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New wave solutions of the Yang-Mills equations with axially symmetric sources

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Abstract

A class of nontrivial axially symmetric wave solutions to the Yang-Mills equations with $SU(2)$ symmetry is studied. It describes transversal non-Abelian waves propagated at constant phase velocities through axially symmetric sources of Yang-Mills fields along their axes. In the considered case, the Yang-Mills equations are reduced to a system of six nonlinear partial differential equations. These equations are studied for a special class of axially symmetric sources satisfying a differential equation of charge conservation. From them partial differential equations of the first order for only the Yang-Mills field strengths are derived. To investigate these equations, a special method is proposed. As a result, their exact solutions are found. Using them, exact formulas for the field strengths and potentials in the non-Abelian waves under examination are obtained.

Keywords: *Electroweak interactions, field potentials and strengths, non-Abelian waves, SU(2) symmetry, Yang-Mills equations.*

1 Introduction

The Yang-Mills field theory is one of the greatest achievements of the XX century, which plays a leading role in modern quantum physics [1–3]. At the same time, the whole area of its applications can concern not only quantum physics but also classical physics [4–6]. To explain this point of view, let us consider powerful fields caused by very big charges and currents. In them not only photons but also massive Z^0 and $W^{\pm}$ bosons could be generated, which are carriers of weak interactions. In such cases, the Maxwell equations may be incorrect since they are applicable to fields for which only photons are their carriers. On the other hand, there are Yang-Mills equations with $SU(2)$ symmetry which are a nonlinear generalization of the linear Maxwell equations and play an important role in various models of electroweak interactions caused by photons and Z^0 and $W^{\pm}$ bosons.
For this reason, the classical Yang-Mills equations with $SU(2)$ symmetry can be also applicable to the case of powerful field sources with very big charges and currents, when Z^0 and $W^{\pm}$ bosons may be generated, along with photons. These equations can be represented in the form [1–3]

$$D_\mu F^{k,\mu\nu} \equiv \partial_\mu F^{k,\mu\nu} + g\varepsilon_{klm} F^{l,\mu\nu} A^m_\mu = (4\pi/c) J^{k,\nu}, \tag{1}$$

$$F^{k,\mu\nu} = \partial^\mu A^{k,\nu} - \partial^\nu A^{k,\mu} - g\varepsilon_{klm} A^{l,\mu} A^{m,\nu}, \tag{2}$$

where $A^{k,\nu}$ and $F^{k,\mu\nu}$ are potentials and strengths of a Yang-Mills field, respectively, $k,l,m = 1,2,3;\ \mu,\nu = 0,1,2,3$, and the summation over repeated indices is implied, $J^{k,\nu}$ are three 4-vectors of source current densities, ε_{klm} is the antisymmetric tensor, $\varepsilon_{123} = 1$, g is the constant of electroweak interactions, D_μ is the Yang-Mills covariant derivative, $\partial_\mu \equiv \partial/\partial x^\mu$, x^μ are space-time coordinates of the Minkowski geometry, $x^0 = ct,\ x^1 = x,\ x^2 = y,\ x^3 = z$, t is time, and x, y, z are spatial rectangular coordinates.
As is well known, the Yang-Mills field strengths $F^{k,\mu\nu}$ satisfy the following identity [1–3]:

$$D_\nu D_\mu F^{k,\mu\nu} \equiv 0 \tag{3}$$

and hence, the Yang-Mills equations (1)–(2) have the consequence

$$D_\nu J^{k,\nu} \equiv \partial_\nu J^{k,\nu} + g\varepsilon_{klm} J^{l,\nu} A_\nu^m = 0\,. \tag{4}$$

From here on, we will consider the field sources $J^{k,\nu}$ of the form

$$J^{1,\nu} = J^\nu, \quad J^{2,\nu} = J^{3,\nu} = 0\,, \tag{5}$$

where J^ν is a classical 4-vector of current densities.
Then from (4) with k=1 and (5), we come to the differential equation of charge conservation

$$\partial_\nu J^\nu = 0\,. \tag{6}$$

It should be noted that the Yang-Mills equations with $SU(2)$ symmetry are covariant under rotations of the vectors $J^{k,\nu}$ in the three-dimensional gauge space [1–3]. That is why the considered case of field sources of form (5) can be extended to sources of the form

$$J^{k,\nu} = \Theta^k V^\nu\,, \tag{7}$$

where Θ^k and V^ν are arbitrary functions.
Indeed, field sources of form (7) can be reduced by gauge rotations to sources of form (5), where we should put $J^\nu = \Theta V^\nu$ and $(\Theta)^2 = (\Theta^1)^2 + (\Theta^2)^2 + (\Theta^3)^2$. Therefore, sources of form (7) satisfy the differential equation (6) of charge conservation, where $J^\nu = \Theta V^\nu$.

It is evident that the Yang-Mills equations (1)–(2) with field sources of form (5) have the following particular Abelian solutions:

$$\begin{gathered} A^{2,\nu} = A^{3,\nu} = 0, \quad F^{2,\mu\nu} = F^{3,\mu\nu} = 0, \quad J^{2,\nu} = J^{3,\nu} = 0, \\ \partial_\mu F^{1,\mu\nu} = (4\pi / c) J^{1,\nu}, \quad F^{1,\mu\nu} = \partial^\mu A^{1,\nu} - \partial^\nu A^{1,\mu}, \end{gathered} \tag{8}$$

where the field potentials $A^{1,\nu}$ and strengths $F^{1,\mu\nu}$ satisfy the Maxwell equations.
At the same time, there exist classes of non-Abelian solutions to the Yang-Mills equations (1)–(2) with the field sources of form (5) and a number of them were found in Refs. [4–6].
Besides, the Lagrangian and energy-momentum tensor of Yang-Mills fields are related to their strengths by the same formulas as those of Maxwell fields. That is why the Yang-Mills equations (1)–(2) with the field sources of form (5) can be regarded as a reasonable nonlinear generalization of the linear Maxwell equations.

One of the important problems is a search for non-Abelian wave solutions to the Yang-Mills equations. It should be noted that in Refs. [7–15] non-Abelian plane waves and their generalizations were studied and in Refs. [16, 17] a class of expanding non-Abelian waves propagated outside their sources was investigated.
In the present paper, we investigate non-Abelian waves propagated through axially symmetric sources of Yang-Mills fields. Consider the Yang-Mills equations (1)–(2) in the case of axially symmetric sources $J^{k,\nu}$ of the following form conforming to (5):

$$\begin{gathered} J^{1,0} = j^0(\eta,\rho), \quad J^{1,1} = xj^1(\eta,\rho), \quad J^{1,2} = yj^1(\eta,\rho), \quad J^{1,3} = j^3(\eta,\rho), \\ J^{2,\nu} = J^{3,\nu} = 0, \\ \eta = \omega x^0 - z, \quad \rho = \sqrt{x^2 + y^2}, \quad \omega = \text{const}, \end{gathered} \tag{9}$$

where j^ν are some functions of the wave phase $\omega x^0 - z$ and the radial coordinate ρ.
These expressions describe waves propagated at the phase velocity ωc through Yang-Mills field sources in the direction of their axis of symmetry z.

Let us seek axially symmetric wave solutions to the Yang-Mills equations (1)–(2) with the considered field sources, given by expressions (9), in the form

$$A^{k,0} = u^k(\eta,\rho), \quad A^{k,1} = xv^k(\eta,\rho), \quad A^{k,2} = yv^k(\eta,\rho), \quad A^{k,3} = w^k(\eta,\rho)\,. \tag{10}$$

Substituting expressions (10) into formula (2) for the field strengths $F^{k,\mu\nu}$, we find

$$\begin{gathered} F^{k,01} = xq^k(\eta,\rho), \quad F^{k,02} = yq^k(\eta,\rho), \quad F^{k,03} = s^k(\eta,\rho), \\ q^k = \omega v^k_\eta + (1/\rho)u^k_\rho - g\varepsilon_{klm}u^l v^m, \quad s^k = \omega w^k_\eta - u^k_\eta - g\varepsilon_{klm}u^l w^m, \end{gathered} \tag{11}$$

where $u^k_\eta \equiv \partial u^k / \partial\eta, \;\; u^k_\rho \equiv \partial u^k / \partial\rho$, and also obtain

$$\begin{gathered} F^{k,12} = 0, \quad F^{k,13} = xh^k(\eta,\rho), \quad F^{k,23} = yh^k(\eta,\rho), \\ h^k = -(1/\rho)w^k_\rho - v^k_\eta - g\varepsilon_{klm}v^l w^m. \end{gathered} \tag{12}$$

Let us now substitute expressions (9)–(12) for the field sources, potentials, and strengths under examination into the Yang-Mills equations (1). Then we come to the following system of equations:

$$\rho q^k_\rho + 2q^k - s^k_\eta - g\varepsilon_{klm}(\rho^2 q^l v^m + s^l w^m) = -(4\pi/c)\,j^0\delta_k\,, \tag{13}$$

$$\omega q^k_\eta + h^k_\eta + g\varepsilon_{klm}(q^l u^m + h^l w^m) = (4\pi/c)\,j^1\delta_k\,, \tag{14}$$

$$\omega s^k_\eta + \rho h^k_\rho + 2h^k + g\varepsilon_{klm}(s^l u^m - \rho^2 h^l v^m) = (4\pi/c)\,j^3\delta_k\,, \tag{15}$$

where

$$\delta_1 = 1, \quad \delta_2 = \delta_3 = 0\,. \tag{16}$$

In section 2, we will study the obtained system of equations (13)–(16) in the case of transversal non-Abelian waves and reduce them to a system of six nonlinear partial differential equations. In section 3, we will examine the six differential equations and find exact formulas for the field strengths in the considered transversal non-Abelian waves. In section 4, we will obtain exact expressions for the field potentials in such non-Abelian waves. In section 5, we will examine a subclass of the obtained wave solutions satisfying an additional differential condition which expresses the conservation of the intrinsic energy of a field source when Yang-Mills field quanta are created inside it.

2 Investigation of transversal non-Abelian waves

Consider the case of transversal non-Abelian waves in which the vector of energy flow is parallel to the direction *z* of their propagation. Then we should put

$$F^{k,03} = s^k = 0\,. \tag{17}$$

From (11) and (17), we have

$$(\omega w^k - u^k)_\eta - g\varepsilon_{klm}u^l w^m = 0\,. \tag{18}$$

Multiplying (18) by $\omega w^k - u^k$ and summing the product over *k*, we find, taking into account the antisymmetry of ε_{klm},

$$\frac{d}{d\eta}\sum_{k=1}^{3}(\omega w^k - u^k)^2 = 0. \tag{19}$$

Let us assume that the considered waves are absent and have zero potentials when $\eta \le 0$. Then, as follows from (19), we have

$$u^k = \omega w^k. \tag{20}$$

From (11), (12), and (20), we obtain

$$q^k = -\omega h^k. \tag{21}$$

Substituting equalities (17), (20), and (21) into equations (13)–(15), we come to the following system of equations:

$$j^0 = \omega j^3, \tag{22}$$

$$\rho h^k_\rho + 2h^k - g\rho^2 \varepsilon_{klm} h^l v^m = (4\pi / c) j^3 \delta_k, \tag{23}$$

$$(1-\omega^2)(h^k_\eta + g\varepsilon_{klm} h^l w^m) = (4\pi / c) j^1 \delta_k. \tag{24}$$

It should be noted that the following sources j^0, j^3 can satisfy the equality (22) which are the sum of the sources j^0_1, j^3_1 and j^0_2, j^3_2 with different signs of the charges of their carriers:

$$\begin{gathered} j^0 = j^0_1 + j^0_2, \quad j^3 = j^3_1 + j^3_2, \\ j^0_1 = \alpha(\eta,\rho)[1+\omega\beta_1(\eta,\rho)], \quad j^0_2 = -\alpha(\eta,\rho)[1-\omega\beta_2(\eta,\rho)], \\ j^3_1 = j^0_1\beta_2, \quad j^3_2 = -j^0_2\beta_1, \end{gathered} \tag{25}$$

where $\alpha(\eta,\rho))$, $\beta_1(\eta,\rho)$, and $\beta_2(\eta,\rho)$ are arbitrary functions.

First, consider the particular case $\omega = \pm 1$. Then from (20)–(24) and (12), we derive

$$u^k = \pm w^k, \quad q^k = \mu h^k, \quad j^0 = \pm j^3, \quad j^1 = 0, \quad \omega = \pm 1, \tag{26}$$

$$\rho h^k_\rho + 2h^k - g\rho^2 \varepsilon_{klm} h^l v^m = (4\pi / c) j^3 \delta_k, \tag{27}$$

$$h^k = -[(1/\rho) w^k_\rho + v^k_\eta + g\varepsilon_{klm} v^l w^m]. \tag{28}$$

Let us put

$$h = \left(\sum_{k=1}^{3}(h^k)^2\right)^{1/2}. \tag{29}$$

Then multiplying equation (27) by h^k and summing the product over *k*, we obtain, using (16) and the antisymmetry of ε_{klm},

$$\rho h_\rho + 2h = (4\pi / c) j^3 h^1 / h. \tag{30}$$

Besides (30), equation (27) gives the relation of the functions v^k to the functions h^k and j^3. As to equation (28), it relates the functions w^k to the functions v^k and h^k.

Taking into account (29), we can put

$$h^1 = h\cos\varphi, \quad h^2 = h\sin\varphi\cos\psi, \quad h^3 = h\sin\varphi\sin\psi\,, \tag{31}$$

where φ and ψ are some differentiable functions.
Then from equation (30) we find that the function h with no singularity at $\rho = 0$ has the form

$$h = \frac{4\pi}{c\rho^2}\int_0^\rho \rho j^3 \cos\varphi\, d\rho\,. \tag{32}$$

Thus, the sought functions h^k can be found by formulas (31) and (32) in the particular case $\omega = \pm 1$. In these formulas φ and ψ are arbitrary differentiable functions of the arguments $\eta = \pm x^0 - z$ and ρ .

Examine now the system of equations (23) and (24), describing the transversal non-Abelian waves under consideration, for arbitrary values of the parameter ω .
Let us use equalities (4)–(6). From them we readily find

$$\partial_\nu J^{1,\nu} = 0, \quad J^{1,\nu} A_\nu^2 = 0, \quad J^{1,\nu} A_\nu^3 = 0\,. \tag{33}$$

Taking into account (9) and (10), from (33) we obtain

$$\omega j_\eta^0 + \rho j_\rho^1 + 2j^1 - j_\eta^3 = 0\,, \tag{34}$$

$$j^0 u^k - \rho^2 j^1 v^k - j^3 w^k = 0, \quad k = 2, 3\,. \tag{35}$$

Owing to (5), we can choose a gauge rotation about the first axis in the gauge space so as to also fulfill equality (35) when $k = 1$.
Then because of (20) and (22), equation (35) acquires the form

$$\rho^2 j^1 v^k + (1-\omega^2)\, j^3 w^k = 0, \quad k = 1, 2, 3\,. \tag{36}$$

Using equality (22) again, we can represent equation (34) as

$$\rho j_\rho^1 + 2j^1 = (1-\omega^2)\, j_\eta^3\,. \tag{37}$$

Let us denote

$$f^k = \rho^2 h^k, \quad a = \rho^2 j^1/(1-\omega^2), \quad b = \rho j^3\,. \tag{38}$$

Using (38), we can rewrite equalities (36) and (37) as

$$a\rho v^k = -b w^k\,, \tag{39}$$

$$a_\rho = b_\eta\,. \tag{40}$$

Let us turn to equations (23) and (24). Using (38) again, we can represent them in the form

$$f_\rho^k - g\rho\varepsilon_{klm} f^l v^m = (4\pi/c) b\delta_k\,, \tag{41}$$

$$f_\eta^k + g\varepsilon_{klm} f^l w^m = (4\pi/c) a\delta_k\,. \tag{42}$$

3 Field strengths in transversal non-Abelian waves

Consider equations (39)–(42). From equation (39), we have

$$v^k = b\sigma^k, \quad w^k = -\rho a\sigma^k, \tag{43}$$

where $\sigma^k = \sigma^k(\eta, \rho)$ are some functions.
Substituting (43) into equations (41) and (42), we obtain

$$f_\rho^k = b\lambda_k, \quad f_\eta^k = a\lambda_k, \tag{44}$$

where

$$\lambda_k = (4\pi / c)\delta_k + g\rho\varepsilon_{klm} f^l \sigma^m . \tag{45}$$

Consider now the expression for the functions h^k in formulas (12). As follows from (38), (43), and the antisymmetry of ε_{klm}, it acquires the form

$$f^k = \rho[(\rho a\sigma^k)_\rho - \rho(b\sigma^k)_\eta]. \tag{46}$$

Let us turn to solving equations (44). From them we easily obtain

$$af_\rho^k - bf_\eta^k = 0. \tag{47}$$

It should be noted that we can find exact solutions to the partial differential equations (47). They are given by the following proposition.

Proposition 3.1: *The partial differential equations (47), where the functions* a and b *are defined in (38) and satisfy equation (40), have the following exact solutions:*

$$f^k = \Phi^k(\xi), \quad \xi = \int_0^\rho b\, d\rho, \quad b = \rho j^3(\eta, \rho). \tag{48}$$

Here $\Phi^k(\xi)$ *are arbitrary differentiable functions which, as follows from (38), satisfy the condition*

$$\Phi^k(0) = 0. \tag{49}$$

Proof. Indeed, from (48) we have, taking into account (40),

$$\xi_\rho = b, \quad \xi_\eta = \int_0^\rho b_\eta d\rho = \int_0^\rho a_\rho d\rho = a, \tag{50}$$

$$f^k = \Phi^k(\xi), \quad f_\rho^k = (d\Phi^k/d\xi)\, b, \quad f_\eta^k = (d\Phi^k / d\xi)\, a, \tag{51}$$

where it is used that because of (38), $a = 0$ when $\rho = 0$.
From formulas (51) we easily come to equalities (47), which is what we set out to prove.

Because of the arbitrariness of the functions Φ^k, the obtained formulas (48) give the general solutions to equations (47), since they are partial differential equations of the first order.
Let us now substitute expressions (48) for the functions f^k and formulas (51) for their derivatives into equations (44)–(45). Then these equations acquire the form

$$d\Phi^k / d\xi - g\rho\varepsilon_{klm}\Phi^l\sigma^m = (4\pi / c)\delta_k, \quad \delta_1 = 1, \ \ \delta_2 = \delta_3 = 0\,, \tag{52}$$

where we have taken (16) into account.
Multiplying equations (52) by Φ^k and then summing them over *k*, we obtain, using the antisymmetry of ε_{klm},

$$\sum_{k=1}^{3}\Phi^k d\Phi^k / d\xi = (4\pi / c)\Phi^1\,. \tag{53}$$

Besides (53), from equations (52) we derive

$$\sigma^2 = \frac{1}{g\rho\Phi^1(\xi)}\frac{d\Phi^3}{d\xi} + \frac{\Phi^2(\xi)}{\Phi^1(\xi)}\sigma^1, \quad \sigma^3 = -\frac{1}{g\rho\Phi^1(\xi)}\frac{d\Phi^2}{d\xi} + \frac{\Phi^3(\xi)}{\Phi^1(\xi)}\sigma^1. \tag{54}$$

The obtained relations (11), (12), (17), (21), (38), (48), and (49) give the following formulas for the field strengths in the considered transversal non-Abelian waves:

$$F^{k,01} = -(\omega x / \rho^2)\Phi^k(\xi), \ \ F^{k,02} = -(\omega y / \rho^2)\Phi^k(\xi), \ \ F^{k,03} = 0, \tag{55}$$

$$F^{k,12} = 0, \ \ F^{k,13} = (x / \rho^2)\Phi^k(\xi), \ \ F^{k,23} = (y / \rho^2)\Phi^k(\xi), \tag{56}$$

$$\Phi^k(0) = 0, \quad \xi = \int_0^{\rho}\rho j^3 d\rho, \quad j^3 = j^3(\omega x^0 - z, \rho)\,, \tag{57}$$

where $\Phi^k(\xi)$ are arbitrary differentiable functions satisfying relation (53) and equal to zero when $\xi = 0$.
Let us put

$$\begin{gathered}\Phi^1 = \Phi\cos\gamma, \ \ \Phi^2 = \Phi\sin\gamma\cos\chi, \ \ \Phi^3 = \Phi\sin\gamma\sin\chi, \\ \Phi = \Phi(\xi), \quad \gamma = \gamma(\xi), \quad \chi = \chi(\xi), \quad \Phi(0) = 0, \quad (\Phi)^2 = \sum_{k=1}^{3}(\Phi^k)^2.\end{gathered} \tag{58}$$

Then from equation (53) we find

$$d\Phi / d\xi = (4\pi / c)\cos\gamma(\xi), \quad \Phi = (4\pi / c)\int_0^{\xi}\cos\gamma(\xi)d\xi\,. \tag{59}$$

Formulas (55)–(59) describe the Yang-Mills field strengths $F^{k,\mu\nu}$ for the considered class of transversal non-Abelian waves propagated at the constant phase velocity ωc through field sources in the direction of their axis of symmetry z. This class is determined by two arbitrary differentiable functions $\gamma(\xi)$ and $\chi(\xi)$, where ξ is defined by the second formula in (57) and depends on the wave phase $\omega x^0 - z$ and coordinate ρ.

4 Field potentials in transversal non-Abelian waves

Let us turn to equations (46). From them and the formula in (48) for the functions f^k, we obtain

$$(\rho a\sigma^k)_\rho - \rho(b\sigma^k)_\eta = \Phi^k(\xi) / \rho\,. \tag{60}$$

Using equality (40), from (60) we easily derive

$$a(\rho\sigma^k)_\rho - b(\rho\sigma^k)_\eta = \Phi^k(\xi)/\rho. \tag{61}$$

Consider expressions (54) and represent them in the form

$$\sigma^k = M^k(\xi)/\rho + N^k(\xi)\sigma^1, \quad k = 2, 3, \tag{62}$$

where

$$M^2(\xi) = \frac{1}{g\Phi^1(\xi)}\frac{d\Phi^3}{d\xi}, \quad M^3(\xi) = -\frac{1}{g\Phi^1(\xi)}\frac{d\Phi^2}{d\xi}, \quad N^k(\xi) = \frac{\Phi^k(\xi)}{\Phi^1(\xi)}. \tag{63}$$

Let us now prove the following property of equations (61).

Proposition 4.1: *Equations (61) with $k = 2, 3$ are consequences of equation (61) with $k = 1$ and equalities (50) and (62)–(63).*

Proof. Consider the derivatives of the functions $\rho\sigma^2$ and $\rho\sigma^3$. From (62) we derive, using equalities (50),

$$\begin{aligned} &(\rho\sigma^k)_\rho = (dM^k/d\xi)\,b + (dN^k/d\xi)\,b\rho\sigma^1 + N^k(\rho\sigma^1)_\rho, \\ &(\rho\sigma^k)_\eta = (dM^k/d\xi)\,a + (dN^k/d\xi)\,a\rho\sigma^1 + N^k\rho\sigma^1_\eta, \quad k = 2, 3. \end{aligned} \tag{64}$$

These equalities give

$$a(\rho\sigma^k)_\rho - b(\rho\sigma^k)_\eta = N^k[a(\rho\sigma^1)_\rho - b(\rho\sigma^1)_\eta], \quad k = 2, 3. \tag{65}$$

From equation (61) with $k = 1$ and equalities (65), we find

$$a(\rho\sigma^k)_\rho - b(\rho\sigma^k)_\eta = N^k\Phi^1/\rho, \quad k = 2, 3. \tag{66}$$

Taking into account the formula for N^k in (63), from (66) we obtain equalities (61) with $k = 2, 3$, which are exactly what we set out to prove.

Consider now equation (61) with $k = 1$ which has the form

$$a(\rho\sigma^1)_\rho - b(\rho\sigma^1)_\eta = \Phi^1(\xi)/\rho. \tag{67}$$

First, let us note that equation (67) becomes trivial in a region where $j^3 = 0$, since in it we have $b = 0$ and $\xi = \text{const}$, as follows from (48).
To solve equation (67) when $j^3 \neq 0$, let us introduce the inverse function

$$\rho = \rho(\eta, \xi) \tag{68}$$

for the function $\xi = \xi(\eta, \rho)$ of form (48):

$$\xi = \int_0^\rho b\,d\rho = \int_0^\rho \rho j^3(\eta, \rho)d\rho \tag{69}$$

in an arbitrary domain where $j^3 > 0$ or $j^3 < 0$.
Then we can put

$$\rho\sigma^1 = P(\eta,\xi), \quad (\rho b)^{-1} = Q(\eta,\xi), \tag{70}$$

where $P(\eta,\xi)$ is an unknown function of the arguments η and ξ and the function $Q(\eta,\xi)$ can be determined from the expressions for ξ and b in (48).
From formulas (50) and the first equality in (70), we find

$$(\rho\sigma^1)_\rho = bP_\xi, \quad (\rho\sigma^1)_\eta = P_\eta + aP_\xi. \tag{71}$$

Therefore,

$$a(\rho\sigma^1)_\rho - b(\rho\sigma^1)_\eta = -bP_\eta. \tag{72}$$

Using this equality, we can represent equation (67) in the form

$$bP_\eta = -\Phi^1(\xi)/\rho. \tag{73}$$

Equation (73) and formulas (70) give

$$P_\eta = -Q(\eta,\xi)\Phi^1(\xi), \quad \sigma^1 = P/\rho = -(1/\rho)\Big(\Phi^1(\xi)\int Q(\eta,\xi)d\eta + P_0(\xi)\Big), \tag{74}$$

where $P_0(\xi)$ is an arbitrary function.
From formulas (62), (63), and (74), we find expressions for the functions σ^k. Using these expressions and formulas (10), (20), (38), (43), and (48), we can determine the Yang-Mills field potentials $A^{k,\nu}$ for the considered class of transversal non-Abelian waves.

5 Transversal wave solutions of the Yang-Mills equations with an additional condition

As is seen from formulas (55)–(59), the obtained Yang-Mills field strengths generated by given field sources of form (9) are not determined uniquely. This can be explained by the fact that the Yang-Mills equations (1)–(2) with field sources of form (5), and in particular of form (9), are not independent. Namely, as follows from formulas (3) and (5)–(6), they are related by the differential identity

$$D_\nu(D_\mu F^{k,\mu\nu} - (4\pi/c)J^{k,\nu}) \equiv 0 \quad \text{when } k=1. \tag{75}$$

That is why we should find an additional equation to uniquely determine the field strengths $F^{k,\mu\nu}$. For this purpose let us introduce the following expressions:

$$\breve{J}^{k,\nu} = J^{k,\nu} - (cg/4\pi)\varepsilon_{klm}F^{l,\mu\nu}A^m_\mu. \tag{76}$$

Then the Yang-Mills equations (1) can be represented as

$$\partial_\mu F^{k,\mu\nu} = (4\pi/c)\breve{J}^{k,\nu}. \tag{77}$$

Since $F^{k,\mu\nu} = -F^{k,\nu\mu}$ and hence $\partial_\nu\partial_\mu F^{k,\mu\nu} = 0$, from (77) we obtain the following differential equations of conservation for the expressions $\breve{J}^{k,\nu}$:

$$\partial_\nu \breve{J}^{k,\nu} = 0. \tag{78}$$

As follows from (76)–(78), the expressions $\tilde{J}^{k,\nu}$ can be interpreted as full field sources which include not only the classical field sources $J^{k,\nu}$ but also Yang-Mills field quanta.

Consider $\tilde{J}^{k,\nu}\tilde{J}_{k,\nu}$. This value should be proportional to the density of the full energy of interactions of the charged particles in a field source and for it the following additional equation was suggested in Refs. [4–6, 17]:

$$\tilde{J}^{k,\nu}\tilde{J}_{k,\nu} = J^{k,\nu}J_{k,\nu}, \tag{79}$$

where $\tilde{J}_{k,\nu} \equiv \tilde{J}^{k}_{\nu}$ and $J_{k,\nu} \equiv J^{k}_{\nu}$.

This equation can be interpreted as a differential condition expressing the conservation of the intrinsic energy of a field source when Yang-Mills field quanta are created inside it [4–6, 17].

Consider now transversal non-Abelian wave solutions satisfying not only the Yang-Mills equations (1)–(2) but also the additional differential condition (79). For this purpose, let us turn to expressions (55)–(56) for the strengths $F^{k,\mu\nu}$ in the transversal non-Abelian waves under examination.

From formulas (55), (56), (50), and (38), we find

$$\begin{gathered} \partial_\mu F^{k,\mu 0} = \omega j^3 d\Phi^k / d\xi, \quad \partial_\mu F^{k,\mu 1} = x j^1 d\Phi^k / d\xi, \\ \partial_\mu F^{k,\mu 2} = y j^1 d\Phi^k / d\xi, \quad \partial_\mu F^{k,\mu 3} = j^3 d\Phi^k / d\xi. \end{gathered} \tag{80}$$

Formulas (77) and (80) give

$$(4\pi / c)^2 \tilde{J}^{k,\nu}\tilde{J}_{k,\nu} = [(\omega^2 - 1)(j^3)^2 - \rho^2 (j^1)^2] \sum_{k=1}^{3} (d\Phi^k / d\xi)^2 \tag{81}$$

and from formulas (9) and (22), we obtain

$$J^{k,\nu}J_{k,\nu} = (\omega^2 - 1)(j^3)^2 - \rho^2 (j^1)^2 . \tag{82}$$

Substituting now expressions (81) and (82) into equation (79), we come to the equation

$$\sum_{k=1}^{3} (d\Phi^k / d\xi)^2 = (4\pi / c)^2 \ . \tag{83}$$

This equation should be added to formulas (58) and (59) for the functions $\Phi^k(\xi)$.

Let us choose the gauge $\chi = \pi/4$ in expressions (58) for $\Phi^k(\xi)$, taking into account the equality $J^{2,\nu} = J^{3,\nu} = 0$ in (9) and hence, the equivalence of the second and third axes in the gauge space. Then substituting formulas (58) for $\Phi^k(\xi)$ into equation (83), we find

$$(\Phi')^2 + (\Phi\gamma')^2 = (4\pi / c)^2, \quad \Phi = \Phi(\xi), \quad \gamma = \gamma(\xi), \quad \chi = \pi / 4 . \qquad . \tag{84}$$

From equations (59) and (84), we obtain

$$\Phi' = (4\pi / c)\cos\gamma, \quad \Phi\gamma' = \pm(4\pi / c)\sin\gamma . \tag{85}$$

The two equations (85) give

$$\Phi' / \Phi = \pm \cot\gamma\, \gamma' . \tag{86}$$

Let us integrate this equation. Then we get

$$\ln|\Phi| = \pm \ln|\sin\gamma| + \text{const} . \tag{87}$$

Choosing the sign '+' in (87) and hence in the second equation in (85), in order to have their nonsingular solution, we find

$$\Phi = D_0 \sin\gamma, \quad D_0 = \text{const} . \tag{88}$$

Then substituting expression (88) for the function $\Phi(\xi)$ into equations (85) and taking into account that the sign '+' has been chosen in the second of them, we obtain

$$\gamma'(\xi) = \frac{4\pi}{cD_0}, \quad \gamma = \frac{4\pi}{cD_0}\xi + D_1, \quad D_1 = \text{const} . \tag{89}$$

As follows from (59), $\Phi(0) = 0$. That is why we set $D_1 = 0$ and from (88) and (89), we get

$$\gamma = \frac{4\pi}{cD_0}\xi, \quad \Phi = D_0 \sin\gamma, \quad D_0 = \text{const} . \tag{90}$$

Substituting now the obtained formulas (90) into expressions (58) for $\Phi^k(\xi)$, using expression (57) for ξ, and taking into account that the gauge $\eta = \pi/4$ was above chosen, we obtain

$$\Phi^1 = \frac{2D}{c}\sin\left(\frac{I}{D}\right), \quad D = \frac{cD_0}{4} = \text{const}, \quad \Phi^2 = \Phi^3 = \frac{\sqrt{2}D}{c}\left[1 - \cos\left(\frac{I}{D}\right)\right], \quad \eta = \frac{\pi}{4}, \tag{91}$$

$$I = 2\pi\int_0^\rho \rho j^3(\eta,\rho)d\rho, \tag{92}$$

where $I = I(\eta,\rho)$ is the source current passing along the axis z through the circle $x^2 + y^2 \le \rho^2$ orthogonal to it.

From formulas (55)–(57) and (91), we find the following expressions for the field strengths $F^{k,\mu\nu}$ in the transversal non-Abelian waves under consideration:

$$\begin{gathered} F^{1,01} = -\frac{2\omega I_{\text{eff}}}{c}\frac{x}{\rho^2}, \quad F^{1,02} = -\frac{2\omega I_{\text{eff}}}{c}\frac{y}{\rho^2}, \quad F^{1,03} = 0, \\ F^{1,12} = 0, \quad F^{1,13} = \frac{2I_{\text{eff}}}{c}\frac{x}{\rho^2}, \quad F^{1,23} = \frac{2I_{\text{eff}}}{c}\frac{y}{\rho^2}, \end{gathered} \tag{93}$$

where

$$I_{\text{eff}} = D\sin\left(\frac{I}{D}\right), \quad I = 2\pi\int_0^\rho \rho j^3(\omega x^0 - z,\rho)d\rho, \tag{94}$$

and

$$F^{2,01} = F^{3,01} = -\frac{\sqrt{2}\omega D}{c}\left[1-\cos\left(\frac{I}{D}\right)\right]\frac{x}{\rho^2},$$
$$F^{2,02} = F^{3,02} = -\frac{\sqrt{2}\omega D}{c}\left[1-\cos\left(\frac{I}{D}\right)\right]\frac{y}{\rho^2}, \quad F^{2,03} = F^{3,03} = 0,$$
$$F^{2,12} = F^{3,12} = 0, \quad F^{2,13} = F^{3,13} = \frac{\sqrt{2}D}{c}\left[1-\cos\left(\frac{I}{D}\right)\right]\frac{x}{\rho^2}, \tag{95}$$
$$F^{2,23} = F^{3,23} = \frac{\sqrt{2}D}{c}\left[1-\cos\left(\frac{I}{D}\right)\right]\frac{y}{\rho^2}.$$

Here D is some constant.
Below we will use the terms 'actual' and 'effective' for the currents $I = I(\omega x^0 - z, \rho)$ and $I_{\text{eff}} = D\sin(I/D)$, respectively, along the axis z through the circle $x^2 + y^2 \le \rho^2$ orthogonal to it.
It should be noted that when $|I| << |D|$, the effective current I_{eff} practically coincides with the actual current I and we have the Maxwell field expressions for the strengths components $F^{1,\mu\nu}$. The value $|D|$ should be a sufficiently large constant. Then the obtained formulas (93)–(95) for the Yang-Mills field strengths $F^{k,\mu\nu}$ can be regarded as a nonlinear generalization of the corresponding Maxwell field expressions when the actual current I is sufficiently large.

Consider equation (77) with $k=1$ and $\nu=3$. Substituting the formulas in (93) for $F^{1,\mu 3}$ into this equation, we obtain

$$\partial I_{\text{eff}} / \partial\rho = 2\pi\rho\, \mathcal{J}^{1,3}, \quad I_{\text{eff}}(\eta,\rho) = 2\pi\int_0^{\rho} \rho\, \mathcal{J}^{1,3}(\eta,\rho)d\rho, \quad \eta = \omega x^0 - z\,. \tag{96}$$

Formulas (96), (76), and (92) signify that the effective current $I_{\text{eff}}(\eta,\rho)$ with the density $\mathcal{J}^{1,3}$ can be interpreted as the full current passing along the axis z through the circle $x^2 + y^2 \le \rho^2$ orthogonal to it, which includes not only the actual current $I(\eta,\rho)$ with the density $J^{1,3} \equiv j^3$ but also the current of Yang-Mills field quanta.

6 Conclusion

We have studied solutions to the Yang-Mills equations with $SU(2)$ symmetry which describe transversal non-Abelian waves propagated at constant phase velocities through axially symmetric field sources. Our investigation of such waves showed that in the case under examination, the Yang-Mills equations could be reduced to a system of six nonlinear partial differential equations. These equations were studied for a special class of field sources satisfying a differential equation of charge conservation.
First, we considered transversal waves propagated at the speed of light in which only three differential equations remained to be solved. After that we studied transversal waves propagated through field sources at arbitrary constant phase velocities. Investigating the six nonlinear partial differential equations describing transversal non-Abelian waves, we found their consequences that were equations for only the Yang-Mills field strengths. It was shown that the found equations for the field strengths had exact solutions. Using them, we obtained formulas for the field strengths in the examined transversal non-Abelian waves. The further analysis allowed us to find expressions for the field potentials in such waves.
The obtained wave solutions to the Yang-Mills equations were not uniquely determined, since they contained an unknown function which could be arbitrary. In order to find unique expressions for the field strengths in the considered waves, an additional equation was suggested. This equation was regarded as a differential condition expressing the conservation of the intrinsic energy of a field source when Yang-Mills field quanta are created inside it. Using the additional condition, we found the unknown function and determined unambiguous expressions for the field strengths in the transversal non-Abelian waves under consideration.

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