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Research paper

Analysis on nonlinear stability of the triangular libration points for radiating and oblate primaries in ER3BP

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Abstract

In this paper we study the non linear stability of the triangular librations points in ER3BP considering both the primaries as radiating and oblate. The study is carried out near the resonance frequency satisfying the conditions $\omega_1 = \omega_2, \omega_1 = 2\omega_2, \omega_1 = 3\omega_2$ in resonance as well as non resonance case. The study is conducted for various values of radiation pressure and oblateness parameters. It is observed that the case $\omega_1 = \omega_2$ corresponds to the boundary region of the stability for the system, further it is examined that the system experiences resonance at $\omega_1 = 2\omega_2, \omega_1 = 3\omega_2$ for different values of radiation pressures and oblateness parameter. In non-resonance case, it is observed that the equilibrium points are stable.In resonance case, for $\mu \le \mu_c = 0.0385209;$ and $\omega_1 = 2\omega_2$ the triangular equilibrium points are unstable. In case, when $\omega_1 = 3\omega_2$ for some values of radiation pressure and oblateness parameter, it is stable and for some of the values of the parameters it is unstable. The model is best suited to the binary systems (Achird, Luyten, α Cen AB, Kruger- 60, Xi- Bootis).

Keywords: *Er3bp; Hamiltonian Functions; Triangular Libration Points; Resonance;Kam Theory.*

1. Introduction

One of the most famous and historically investigated chaotic dynamical systems is the elliptical restricted problem of three bodies (ER3BP) due to its application in dynamical astronomy, celestial mechanics and space mechanics. It constitute a major source of interesting theories in lunar and planetary sciences .The ER3BP describes the three dimensional motion of small particle called the infinitesimal mass under the gravitational force of two finite bodies called primaries around their common centre of mass. A typical example of the ER3BP is the motion of an asteroid under the gravitational attraction of the Sun and the Jupiter. The orbital motion of the Earth- Moon System around the Sun and the motion of thousands of astronomical and astrophysical bodies such as satellites, binary pulsars and the Moon have attracted attention and many studies has been conducted out in the model of ER3BP. The equilibrium points that appeared in ERTBP are very important for astronautical applications as space station can be easily located at these points due to minimum consumption of fuel, which is required for station keeping.

Many researchers considered the primaries and the third body to be either a point mass or spherical but later on it has been found that in general, celestial or stellar bodies are sufficiently oblate in shape which plays a significant role in restricted three body problem. Some of the planets such as Earth, Saturn, Jupiter and stars namely Achernar, Regulus, Vega, Alfa Arae and Altair are oblate showing the deviated behavior from spherical shape having a significant impact in restricted three body problem. Since, the rotation of stars is very fast, equatorial bulges is formed and are responsible for oblate shape of neutrons stars, pulsars, white and brown dwarf stars. Many stars belong to binary system; hence the motion of a particle in the field of double star has created special interest in space dynamics. High mass X-ray binaries after common – envelope evolution and spiral in, produces double neutron star binaries. Fast spinning "recycled pulsars in binary systems provides good source to study the orbital kinematics. Einstein theory of relativity and theoretical gravitational waves can be applied to the double pulsars where warped space time due to shift of intense masses is extremely rare. [1]. It is found that pulsars and their orbiting companions are compact enough to be treated as two point masses. Along with [2,3] many others researchers found that some planets exist outside our solar system and confirmed that PSR B1257+12 and PSR B1620-26 possess extra solar planets orbiting them. In the stellar system, a planet moving in the field of binary star forms three body problem; providing a premise to study fundamental physics and to test alternative gravities as given by [4,5]. A planet in the neighborhood of any of the following PSR B1534+12, PSR B1913+16, PSR B1620+26, PSR 1257+13, PSR J1022+1002, PSR J1518+4904, PSR B1534+12, PSR B2127+11C, PSR 2303+46 and others provide an excellent models for elliptical restricted three body problem.

The nonlinear stability of an elliptical or circular restricted three body problem of a Hamiltonian system is generally performed by using Kolmogorov-Arnold-Moser (KAM) theorem in non-resonance case and by Markeev theorem in resonance case.

Taking one of the bodies as radiating, the nonlinear stability of the triangular equilibrium points for resonance and non-resonance conditions was studied by [6]. Considering both the bodies as radiating in circular restricted three body problem, [7], [8] examined the stability of the triangular equilibrium points for non-resonance as well as resonance case. [9] Discussed the nonlinear stability of the triangular equilibrium points in circular restricted three bodies, considering bigger primary as a source of radiation.

The nonlinear stability of the triangular Lagrangian points, considering the bigger primary as oblatespheroid in circular case was examined by [10].
The ER3BP are studied by [11-26]. The present paper investigates the stability of the infinitesimal mass about the triangular equilibrium points in both resonance and non-resonance case satisfying $\omega_1=\omega_2, \omega_1=2\omega_2, \omega_1=3\omega_2$ in ER3BP The study is carried out at various values of radiation pressures and oblateness parameter.
This paper has been organized in various sections:
Section 1 gives introduction, section 2 describes the equations of motion of the problem, section 3 deals with characteristics roots and first order stability of the triangular equilibrium points. The existence of resonance is discussed in section 4, while section 5 deals with normalization and higher order stability of the libration points in non-resonance case. Stability in resonance case is discussed in section 6. Finally section 7 summarizes the discussion and conclusion of the paper.

2. Equations of motion

The differential equations of the motion of the infinitesimal mass in elliptical restricted three body problem under radiating primaries in pulsating system as given by Narayan and Shrivastava [27] is:

$$x'' - 2y' = \varphi\, \Omega_x ;$$

$$y'' - 2x' = \varphi\, \Omega_y \tag{1}$$

where the force function is defined as:

$$\Omega = \frac{x^2+y^2}{2} + \frac{1}{1+3\left(\frac{A_1+A_2}{2}\right)} \times \left[\frac{(1-\mu)Q_1}{r_1} + \frac{\mu Q_2}{r_2} + \frac{(1-\mu)Q_1A_1}{2r_1^3} + \frac{\mu A_2 Q_2}{2r_2^3}\right] \tag{2}$$

and

$$r_1^2 = (x+\mu)^2 + y^2 ;$$

$$r_2^2 = (x-1+\mu)^2 + y^2 ; \tag{3}$$

$$\varphi = \frac{1}{(1+e\cos f)} \tag{4}$$

$Q_i = 1-\beta_i, (i=1,2)$ The radiation pressure, f a true anomaly of the primariesand A_1, A_2 the oblateness parameter.
The coordinates of the triangular equilibrium points L_4 and L_5 as given by: Narayan and Shrivastava [27] is:

$$x_0 = \frac{1}{2} - \mu + \frac{A_1}{2} - \frac{A_2}{2} + \frac{\beta_2}{3} - \frac{\beta_1}{3} - \frac{1}{2}A_1\beta_1 + \frac{1}{2}A_2\beta_2;$$

$$y_0 = \pm\frac{\sqrt{3}}{2}\left[1 - \frac{A_1}{3} - \frac{A_2}{3} - \frac{2\beta_1}{9} - \frac{2\beta_2}{9} - \frac{1}{3}A_1\beta_1 - \frac{1}{3}A_2\beta_2\right]. \tag{5}$$

3. Characteristics roots and first order stability of the triangular equilibrium points

The stability of the elliptical restricted three body problem is restricted to planar case only. It is sufficient to study the stability of the triangular equilibrium points about L_4 due to similar of L_4 and L_5 .The Hamiltonian as described by Narayan and Shrivastava [27] is given by:

$$H = \frac{p_1^2+p_2^2}{2} + (p_1q_2 - p_2q_1) + \frac{(q_1^2+q_2^2)e\cos f}{2(1+e\cos f)} - \frac{1}{(1+e\cos f)\left(1+3\frac{A_1+A_2}{2}\right)} \times \left\{\left\{\frac{(1-\mu)Q_1}{r_1} + \frac{\mu Q_2}{r_2} + \frac{(1-\mu)Q_1A_1}{2r_1^3} + \frac{\mu A_2 Q_2}{2r_2^3}\right\}\right\} \tag{6}$$

Now expanding the Hamiltonian function given by equation (6) in the powers of p_i and q_i as:

$$H = \sum_{K=0}^{\infty} H_K = H_0 + H_1 + H_2 + H_3 + H_4 + \ldots\ldots\ldots\ldots$$

where

$$H_0 = H(\xi, \eta, p_\xi, p_\eta) = \text{constant}; \quad H_1 = 0 .$$

and by substituting $Q_i = 1-\beta_i, (i=1,2)$ we have:

$$H = \frac{p_1^2+p_2^2}{2} + (p_1q_2 - p_2q_1) + \frac{(q_1^2+q_2^2)e\cos f}{2(1+e\cos f)} - \frac{1}{(1+e\cos f)\left(1+3\frac{A_1+A_2}{2}\right)} \times \left\{\left\{\frac{(1-\mu)(1-\beta_1)}{r_1} + \frac{\mu(1-\beta_2)}{r_2} + \frac{(1-\mu)(1-\beta_1)A_1}{2r_1^3} + \frac{\mu A_2(1-\beta_2)}{2r_2^3}\right\}\right\} \tag{7}$$

Further by using Taylor's theorem and equating the coefficients of 2nd, 3rd, 4th order from the above equation (7) we have:

$$H_2=\frac{p_1{}^2+p_2{}^2}{2}+\left(p_1q_2-p_2q_1\right)+ e\cos f\,\frac{\left(q_1{}^2+q_2{}^2\right)}{2\left(1+e\cos f\right)}-\frac{1}{\left(1+e\cos f\right)\left\{1+3\left(\frac{A_1+A_2}{2}\right)\right\}}\times$$

$$\left[\begin{array}{l}\left(\begin{array}{l}-\frac{1}{8}+\frac{9}{8}A_1-\frac{9}{16}A_2-\frac{73}{24}A_1\beta_1+\frac{11}{48}A_2\beta_2\\ -\frac{27}{16}\mu A_1+\frac{27}{16}\mu A_2+\frac{17}{8}\mu\beta_1-\frac{7}{9}\mu\beta_2\\ +\frac{1}{2}\beta_2-\frac{13}{8}\beta_2\end{array}\right)q_1{}^2+\\ \sqrt{3}\left(\begin{array}{l}\frac{3}{4}+\frac{17}{4}A_1+\frac{7}{8}A_2+\frac{139}{24}A_1\beta_1+\\ \frac{15}{8}A_2\beta_2-\frac{41}{8}\mu A_1-\frac{41}{8}\mu A_2-\frac{1}{6}\mu\beta\\ 1+\mu\beta_2\\ -\frac{3\sqrt{3}}{2}\mu-\frac{1}{3}\beta_2-\frac{7}{12}\beta_1\end{array}\right)q_1q_2+\\ \left(\begin{array}{l}\frac{5}{8}+\frac{27}{8}A_1+\frac{21}{16}A_2+\frac{107}{16}A_1\beta_{11}+\frac{11}{16}A_2\beta_2\\ -\frac{33}{16}\mu A_1+\frac{33}{16}\mu A_2-\frac{7}{8}\mu\beta_1+\frac{11}{8}\mu\beta_2-\\ \frac{1}{2}\beta_2+\frac{7}{8}\beta_1\end{array}\right)q_2{}^2\end{array}\right]$$

After further simplification equation (8) can be rewritten as:

$$H_2=\frac{p_1^2+p_2^2}{2}+(p_1q_2-p_2q_1)+\frac{e\cos f}{2(1+e\cos f)\left\{1+3\left(\frac{A_1+A_2}{2}\right)\right\}}(q_1^2+q_2^2)-\frac{1}{2(1+e\cos f)}\times(H_{20}q_1^2-H_{11}q_1q_2-H_{02}q_2^2) \tag{8}$$

where

$$H_{20}=-\left(\begin{array}{l}\frac{-1}{8}+\frac{9}{8}A_1-\frac{9}{16}A_2-\frac{73}{24}A_1\beta_1+\frac{11}{48}A_2\beta_2\\ -\frac{27}{16}\mu A_1+\frac{27}{16}\mu A_2+\frac{17}{8}\mu\beta_1-\frac{7}{9}\mu\beta_2\\ +\frac{\beta_2}{2}-\frac{13\beta_1}{8}\end{array}\right) \tag{9}$$

$$H_{11}=\sqrt{3}\left(\begin{array}{l}\frac{3}{4}+\frac{17}{4}A_1+\frac{7}{8}A_2+\frac{139}{24}A_1\beta_1+\\ \frac{15}{8}A_2\beta_2-\frac{41}{8}\mu A_1-\frac{41}{8}\mu A_2-\frac{1}{6}\mu\beta_1\\ +\mu\beta_2-\frac{3\mu}{2}-\frac{\beta_2}{3}-\frac{7\beta_1}{12}\end{array}\right) \tag{10}$$

$$H_{02}=\left(\begin{array}{l}\frac{5}{8}+\frac{27}{8}A_1+\frac{21}{16}A_2+\frac{107}{16}A_1\beta_1+\frac{11}{16}A_2\beta_2\\ -\frac{33}{8}\mu A_1+\frac{33}{16}\mu A_2-\frac{7}{8}\mu\beta_1+\frac{11}{8}\mu\beta_2-\frac{\beta_2}{2}\\ +\frac{7\beta_1}{8}\end{array}\right) \tag{11}$$

Also,

$$H_3=\frac{-1}{6(1+e\cos f)[1+3(A_1+A_2)/2]}\times$$

$$\left[\begin{array}{l}q_1^3\times\\ \left(\begin{array}{l}\frac{21}{8}+\frac{15A_1}{4}-\frac{295}{16}A_2-\frac{93\beta_1}{4}-\frac{65\beta_2}{6}\\ -\frac{365\beta_1A_1}{24}-\frac{155\beta_1A_2}{16}+\\ \mu\left(\begin{array}{l}-\frac{57}{8}-\frac{15A_1}{4}-\frac{305A_2}{16}-\frac{29\beta_1}{4}+\\ \frac{51\beta_2}{8}-\frac{365\beta_1A_1}{24}+\frac{85\beta_1A_2}{16}-\frac{17\beta_2A_1}{8}\\ -\frac{5\beta_2A_2}{3}\end{array}\right)\end{array}\right)\\ +3q_1{}^2q_2\times\\ \left(\begin{array}{l}\frac{-3}{8}+\frac{21A_1}{8}-\frac{61}{8}A_2-\frac{113\beta_1}{24}+\frac{7\beta_2}{4}-\\ \frac{53\beta_1A_1}{24}+\frac{61\beta_1A_2}{8}+\\ \mu\sqrt{3}\left(\begin{array}{l}\frac{17A_1}{16}+\frac{9A_2}{16}+\frac{47\beta_1}{6}-\frac{13\beta_2}{4}-\\ \frac{53\beta_1A_1}{24}-\frac{113\beta_1A_2}{24}-\\ \frac{91\beta_2A_1}{16}-\frac{\beta_2A_2}{48}\end{array}\right)\end{array}\right)\\ +3q_1q_2{}^2\times\\ \left(\begin{array}{l}\frac{-33}{8}-\frac{315A_1}{8}+\frac{5}{12}A_2+\frac{149\beta_1}{24}+\frac{5\beta_2}{8}\\ -\frac{265\beta_1A_1}{8}-\frac{5\beta_1A_2}{12}+\\ \mu\left(\begin{array}{l}\frac{33}{4}-\frac{245A_1}{8}+\frac{263A_2}{8}+\frac{77\beta_1}{48}-\frac{311\beta_2}{16}-\\ \frac{55\beta_1A_1}{4}+\frac{3805\beta_1A_2}{96}+\frac{55\beta_2A_1}{8}\\ -\frac{1441\beta_2A_2}{16}\end{array}\right)\end{array}\right)+\\ q_2^3\times\\ \left(\begin{array}{l}\frac{-9}{8}-\frac{51A_1}{4}-\frac{69}{16}A_2-\frac{13\beta_1}{8}-\frac{11\beta_2}{4}+\\ \frac{151\beta_1A_1}{16}+\frac{65\beta_1A_2}{16}\\ +\mu\left(\begin{array}{l}\frac{195A_1}{16}-\frac{33A_2}{2}+\frac{81\beta_1}{16}+\frac{31\beta_2}{8}-\\ \frac{151\beta_1A_1}{16}+\frac{117\beta_1A_2}{32}-\frac{317\beta_2A_1}{32}\\ +\frac{5\beta_2A_2}{4}\end{array}\right)\end{array}\right)\end{array}\right] \tag{13}$$

$$H_4=\frac{-1}{24(1+e\cos f)}\times[1+3(A_1+A_2)/2]\times$$

$$\left[q_1^4\times\left(\begin{array}{l}\frac{-111}{16}-\frac{1155A_1}{16}+\frac{9815}{16}A_2+\frac{397\beta_1}{2}-\\ \frac{25\beta_2}{2}-\frac{1235\beta_1A_1}{16}-\frac{1185\beta_1A_2}{2}\\ +\mu\left(\begin{array}{l}-\frac{1575}{16}-\frac{395A_1}{48}-\frac{29545A_2}{48}+\frac{9269\beta_1}{16}\\ +\frac{439\beta_2}{144}-\frac{1234\beta_1A_1}{16}-\frac{6325\beta_1A_2}{12}\\ +\frac{6755\beta_2A_1}{48}+\frac{605\beta_2A_2}{48}\end{array}\right)\end{array}\right)\right.$$

$$+4\sqrt{3}q_1^3q_2\times\left(\begin{array}{l}\frac{-75}{16}-\frac{281A_1}{16}-\frac{21}{2}A_2+\frac{5\beta_1}{16}+\\ \frac{21\beta_2}{6}+\frac{351\beta_1A_1}{16}+\frac{21\beta_1A_2}{2}\\ +\mu\left(\begin{array}{l}\frac{75}{8}+\frac{5669A_1}{208}-\frac{117A_2}{8}-\frac{235\beta_1}{48}\\ -\frac{6121\beta_2}{560}-\frac{7\beta_1A_1}{2}+\frac{319\beta_1A_2}{32}+\\ \frac{2357\beta_2A_1}{156}-\frac{83\beta_2A_2}{4}\end{array}\right)\end{array}\right)$$

$$+6q_1^2q_2^2\times\left(\begin{array}{l}\frac{123}{16}+\frac{4865A_1}{48}-\frac{105}{8}A_2+\frac{235\beta_1}{12}\\ +\frac{75\beta_2}{12}-\frac{4295\beta_1A_1}{48}+\frac{105\beta_1A_2}{8}\\ +\mu\left(\begin{array}{l}-\frac{123}{8}-\frac{5735A_1}{48}+\frac{1601A_2}{16}-\frac{421\beta_1}{48}\\ -\frac{31\beta_2}{8}+\frac{4865\beta_1A_1}{48}+\frac{245\beta_1A_2}{32}\\ +\frac{45\beta_2A_1}{8}-\frac{539\beta_2A_2}{4}\end{array}\right)\end{array}\right)$$

$$+4\sqrt{3}q_1q_2^3\times\left(\begin{array}{l}\frac{135}{16}+\frac{1531A_1}{16}+\frac{15}{4}A_2-\frac{69\beta_1}{16}-\\ \frac{21\beta_2}{5}+\frac{23\beta_1A_1}{2}-\frac{15\beta_1A_2}{4}\\ +\mu\left(\begin{array}{l}-\frac{135}{8}-\frac{337A_1}{2}-\frac{8997A_2}{112}+\frac{259\beta_1}{16}+\\ \frac{293\beta_2}{40}+\frac{1391\beta_1A_1}{16}-\\ \frac{3005\beta_1A_2}{32}+\frac{1315\beta_2A_1}{116}-\frac{175\beta_2A_2}{48}\end{array}\right)\end{array}\right)$$

$$\left.+q_2^4\left(\begin{array}{l}\frac{9}{16}+\frac{1125A_1}{16}-\frac{1575}{32}A_2+\frac{1781\beta_1}{16}\\ -\frac{45\beta_2}{2}-\frac{1045\beta_1A_1}{8}+\frac{1575\beta_1A_2}{32}\\ +\mu\left(\begin{array}{l}-\frac{495A_1}{8}-\frac{3167A_2}{32}-\frac{1917\beta_1}{16}+26\beta_2\\ +\frac{1045\beta_1A_1}{8}-\frac{14765\beta_1A_2}{96}\\ -\frac{1325\beta_2A_1}{32}-\frac{6775\beta_2A_2}{48}\end{array}\right)\end{array}\right)\right] \tag{14}$$

The characteristics equation can be written in following form as:

$$\lambda^4+(1+e^2)\lambda^2+\left\{\frac{27\mu(1-\mu)}{4}\right\}\times\left\{1-\frac{125}{6}A_1+\frac{40}{3}A_2+\frac{5}{6}\beta_1-\frac{35}{18}\beta_2\right\}=0$$

If ω1 and ω2 be the frequencies and putting $\lambda^2=-\omega^2$ then

$$\omega_1^2=-\lambda_{1,2}^2=\frac{1}{2}\left[(1+e^2)+\left\{\begin{array}{l}(1+e^2)^2-27\mu(1-\mu)\times\\ \left(\begin{array}{l}1-\frac{125}{6}A_1+\frac{40}{3}A_2\\ +\frac{5}{6}\beta_1-\frac{35}{18}\beta_2\end{array}\right)\end{array}\right\}^{1/2}\right] \tag{15}$$

$$\omega_2^2=-\lambda_{3,4}^2=\frac{1}{2}\left[(1+e^2)-\left\{\begin{array}{l}(1+e^2)^2-27\mu(1-\mu)\times\\ \left(\begin{array}{l}1-\frac{125}{6}A_1+\frac{40}{3}A_2\\ +\frac{5}{6}\beta_1-\frac{35}{18}\beta_2\end{array}\right)\end{array}\right\}^{1/2}\right] \tag{16}$$

The correlation between μ and ω1, ω2 is shown in figs. 1-5 for different values of radiation pressure and oblateness parameter. It is found that ω1 increases with increasing μ whereas ω2 decreases, and becomes equal to the critical value which is shown in the figures.

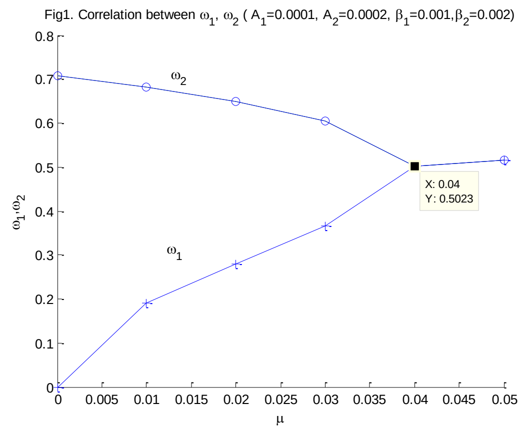

Fig. 1: Correlation between Ω1 and Ω2 for A1=0.0001, A2=0.0002, B1=0.001, B2=0.002.

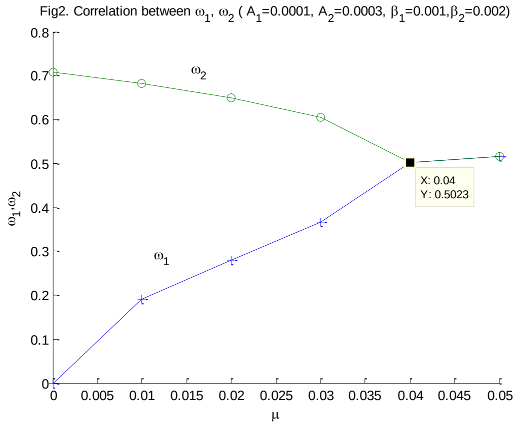

Fig. 2: Correlation between Ω1 and Ω2 for A1=0.0001, A2=0.0003, B1=0.001, B2=0.002.

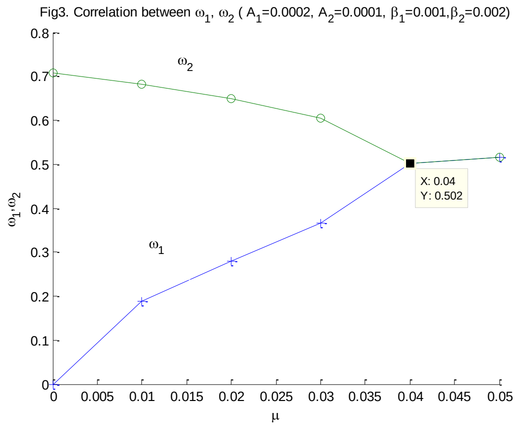

Fig. 3: Correlation between Ω1 and Ω2 for A1=0.0002, A2=0.0001, B1=0.001, B2=0.002.

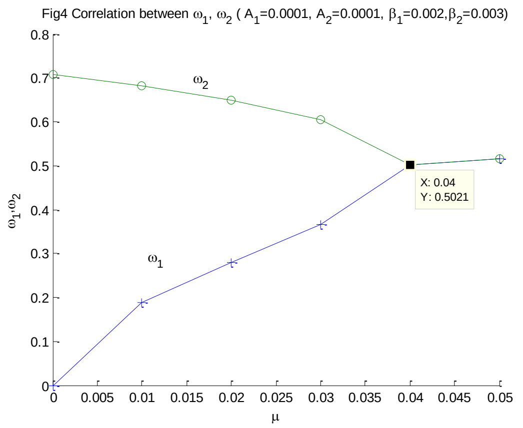

Fig. 4: Correlation between Ω1 and Ω2 for A1=0.0001, A2=0.0001, B1=0.002, B2=0.003.

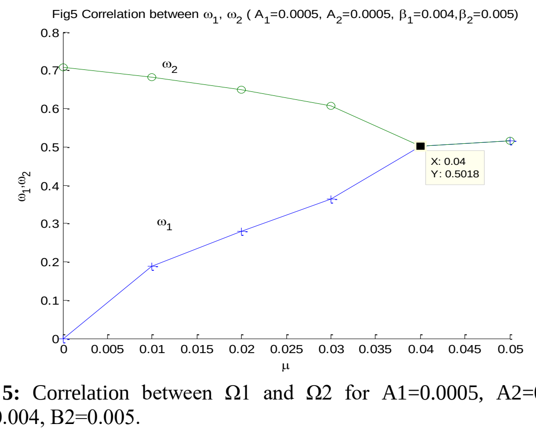

Fig. 5: Correlation between Ω1 and Ω2 for A1=0.0005, A2=0.0005, B1=0.004, B2=0.005.

4. Existence of resonance in elliptical cases

In order to discuss the existence of resonance we consider the following three cases:

case1. When ω1= ω2

i.e. $\lambda_{1,2}^2=\lambda_{3,4}^2$

Solving we obtain:

$$(1+e^2)^2-27\mu(1-\mu)(1+e^2)\left(1-\frac{125}{6}A_1+\frac{40}{3}A_2+\frac{5}{6}\beta_1-\frac{35}{18}\beta_2\right)\geq 0 \qquad (17)$$

For equality we have:

$$\mu^2-\mu+\frac{1-\frac{3e^4}{4}}{27}\times 1+\frac{125}{6}A_1-\frac{40}{3}A_2 -\frac{5}{6}\beta_1+\frac{35}{18}\beta_2=0 \qquad (18)$$

Since $\mu\leq\frac{1}{2}$, the positive sign is inadmissible. Hence the region of stability in first approximation can be written as:
Thus, the value of μ responsible for stable equilibrium points is given by:

$$\mu_{01}=0.0385208965+0.83601A_1+0.535048A_2 -0.0334405\beta_1+0.0780279\beta_2+ e^4(0.01618+1.254A_1-.80251A_2- 0.50159\beta_1+0.117023\beta_2) \qquad (19)$$

From equation (19) it is clear that, when A1= A2= β1= β2=e=0, $\mu_c=0.0385208965$

It is obvious that the case ω1= ω2 usually corresponds to a boundary of the region of stability of the system.

case2. When ω1= 2ω2

i.e. $\lambda_{1,2}^2=4\lambda_{3,4}^2$

Solving for μ the resonance value is obtained as:

$$\mu_{02}=0.0242939-0.519045A_1+0.33219A_2+ 0.20762\beta_1-0.0484445\beta_2+e^4(0.19635+ .409165A_1-0.26185A_2-0.0163665\beta_1+ 0.0381885\beta_2 \qquad (20)$$

case3. When ω1= 3ω2

i.e. $\lambda_{1,2}^2=9\lambda_{3,4}^2$

Solving for μ the resonance value is obtained as:

$$\mu_{03}=0.0135160+20.270A_1-12.9729A_2- 0.81081\beta_1+3.7837\beta_2+e^4(0.75+15.625A_1 -10A_2-0.625\beta_1+1.45833\beta_2)$$

5. Normalization and higher order stability of the libration points in non-resonance case

In order to investigate the stability the Hamiltonian H is normalized by Birkhoff's method to the following form:

$$H=\omega_1 t_1-\omega_2 t_2+c_{20}t_1^2+c_{11}t_1t_2 +c_{02}t_2^2+o(t_1+t_2)^{5/2} \tag{21}$$

Where

$$2t_i=p_i^2+q_i^2, i=1,2$$

ω_1,ω_2 is defined by equation (16).

It is well known fact that if , H2 is of a positive definite form, then the equilibrium position is stable by virtue of Liapunov theorem [28] .Otherwise the problem of stability is solved by KAM theorem as given by Arnold [29,30]. Considering the canonical transformation ofvariations given by [6]:

$$(q_1,q_2,p_1,p_2)=(q'_1,q'_2,p'_1,p'_2)N \tag{23}$$

where N is defined as:

$$N=\begin{bmatrix} a_1 & a_1c_1 & -a_1c_1 & a_1\left(1-\omega_1^2 b_1\right) \\ a_2 & a_2c_2 & -a_2c_2 & a_2\left(1-\omega_2^2 b_2\right) \\ 0 & a_1 b_1 & a_1(1-b_1) & a_1c_1 \\ 0 & -a_2b_2 & -a_2(1-b_2) & -a_2c_2 \end{bmatrix} \tag{24}$$

where

$$a_1=\frac{1}{2}\left(\frac{2l_1}{\omega_1^2-1/2}\right)^{1/2} \quad a_2=\frac{1}{2}\left(\frac{2l_2}{\omega_2^2-1/2}\right)^{1/2};$$

$$l_1=1+\omega_1^2+2H_{02}; \quad l_2=1+\omega_2^2+2H_{02}$$

$$b_1=\frac{2}{l_1}; \quad b_2=\frac{2}{l_2}; \quad c_1=\frac{-H_{11}}{l_1}; c_2=\frac{-H_{11}}{l_2} \tag{25}$$

The Hamiltonian H defined in equation (7) is reduced to the following form by help of the transformation given by (23).

$$H=\frac{1}{2}\left(p_1'^2+\omega_1^2 q_1'^2\right)-\frac{1}{2}\left(p_2'^2+\omega_2^2 q_2'^2\right) +\sum_{\alpha+\gamma=3}^{\infty} h_{\alpha_1\alpha_2\gamma_1\gamma_2} q_1'^{\alpha_1} q_2'^{\alpha_2} p_1'^{\gamma_1} p_2'^{\gamma_2} \tag{26}$$

where

$\alpha=\alpha1+\alpha2$; $\gamma=\gamma1+\gamma2$

The Hamiltonian H3 and H4 can be expanded as follows,

$$H_3=H_{0003}p_2^3+H_{0030}p_1^3+H_{0300}q_2^3+H_{3000}q_1^3+ H_{2100}q_1^2q_2+H_{2010}q_1^2p_1+\dots\dots\dots+H_{1011}q_1p_1p_2$$

$$H_4=H_{0004}p_2^4+H_{0040}p_1^4+H_{0400}q_2^4 +H_{4000}q_1^4+\dots\dots\dots+H_{1111}q_1q_2p_1p_2 \tag{27}$$

The coefficients of third and fourth terms of of $h_{\alpha_1\alpha_2\gamma_1\gamma_2}$ and $h'_{\alpha_1\alpha_2\gamma_1\gamma_2}$ are given in the appendices (1).

For further investigation the Hamiltonian is reduced to a more convenient form, by the canonical transformation defined below as:

$$q'_1=\frac{1}{2}q''_1+\frac{i}{\omega_1}p''_1; \quad p'_1=\frac{1}{2}i\omega_1 q''_1+p''_1;$$

$$q'_2=-\frac{1}{2}q''_2+\frac{1}{\omega_2}p''_2; \quad p'_2=-\frac{1}{2}i\omega\ q''_2+ip''_2. \tag{28}$$

Thus, the Hamiltonian (26) is reduced as:

$$H=i\omega_1 q_1'' p_1''+i\omega_2 q_2'' p_2''+ \sum_{\alpha+\gamma=3}^{\infty} h'_{\alpha_1\alpha_2\gamma_1\gamma_2} q_1''^{\alpha_1} q_2''^{\alpha_2} p_1''^{\gamma_1} p_2''^{\gamma_2} \tag{29}$$

The other coefficients of third order terms are obtained by the formula and are given in appendices (2, 3).

$$h'_{\alpha_1\alpha_2\gamma_1\gamma_2}=(x_{\alpha_1\alpha_2\gamma_1\gamma_2}+iy_{\alpha_1\alpha_2\gamma_1\gamma_2}) \times\left(\frac{-\omega_1}{2}\right)^{\gamma_1-\alpha_1}\left(\frac{-\omega_2}{2}\right)^{\gamma_2-\alpha_2} \tag{30}$$

All the third order terms from the Hamiltonian (29) is nullified by using Birkhoff's transformation $(q_j'',p_j'')\to(q_j''',p_j''')$, provided third order resonance does not occur. This transformation is introduced by means of the generating function which is given as follows:

$$s=q_1''p_1'''+q_2''p_2'''+s_3+s_4; \tag{31}$$

Where

$$q_i'''=q_i''+\frac{\partial s_3}{\partial p_i'''}+\frac{\partial s_4}{\partial p_i'''};$$

$$p_i''=p_i'''+\frac{\partial s_3}{\partial q_i''}+\frac{\partial s_4}{\partial q_i''}; \quad (i=1,2) \tag{32}$$

Using equation (29) and (32); expanding and equating the terms of the same degree on the two sides, we obtain:

$$H'_2(q_1'',q_2'',p_1''',p_2''')=H_2(q_1'',q_2'',p_1''',p_2''');$$

$$H'_3(q_1'',q_2'',p_1''',p_2''')=\sum_{i=1}^{2}\left[-\frac{\partial s_3}{\partial p_i'''}\frac{\partial H'_2}{\partial p_i'''}+\frac{\partial s_3}{\partial q_i''}\frac{\partial H_2}{\partial p_i'''}\right] +H_3(q_1'',q_2'',p_1''',p_2''');$$

$$\sum_{i=1}^{2}\left[-\frac{\partial s_4}{\partial p_i'''}\frac{\partial H'_2}{\partial q_i'''}+\frac{\partial s_4}{\partial q_i''}\frac{\partial H_2}{\partial p_i''}\right]+K_4=0;$$

$$H'_4=\sum_{i=1}^{2}\left[-\frac{\partial s_3}{\partial p_i'''}\frac{\partial H'_3}{\partial q_i''}+\frac{\partial s_3}{\partial q_i''}\frac{\partial H_3}{\partial p_i'''}\right]+H_4-K_4. \tag{33}$$

Where K4 is the term other than the homogeneous ones in q1p1 and q2p2. By application of implicit function theorem, inEquation (33) the new variables $q_1'''$and p_2''' can be replaced by $q_1''$and p_2'' on both sides of equation (41). Considering autonomous system we have, $\frac{\partial s_3}{\partial t} = \frac{\partial s_4}{\partial t} = 0$.

If we put

$$H_3 = \sum_{\alpha+\gamma=3}^{\infty} h_{\alpha_1\alpha_2\gamma_1\gamma_2} q_1'''^{\alpha_1} q_2'''^{\alpha_2} p_1'''^{\gamma_1} p_2'''^{\gamma_2}$$

And

$$s_3 = \sum_{\alpha+\gamma=3}^{\infty} g_{\alpha_1\alpha_2\gamma_1\gamma_2} q_1'''^{\alpha_1} q_2'''^{\alpha_2} p_1'''^{\gamma_1} p_2'''^{\gamma_2} \tag{34}$$

Equation (33) yields:

$$g_{\alpha_1,\alpha_2\gamma_1,\gamma_2} = \frac{ih'_{\alpha_1,\alpha_2\gamma_1,\gamma_2}}{(\alpha_1-\gamma_1)\omega_1 + (\alpha_2-\gamma_2)\omega_2} \tag{35}$$

The new Hamiltonian inclusive of the fourth order terms can be found with help of equation (33) which is given as follows:

$$\begin{aligned} H' &= i\omega_1 q_1''' p_1''' + i\omega_2 q_2''' p_2''' - c_{20}(q_1''' p_1''')^2 \\ &\quad + c_{11}(q_1''' p_1''')(q_2''' p_2''') - c_{02}(q_2''' p_2''')^2 + \ldots \end{aligned} \tag{36}$$

Where

$$\begin{aligned} K_4 &= H_4(q_1''', q_2''', p_1''', p_2''') - h'_{2020} q_1'''^2 p_1'''^2 \\ &\quad - h'_{1111}(q_1''' p_1''')(q_2''' p_2''') - h'_{0202} q_2'''^2 p_2'''^2; \end{aligned}$$

And c20, c11, c02 are given in appendices.

Now we apply KAM- theorem which is stated as follows:

If the Hamiltonian of the perturbed motion is such that is satisfy the mentioned given conditions as:

1) If the characteristics equation of the system H2 has pure imaginary roots such that it holds the condition

$$n_1\omega_1 + n_2\omega_2 \neq 0; \text{ when } |n_1| + |n_2| \leq 4$$
$$(n_j \text{ are integers})$$

2) $$D = c_{20}\omega_2^2 + c_{11}\omega_1\omega_2 + c_{02}\omega_1^2 \neq 0$$

If both the conditions are satisfied then the equilibrium points are stable.

The value of D is calculated with the help of following formula:

$$D = c_{20}\omega_2^2 + c_{11}\omega_1\omega_2 + c_{02}\omega_1^2;$$

Fig 6-9 shows the values D for different values of radiation pressures and oblateness parameter. It is observed that D ≠0, for any values of oblateness parameter. Hence, it is found that the equilibrium points are stable.Fig 10 shows the values of D for different values of eccentricity, taking oblateness parameter as fixed and by varying radiation parameter. Again, it is found that D≠0. Hence, the equilibrium points are stable.

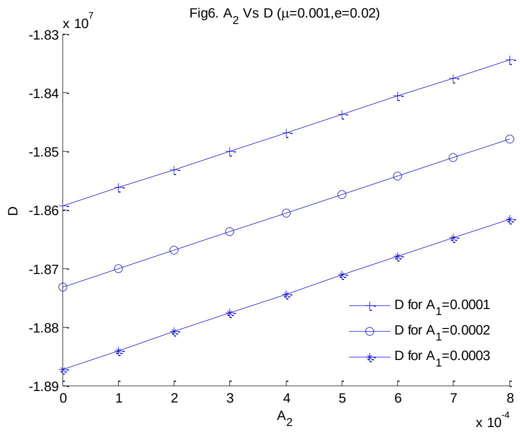

Fig. 6: A2 Vs D for M=0.001, E=0.02.

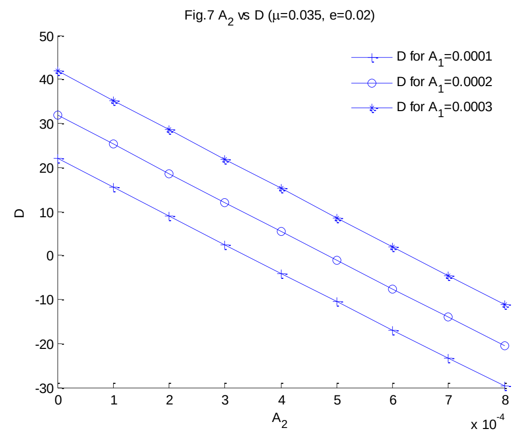

Fig. 7: A2 Vs D for M= 0.035, E=0.02.

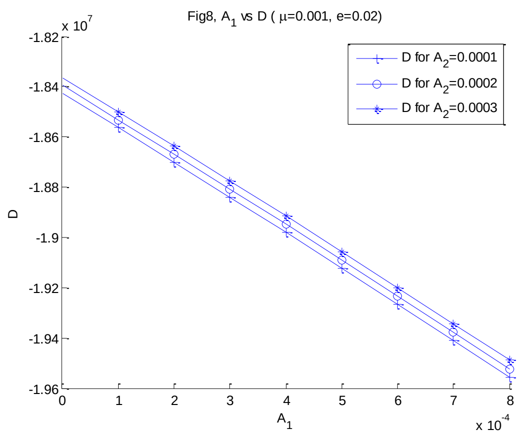

Fig. 8: A1 Vs D for M= 0.001, E=0.02.

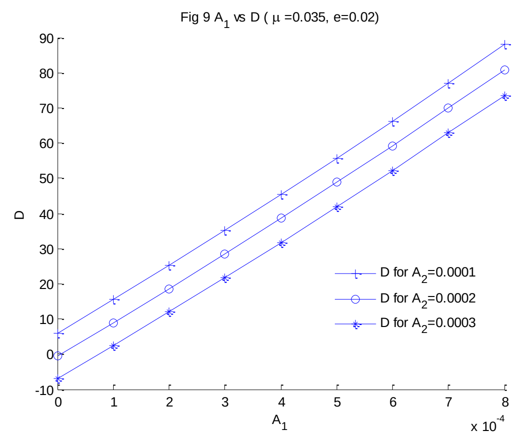

Fig. 9: A1 Vs D for M= 0.035, E=0.02.

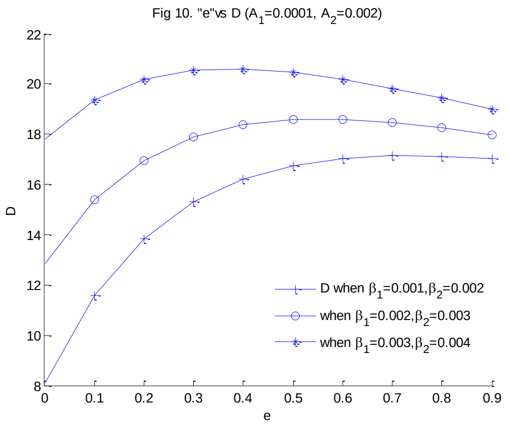

Fig. 10: "E" VsD for A1=0.0001, A2=0.002.

6. Stability in the resonance case

Case 1) ω1= 2ω2

In the resonance case ω1= 2ω2, the Birkhoffs transformation $(q"_j, p"_j) \rightarrow (Q_j, P_j)$ is used. It is not possible to cancel whole $\bar{H}_3$ of the Hamiltonian H .In this case $\bar{H}_3$ retains two resonant terms with coefficients h'_{1002} and h'_{0210}. Thus the Normalised form of the Hamiltonian is written as;

$$\bar{H} = i(\omega_1 Q_1 P_1 + \omega_2 Q_2 P_2) + h'_{1002}\, Q_1 P_2^2 + h'_{0210}\, Q_2^2 P_1 + \ldots.. \quad (37)$$

For further investigation following two canonical transformation is used as:

$$Q_1 = \frac{1}{(\omega_1)^{1/2}}(Q_1^0 - iP_1^0);$$

$$Q_2 = \frac{1}{(\omega_2)^{1/2}}(iQ_2^0 - P_2^0); \quad P_1 = \frac{(\omega_1)^{1/2}}{2}(-iQ_1^0 + P_1^0);$$

$$P_2 = \frac{(\omega_2)^{1/2}}{2}(iQ_2^0 - iP_2^0);$$

And

$$Q_1^0 = (2r_1)^{1/2}\sin(\varphi_1 - \theta_1) \quad (38)$$

$$P_1^0 = (2r_1)^{1/2}\cos(\varphi_1 - \theta_1); \quad Q_2^0 = (2r_2)^{1/2}\sin\varphi_2 \quad (39)$$

$$P_2^0 = (2r_1)^{1/2}\cos\varphi_2$$

where

$$\sin\theta_1 = \frac{y_{1002}}{(x_{1002}^2 + y_{1002}^2)^{1/2}}$$

$$\cos\theta_1 = \frac{x_{1002}}{(x_{1002}^2 + y_{1002}^2)^{1/2}} \quad (40)$$

The Hamiltonian (29) is reduced to the following polar form as:

$$\bar{H} = 2\omega_2 r_1 - \omega_2 r_2 - \sqrt{(x_{1002}^2 + y_{1002}^2)\omega_2 r_2} \times \sqrt{r_1}\sin(\varphi_1 + \varphi_2) + o((r_1 + r_2)^2) \quad (41)$$

The nonlinear stability of the equilibrium points is determined by Markeev [31]. Accordingly, if $D_1 = c_{20} + 2c_{11} + 4c_{02} \neq 0$ and $D_2 = (x_{1002}^2 + y_{1002}^2) = 0;$ hold simultaneously then the equilibrium points are stable otherwise unstable.

From Figs. 11-14, it is clear that for no values of the radiation pressure and oblateness parameter the expression $D_1 \neq 0$ and $D_2 = 0$ are true simultaneously. This clarify that the equilibrium points are unstable. Fig 15 depicts the value of $D_1$and D_2 by considering oblateness parameterasconstant and by varyingthe eccentricity. Hence, it followsthat the motion is unstable in theresonance caseω1= 2ω2.

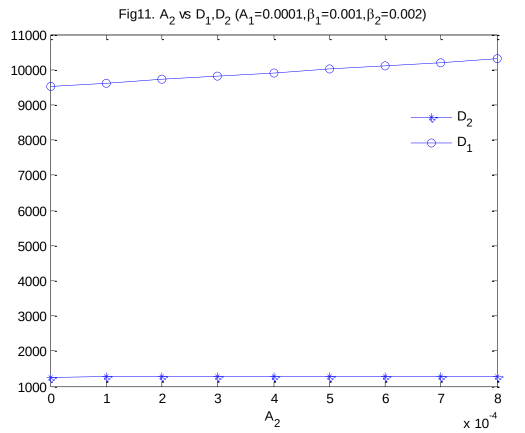

Fig. 11: A2 VsD1, D2for A1=0.0001, B1=0.001, B2 =0.002.

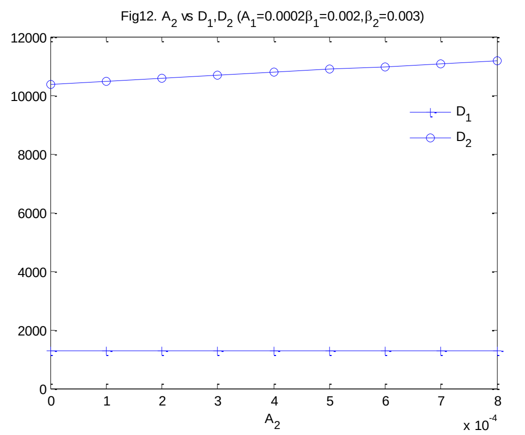

Fig. 12: A2 VsD1, D2for A1=0.0002, B1=0.002, B2 =0.003.

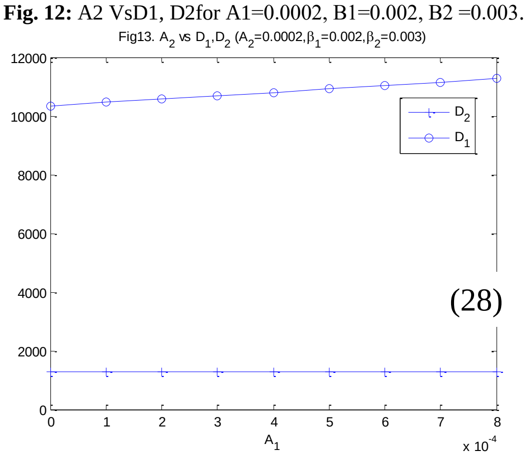

Fig. 13: A2 VsD1, D2 ForA1=0.0001, B1=0.001, B2 =0.002.

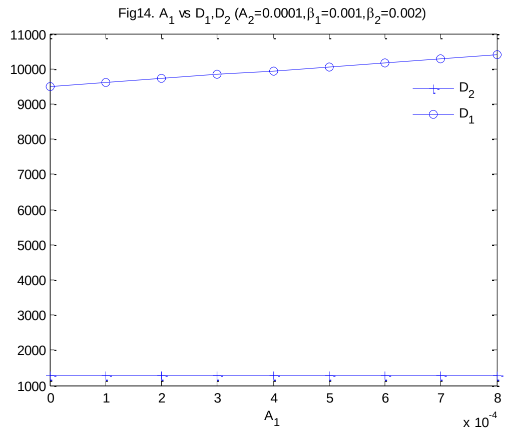

Fig. 14: A1 VsD1, D2 for A2=0.0001, B1=0.001, B2 =0.002

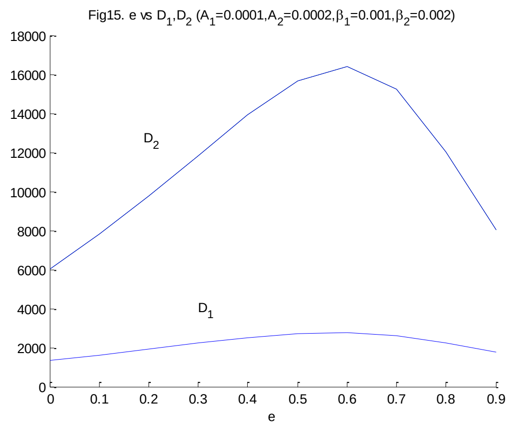

Fig. 15: "e" vs D1,D2 for A1=0.0001,A2=0.0002, β1=0.001, β2 =0.002.

Case (2) ω1= 3ω2.

In the resonance case ω1= 3ω2, it is possible to cancel whole of $\bar{H}_3$ of theHamiltonian $\bar{H}$ but $\bar{H}_4$ retains the resonance terms and the terms with the same degree of canonical variables. Thus the normalized form of the Hamiltonian is written as:

$$\bar{H} = i(\omega_1 Q_1 P_1 + \omega_2 Q_2 P_2) - c_{20} Q_1^2 P_1^2 + c_{11} Q_1 P_1 Q_1 P_2 - c_{02} Q_2^2 P_2^2 + h'_{1003} Q_1 P_2^3 + h'_{0310} Q_2^3 P_1 + \ldots.. \quad (42)$$

Again, applying the canonical transformation by means of polar coordinates given as:

$$Q_1^0 = (2r_i)^{1/2} \sin(\varphi_i - \theta_i)$$

$$P_1^0 = (2r_i)^{1/2} \cos(\varphi_i - \theta_i); \quad (i = 1, 2) \quad (43)$$

Where

$$\theta_2 = 0$$

$$\sin\theta_1 = \frac{x_{1003}}{(x_{1003}^2 + y_{1003}^2)^{1/2}}$$

$$\cos\theta_1 = \frac{-y_{1003}}{(x_{1003}^2 + y_{1003}^2)^{1/2}} \quad (44)$$

The polar form of the normalized Hamiltonian reduces to the following form:

$$\bar{H} = 3\omega_2 r_1 - \omega_2 r_2 + c_{20} r_1^2 + c_{11} r_1 r_2 + c_{02} r_2^2 + \frac{\omega_2}{3} \times \{3(\sqrt{(x_{1003}^2 + y_{1003}^2)})\}.r_2 (r_1 r_2)^{1/2} \sin(\varphi_1 + 3\varphi_2) + o((r_1 + r_2)^{5/2}) \quad (45)$$

For sake of simplicity we denote

$$a = c_{20} + 3c_{11} + 9c_{02};$$
$$d = 3\omega_2 (x_{1003}^2 + y_{1003}^2)^{1/2} \quad (46)$$

The stability problem can be resolved by application of the Markeev theorem.

Accordingly if;

1) $(x_{1003}^2 + y_{1003}^2) \neq 0$ and $d \geq |a|$ are simultaneously satisfied then the equilibrium points are unstable and if $d < |a|$ then it is stable.
2) When $(x_{1003}^2 + y_{1003}^2) = 0$ and $d = 0$ are satisfied simultaneously then also, it is stable.
3) If $(x_{1003}^2 + y_{1003}^2) = 0$ and $d \neq 0$ then the stability question is decided by the analysis of higher order terms in the normal form.

From figs 16, 17 and 19 it is clear that, $(x_{1003}^2 + y_{1003}^2) \neq 0$ and the condition $d < |a|$ are satisfied, which states that the equilibrium points are stable. From Fig. 18 it is observed that for some combination of oblateness parameters and radiation parameters either $d < |a|$ or $d \geq |a|$.This states that for some cases it is stable and for some it is notstable. Fig 20 depicts the values of d and $|a|$ by varying the eccentricity. Here, it is observed that $d < |a|$, hence equilibrium points are stable.

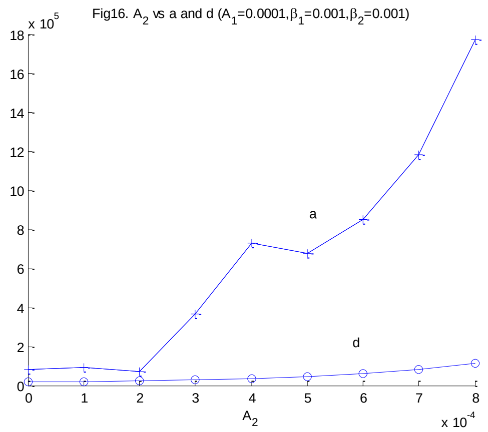

Fig. 16: A2 vs 'a' and'd' for A1=0.0001, β1=0.001, β2 =0.001.

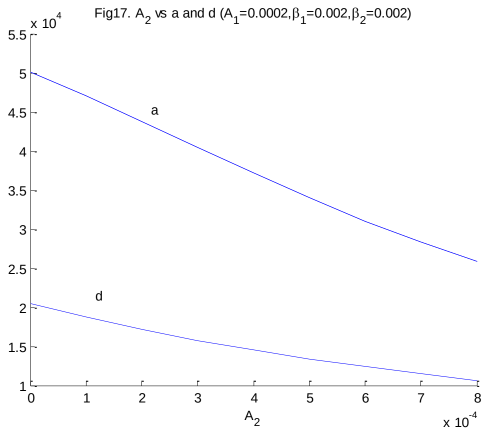

Fig. 17: A2 Vs ‘A’ and ‘D’ For A1=0.0002, B1=0.002, B2 =0.002.

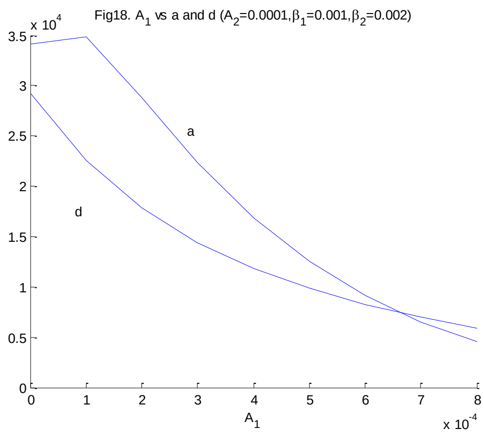

Fig. 18: A1 Vs ‘A’ and‘D’ For A2=0.0001, B1=0.001, B2 =0.002.

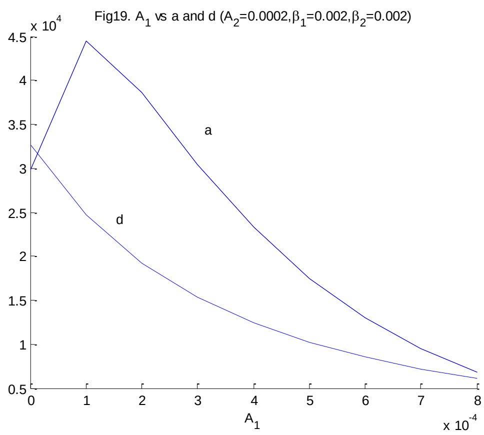

Fig. 19: A1 Vs ‘A’ and ‘D’ for A2=0.0001, B1=0.002, B2 =0.002.

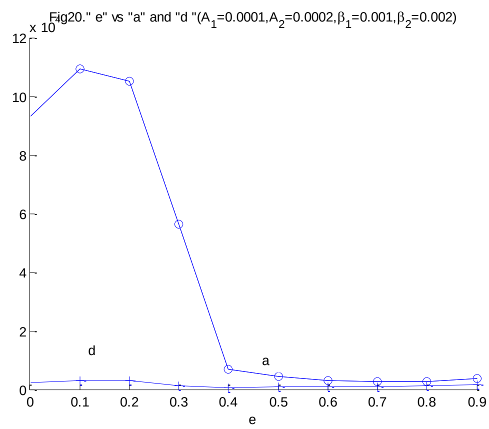

Fig. 20: “E" Vs "A" And "D” For A1=0.0001, A2=0.0002, B1 =0.001, B2=0.002

7. Discussion and conclusion

The stability of the triangular equilibrium points in ER3BP is investigated considering both the primaries are radiating and oblate; under both non resonance and resonance case. Recently [32] studied nonlinear stability of the triangular points considering both the primaries as radiating in ER3BP in non-resonance case .It was found that the except for some values of radiation pressure the triangular points are stable. The resonance condition was also studied by [33] considering only radiating primaries and observed that the motion is stable for third order resonance but unstable for fourth order resonance. [34] Studied the nonlinear stability in CR3BP taking radiating and oblate primaries in resonance case. It was observed that in resonance cases ω1= 2ω2 the motion is unstable while for ω1= 3ω2, motion is stable for all values of radiation pressures taken. [35] studied the nonlinear stability in CR3BP in non-resonance case considering both the primaries as oblate luminous spheroid and found that except few cases considered, the triangular points are stable .It was observed that in general the stability character remains same even if oblateness factor is considered apart from radiation factor in circular cases. .

The following observation has been made regarding the nonlinear stability in both resonance as well as non-resonance case considering the luminous oblate spheroid in ER3BP.

It is found that $\omega_1 = \omega_2$ corresponds to the boundary region of the stability for the system, whereas the other two cases $\omega_1 = 2\omega_2$, $\omega_1 = 3\omega_2$ correspond to the resonant cases.

It is noticed that ω1 increases with increasing μ whereas ω2 decreases, and becomes equal to the critical value (i.e. μ = μc).

In non-resonance case, there in no cases where D=0. Hence it is found that theequilibrium points are stable.

In resonance case ω1= 2ω2, it is clear from Fig 11-15 that in no case $D_1 \neq 0$ and $D_2 = 0$ holds simultaneously true. Hence it followsthat the motion is unstable.

In resonance case ω1= 3ω2, for some cases it is stable and for some it is unstable.

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Legends:

1) Correlation between ω_1 and ω_2 for A_1=0.0001, A_2=0.0002, β_1=0.001, β_2=0.002.
2) Correlation between ω_1 and ω_2 for A_1=0.0001, A_2=0.0003, β_1=0.001, β_2=0.002.
3) Correlation between ω_1 and ω_2 for A_1=0.0002, A_2=0.0001, β_1=0.001, β_2=0.002.
4) Correlation between ω_1 and ω_2 for A_1=0.0001, A_2=0.0001, β_1=0.002, β_2=0.003.
5) Correlation between ω_1 and ω_2 for A_1=0.0005, A_2=0.0005, β_1=0.004, β_2=0.005.
6) A_2Vs D for μ=0.001, e=0.02.
7) A_2Vs D for μ= 0.035, e=0.02.
8) A_1Vs D for μ= 0.001, e=0.02.
9) A_1Vs D for μ= 0.035, e=0.02.
10) "e" vs D for A_1=0.0001, A_2=0.002.
11) A_2vs D_1, D2 for A_1=0.0001, β_1=0.001, β_2 =0.002.
12) A_2vs D_1, D2 for A_1=0.0002, β_1=0.002, β_2 =0.003.
13) A_2vs D_1, D2 for A_1=0.0001, β_1=0.001, β_2 =0.002.
14) A_1vs D_1, D2 for A_2=0.0001, β_1=0.001, β_2 =0.002.
15) evs D_1,D_2 for A_1=0.0001,A_2=0.0002, β_1=0.001, β_2 =0.002.
16) A_2vs 'a' and'd' forA_1=0.0001, β_1=0.001, β_2 =0.001.
17) A_2vs 'a' and'd' forA_1=0.0002, β_1=0.002, β_2 =0.002.
18) A_1vs 'a' and'd' forA_2=0.0001, β_1=0.001, β_2 =0.002.
19) A_1vs 'a' and'd' forA_2=0.0001, β_1=0.002, β_2 =0.002.
20) "e" vs "a" and "d" for (A_1=0.0001,A_2=0.0002, β_1 =0.001 ,β_2=0.002.