i}^2}\right]\frac{dn_1}{dt}$$

$$\left(\frac{d^2\Omega_1}{dt^2}\right)_S = \frac{1}{2}\left[\Sigma_{i=2}^{5} m_i \alpha_{1i} C(\alpha_{1i})\right]\frac{dn_1}{dt}$$

$$\left(\frac{d^2\Omega_3}{dt^2}\right)_S = \frac{1}{2}\left[\Sigma_{i=1}^{2} m_i C(\alpha_{i3}) + \Sigma_{i=4}^{6} m_i \alpha_{3i} C(\alpha_{3i})\right]\frac{dn_3}{dt}$$

$$\left(\frac{d^2\varpi_3}{dt^2}\right)_S = 2\left[\Sigma_{i=1}^{2} m_i B(\alpha_{i3}) + \Sigma_{i=4}^{6} m_i \alpha_{3i} B(\alpha_{3i})\right]\frac{dn_3}{dt}$$

$$\left(\frac{d^2\lambda_3}{dt^2}\right)_S = 2\left[\Sigma_{i=1}^{2} \begin{matrix} -m_i\left\{\alpha_{i3}\dfrac{dA(\alpha_{i3})}{d\alpha_{i3}} + A(\alpha_{i3})\right\} \\ + \Sigma_{i=4}^{6} m_i \alpha_{3i}^2 \dfrac{dA(\alpha_{3i})}{d\alpha_{3i}} \end{matrix}\right]\frac{dn_3}{dt} \tag{3}$$

(We are not considering any changes in semi major axis of any satellites)

And

$$\left(\frac{d^2\varphi}{dt^2}\right)_S = \left(\frac{d^2\lambda_1}{dt^2}\right)_S - 2\left(\frac{d^2\lambda_3}{dt^2}\right)_S + \frac{1}{2}\left[\left(\frac{d^2\Omega_1}{dt^2}\right)_S + \left(\frac{d^2\Omega_3}{dt^2}\right)_S\right]$$
$$\left(\frac{d^2\sigma}{dt^2}\right)_S = \frac{1}{2}\left(\frac{d^2\Omega_1}{dt^2}\right)_S - \frac{3}{2}\left(\frac{d^2\Omega_3}{dt^2}\right)_S + \left(\frac{d^2\varpi_3}{dt^2}\right)_S \tag{4}$$

Now we will find the terms due to Oblateness of Saturn. Saturn's gravitational momenta are quite important so that we have, in order to get the full variations of the mean longitudes, nodes and pericentres due to the oblateness, taken into account the lowest – degree terms with $J_2, J_4, J_2^2, J_6, J_2J_4$ and J_2^3 (See Table 2) as a factor (see [10]). Values of α_{ij}, $A(\alpha_{ij})$, $B(\alpha_{ij})$, $C(\alpha_{ij})$, $\dfrac{dA(\alpha_{ij})}{d\alpha_{ij}}$ for Every Pair (i, j), ($i < j$) Involving Mimas, Enceladus, Tethys, Dione, Rhea and Titan are given in Table 3. The other terms are taken constant. We then get (a_e is the equatorial radius of Saturn).

$$\left(\frac{d^2\Omega_3}{dt^2}\right)_o = \left[\begin{matrix} \frac{-3}{2}J_2\left(\frac{a_e}{a_3}\right)^2\left(1+2e_3^2-2\gamma_3^2\right) \\ +a_3^{-4}\left(\frac{15}{4}J_4-\frac{45}{8}J_2^2\right)\end{matrix}\right]\frac{dn_3}{dt} - 6J_2n_3\left(\frac{a_e}{a_3}\right)^2\left(e_3\frac{de_3}{dt}-\gamma_3\frac{d\gamma_3}{dt}\right) \quad (5)$$

$$\left(\frac{d^2\lambda_1}{dt^2}\right)_o = \left[\begin{matrix} 3J_2\left(\frac{a_e}{a_1}\right)^2(1+\frac{7}{4}e_1^2-7\gamma_1^2) \\ +a_1^{-4}(-\frac{15}{4}J_4+\frac{160}{11}J_2^2)\end{matrix}\right]\frac{dn_1}{dt} - 42J_2n_1\gamma_1\left(\frac{a_e}{a_1}\right)^2\frac{d\gamma_1}{dt} \quad (6)$$

$$\left(\frac{d^2\lambda_3}{dt^2}\right)_o = \left[\begin{matrix} 3J_2\left(\frac{a_e}{a_3}\right)^2(1+\frac{7}{4}e_3^2-7\gamma_3^2) \\ +a_3^{-4}(-\frac{15}{4}J_4+\frac{160}{11}J_2^2)\end{matrix}\right]\frac{dn_3}{dt} - 42J_2n_3\gamma_3\left(\frac{a_e}{a_3}\right)^2\frac{d\gamma_3}{dt} \quad (7)$$

$$\left(\frac{d^2\varphi}{dt^2}\right)_o = \left(\frac{d^2\lambda_1}{dt^2}\right)_o - 2\left(\frac{d^2\lambda_3}{dt^2}\right)_o + \frac{1}{2}\left\{\left(\frac{d^2\Omega_1}{dt^2}\right)_o + \left(\frac{d^2\Omega_3}{dt^2}\right)_o\right\} \quad (8)$$

$$\left(\frac{d^2\sigma}{dt^2}\right)_o = \frac{1}{2}\left(\frac{d^2\Omega_1}{dt^2}\right)_o - \frac{3}{2}\left(\frac{d^2\Omega_3}{dt^2}\right)_o + \left(\frac{d^2\varpi_3}{dt^2}\right)_o \quad (9)$$

Table 1: Analytical Expressions of the Functions $f_i(\alpha_{13})$ for the Arguments φ, $\psi+\varphi$, ψ, $\psi-\varphi$, χ, with Their Value for $\alpha_{13}=0.6306386$ (See [11])

Argument	i	$f_i(\alpha)$	$f_i(\alpha_{13})$
φ	0	$-\alpha_{13}b_{3/2}^3(\alpha_{13})$	−1.65088068
$\psi+\varphi$	1	$3\alpha_{13}b_{3/2}^4(\alpha_{13})+\frac{1}{4}(\alpha_{13})^2\frac{d}{d\alpha}b_{3/2}^4(\alpha_{13})$	5.23786953
ψ	2	$2\alpha_{13}b_{3/2}^2(\alpha_{13})+\frac{1}{2}(\alpha_{13})^2\frac{d}{d\alpha}b_{3/2}^2(\alpha_{13})$	9.70821605
$\psi-\varphi$	3	$-\alpha_{13}b_{3/2}^2(\alpha_{13})+\frac{1}{4}(\alpha_{13})^2\frac{d}{d\alpha}b_{3/2}^2(\alpha_{13})$	0.22188903
χ	4	$\frac{1}{2}\alpha_{13}b_{3/2}^3(\alpha_{13})=-f_0(\alpha_{13})$	0.82544034

Table 2 : Parameters of Mimas, Enceladus, Tethys, Dione, Rhea, Titan and Saturn $M_{i,i=1-6}$, M and M_S are the Masses of the Considered Satellite, Saturn and the Sun, Respectively

	$m=\frac{M_{i,i=1-6}}{M}$	n (rad/yr)	i(deg)	M_S/M	a_e (km)	E	J_2	J_4	J_6
Mimas	6.34×10^{-8}	2422.44	1.62	-	-	0.0194			
Enceladus	0.15×10^{-6}								
Tethys	1.06×10^{-6}	1213.17	1.093	-	-	0.009			
Dione	1.963×10^{-6}								
Rhea	4.32×10^{-6}	-	-	-	-	-			
Titan	236.638×10^{-6}								
Saturn	-	-	-	3498.79	60330	-	0.012 98	0.000 915	0.000 095

Table 3: Values of $\alpha_{ij,}$ $A(\alpha_{ij})$, $B(\alpha_{ij})$, $C(\alpha_{ij})$, $\frac{dA(\alpha_{ij})}{d\alpha_{ij}}$ for Every Pair (i, j), (i < j) Involving Mimas, Enceladus, Tethys, Dione, Rhea and Titan. (α_{ij}) (See [3] and [4])

i – j	$\alpha_{ij,}$	$A(\alpha_{ij})$	$B(\alpha_{ij})$	$C(\alpha_{ij})$	$\frac{dA(\alpha_{ij})}{d\alpha_{ij}}$
1-2	0.78026	1.2473	1.3674	-5.4695	1.0996
1-3	0.63064	1.1306	0.38952	-1.5581	0.55451
1-4	0.49258	1.0706	0.15366	-0.61463	0.33609
1-5	0.35283	1.0335	0.059940	-0.23976	0.20479
1-6	0.15223	1.0059	0.0090810	-0.036324	0.078148
2-3	0.80824	1.2807	1.8535	-7.4138	1.2978
3-4	0.78108	1.2482	1.3790	-5.5159	1.1047
3-5	0.55948	1.0960	0.23856	-0.95425	0.42538
3-6	0.24140	1.0151	0.024460	-0.097840	0.12912

4. Numerical integration and surface of sections

Our equations were integrated backwards in time. The initial conditions are taken from Vienne and Duriez [3,4] for J2000. $i_1=1.62^o, i_3=1.093^o$ and $e_3=0.0008$ are taken to be fixed. Here we have integrated it for -62.8×14000 yrs.

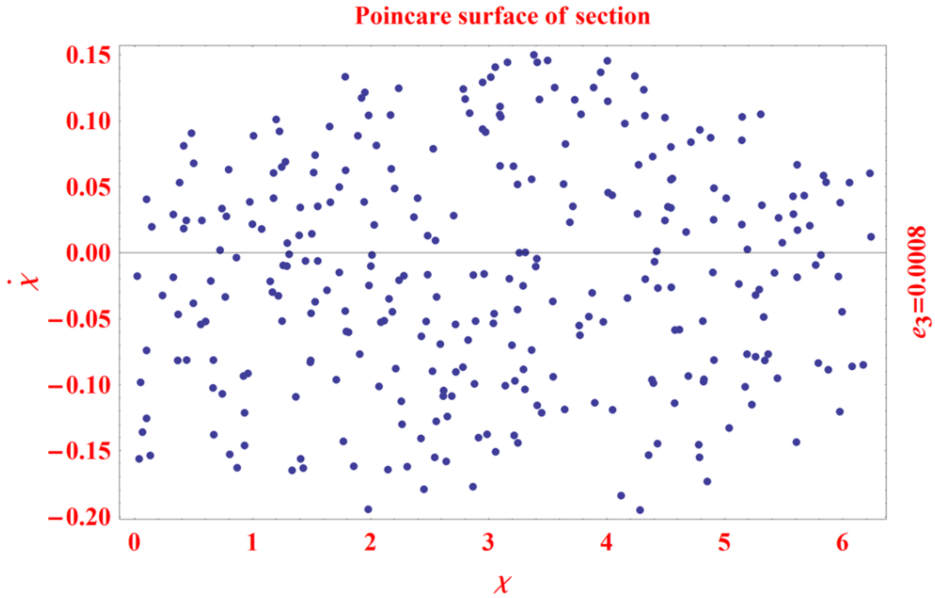

Fig. 4.1: Poincare Surface of Section at $i_3^2, i_1^2e_3, i_1i_3e_3$ and $i_3^2e_3$ Resonances with Oblateness of Saturn. Secular Resonances of All Inner Satellites Have Been Considered at $i_1=1.62^o, i_3=1.093^o$ and $e_3=0.0008$.

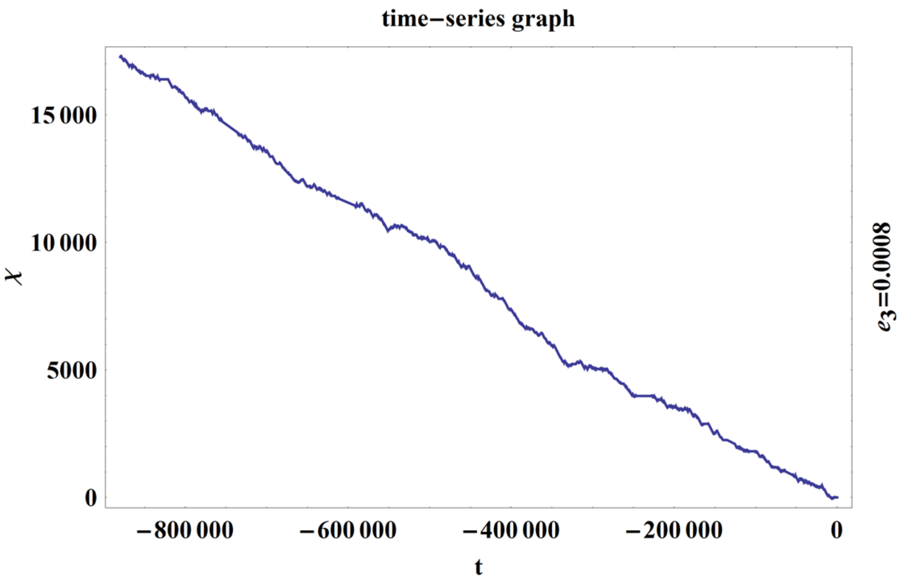

Fig. 4.2: Time-Series Graph at $i_3^2, i_1^2e_3, i_1i_3e_3$ and $i_3^2e_3$ Resonances with Oblateness of Saturn. Secular Resonances of All Inner Satellites Have Been Considered at $i_1=1.62^o, i_3=1.093^o$ and $e_3=0.0008$.

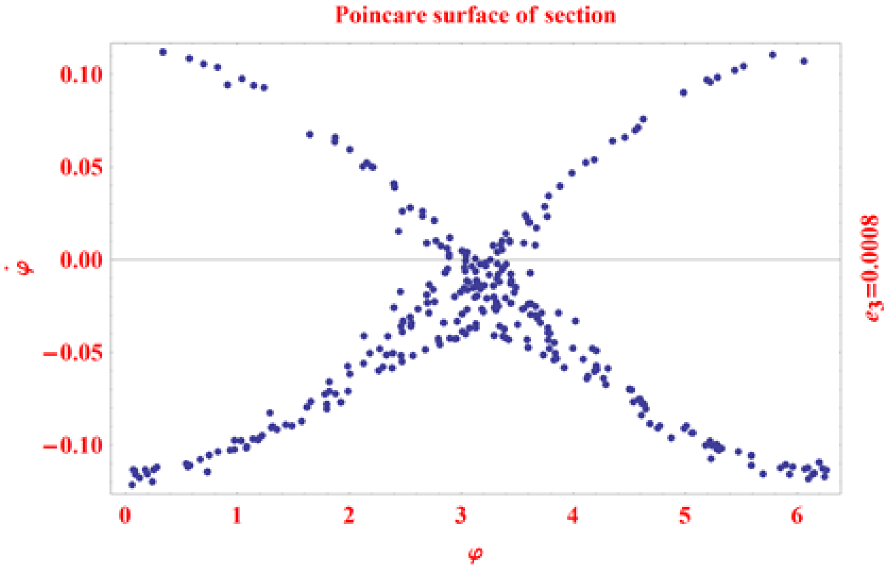

Fig. 4.3: Poincare Surface of Section at $i_1i_3, i_1^2e_3, i_1i_3e_3$ and $i_3^2e_3$ Resonances with Oblateness of Saturn. Secular Resonances of All Inner Satellites Have Been Considered at i_1=1.62^o,i_3=1.093^{o}and e_3=0.0008 .

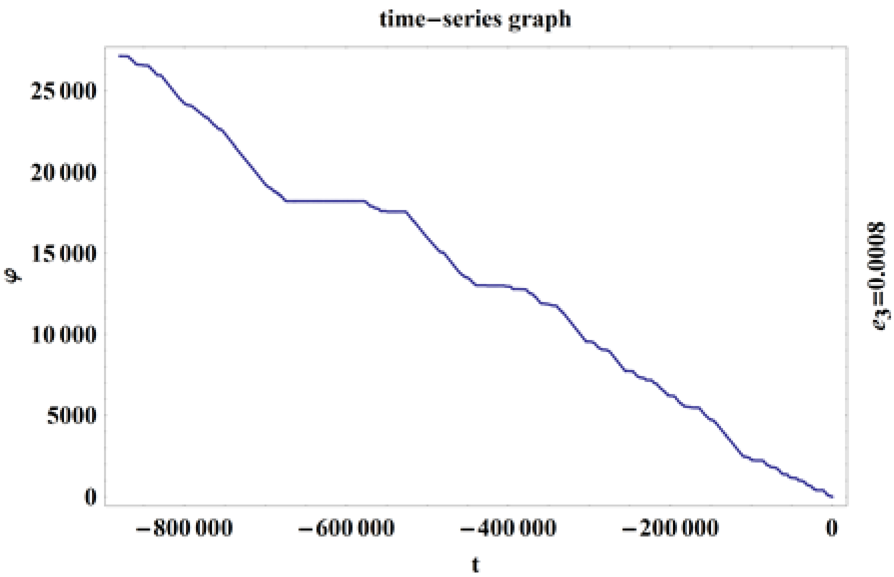

Fig. 4.4: Time-Series Graph at $i_1i_3, i_1^2e_3, i_1i_3e_3$ and $i_3^2e_3$ Resonances with Oblateness of Saturn. Secular Resonances of All Inner Satellites Have Been Considered at i_1=1.62^o,i_3=1.093^{o}and e_3=0.0008 .

5. Discussion

Vienne and Duriez [3] discovered the 200 year period in the mean longitude of Mimas. Champenois and Vienne [5,6] have investigated the role of 200 year long period on the dynamics of Mimas-Tethys system when considered to be presently trapped in i_1i_3 resonance with probability of capture 0.76 at 2:4 commensurability. They found that the sources of this period are both the oblateness of Saturn and the interaction between its six inner satellites. They also realized that considering an eccentric orbit of Tethys upsets the earlier vision of dynamics of the Mimas-Tethys system.
Here we have analyzed that the system was more chaotic if considered to be (at presently it is) locked in i_1i_3 resonance compared with i_3^2 resonance, we considered the effect of Saturns oblateness and the interaction of its six inner satellites , by the help of Poincare surface of section and time-series graphs which confirms the earlier results too. Jha and Agrawal [5] also got the same result without considering the secular resonance of all inner satellite and Saturn's oblateness.

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