International Journal of Advanced Mathematical Sciences, 1 (xx) (2013) xxx-xxx

www.sciencepubco.com/index.php/IJAMS

Unboundedness of classical global solutions of parabolic equations with forcing terms

Norio Yoshida

Department of Mathematics, University of Toyama, Toyama, Japan
E-mail: nori@sci.u-toyama.ac.jp

Abstract

Semilinear parabolic equations with forcing terms are discussed, and sufficient conditions for every classical global solution of boundary value problems to be unbounded on a cylindrical domain in $\mathbb{R}^{n+1}$. The approach used is to reduce the multi-dimensional problems to one-dimensional problems for first-order ordinary differential inequalities.

Keywords: *Boundary value problem, classical global solution, forcing term, parabolic equations, unboundedness of solutions.*

1 Introduction

Since the pioneering work of McNabb [14], unboundedness of classical global solutions of parabolic equations or systems has been developed via Picone identity by numerous authors. We refer the reader to Dunninger [4], Jaroš, Kusano and Yoshida [7, 8], Kusano and Narita [13] and McNabb [14] for parabolic equations, and to Chan [1], Chan and Young [2], Kobayashi and Yoshida [9], Kuks [11] for parabolic systems. In particular, we mention the paper [12] by Kuks and Svirchevskii which deals with parabolic difference equations. However, all of them pertain to time-dependent parabolic equations or systems without forcing terms. The approach used is to utilize the Picone identity or Picone-type inequality for elliptic operators with time-dependent coefficients.

We are concerned with the parabolic equation with forcing term

$$\frac{\partial u}{\partial t} - a(t)\Delta u + p(x,t)\varphi(u) = f(x,t), \quad (x,t) \in G \times (0,\infty), \tag{1}$$

where Δ denotes the Laplacian in $\mathbb{R}^n$ and G is a bounded domain in $\mathbb{R}^n$ with piecewise smooth boundary ∂G.

We consider two kinds of boundary conditions:

(B_1) $u = \psi_1$ on $\partial G \times (0,\infty)$,

(B_2) $\dfrac{\partial u}{\partial \nu} + \mu u = \psi_2$ on $\partial G \times (0,\infty)$,

where $\psi_1,\ \psi_2 \in C(\partial G \times [0,\infty); \mathbb{R})$, $\mu \in C(\partial G \times [0,\infty); [0,\infty))$, and ν denotes the unit exterior normal vector to ∂G.

It is assumed that:

(H_1) $a(t) \in C([0,\infty); [0,\infty))$ and $a(t)$ is bounded from above, i.e., there exists a constant $M_1 > 0$ such that

$$0 \le a(t) \le M_1, \quad t \in [0,\infty);$$

(H_2) $p(x,t) \in C(\overline{G} \times [0,\infty); [0,\infty))$ and $p(x,t)$ is bounded from above, i.e., there exists a constant $M_2 > 0$ such that

$$0 \le p(x,t) \le M_2, \quad (x,t) \in \overline{G} \times [0,\infty);$$

(H_3) $\varphi(\xi) \in C(\mathbb{R}; \mathbb{R})$, $\varphi(\xi) > 0$ and $\varphi(-\xi) = -\varphi(\xi)$ for $\xi > 0$, and $\varphi(\xi)$ is nondecreasing in $\mathbb{R}$;

(H_4) $f(x,t)$ is a real-valued continuous function on $\overline{G} \times [0,\infty)$;

(H_5) μ is bounded from above, i.e., there exists a constant $M_3 > 0$ such that

$$0 \le \mu \le M_3 \quad \text{on } \partial G \times [0,\infty).$$

Existence and uniqueness of classical solution of the initial-boundary value problem

$$\begin{cases} \dfrac{\partial u}{\partial t} - a\Delta u + p(x)u = f(x,t), \quad (x,t) \in G \times (0,\infty), \\ u(x,0) = u_0(x) \ (x \in G) \quad \text{(initial condition)}, \\ (\mathrm{B}_i) \ (i = 1,2) \quad \text{(boundary condition)} \end{cases}$$

are studied in Itô [5,6] under some additional hypotheses on $p(x)$ and $f(x,t)$, where a is a positive constant.

There appears to be no known unboundedness results for parabolic equations with forcing terms. The objective of this paper is to derive sufficient conditions for every classical solution u of the boundary value problems (1), (B_i) ($i = 1,2$) to be unbounded on $\overline{G} \times [0,\infty)$. The method used here is an adaptation of that used in Kobayashi and Yoshida [10].

In Section 2 we reduce unboundedness problem for (1) to that for ordinary differential inequalities of first order. In Section 3 we derive sufficient conditions for every solution of the first-order ordinary differential inequalities to be unbounded from below, and we provide unboundedness results for (1) in Section 4. We present conclusion of this paper in Section 5.

2 Reduction to ordinary differential inequalities

In this section we reduce our multi-dimensional unboundedness problems to one-dimensional problems for ordinary differential inequalities of first order.

It is known that the first eigenvalue λ_1 of the eigenvalue problem

$$\begin{aligned} -\Delta w &= \lambda w \quad \text{in } G, \\ w &= 0 \quad \text{on } \partial G \end{aligned}$$

is positive and the corresponding eigenfunction $\Phi(x)$ may be chosen so that $\Phi(x) > 0$ in G (see Courant and Hilbert [3]).

Theorem 2.1 *Every classical solution u of the boundary value problem* (1), (B_1) *is unbounded on $\overline{G} \times [0,\infty)$ if for any constant $\tilde{K} > 0$, all solutions $y(t)$ of the first-order ordinary differential inequalities*

$$y' - \lambda_1 M_1 C_\Phi \tilde{K} - M_2 C_\Phi \varphi(\tilde{K}) \le -a(t)\Psi_1(t) + F_1(t), \quad t > 0, \tag{2}$$

$$y' - \lambda_1 M_1 C_\Phi \tilde{K} - M_2 C_\Phi \varphi(\tilde{K}) \le -\big(-a(t)\Psi_1(t) + F_1(t)\big), \quad t > 0 \tag{3}$$

are not bounded from below, where

$$\begin{aligned} C_\Phi &= \int_G \Phi(x)\, dx, \\ \Psi_1(t) &= \int_{\partial G} \psi_1 \frac{\partial \Phi(x)}{\partial \nu}\, dS, \\ F_1(t) &= \int_G f(x,t)\Phi(x)\, dx. \end{aligned}$$

Proof. Suppose to the contrary that there exists a solution u of the boundary value problem (1), (B_1) which is bounded on $\overline{G} \times [0,\infty)$. Then there is a constant $K > 0$ such that

$$|u(x,t)| \le K, \quad (x,t) \in \overline{G} \times [0,\infty),$$

or

$$-K \le u(x,t) \le K, \quad (x,t) \in \overline{G} \times [0,\infty).$$

First we consider the case where

$$-K \le u(x,t), \quad (x,t) \in \overline{G} \times [0,\infty).$$

We see from the hypothesis (H_3) that

$$\varphi(u) \ge \varphi(-K) = -\varphi(K)$$

and hence

$$p(x,t)\varphi(u) \ge -p(x,t)\varphi(K) \ge -M_2\varphi(K) \tag{4}$$

in light of the hypothesis (H_2). Combining (1) with (4) yields

$$\frac{\partial u}{\partial t} - a(t)\Delta u - M_2\varphi(K) \le f(x,t), \quad (x,t) \in G \times (0,\infty). \tag{5}$$

Multiplying (5) by $\Phi(x)$ and then integrating over G, we obtain

$$\frac{d}{dt}\int_G u\Phi(x)\,dx - a(t)\int_G (\Delta u)\Phi(x)\,dx - M_2\varphi(K)\int_G \Phi(x)\,dx \le \int_G f(x,t)\Phi(x)\,dx, \quad t > 0. \tag{6}$$

It follows from Green's formula that

$$\begin{aligned} \int_G (\Delta u)\Phi(x)\,dx &= \int_{\partial G}\left[\Phi(x)\frac{\partial u}{\partial \nu} - u\,\frac{\partial \Phi(x)}{\partial \nu}\right] dS + \int_G u\Delta\Phi(x)\,dx \\ &= -\int_{\partial G}\psi_1\frac{\partial \Phi(x)}{\partial \nu}\,dS - \lambda_1\int_G u\,\Phi(x)\,dx \\ &= -\Psi_1(t) - \lambda_1 U(t), \end{aligned} \tag{7}$$

where $U(t) = \int_G u\,\Phi(x)\,dx$. Combining (6) with (7), we derive

$$U'(t) + \lambda_1 a(t)U(t) - M_2 C_\Phi \varphi(K) \le -a(t)\Psi_1(t) + F_1(t), \quad t > 0. \tag{8}$$

We easily check that

$$U(t) = \int_G u\Phi(x)\,dx \ge -KC_\Phi, \tag{9}$$

that is, $U(t)$ is bounded from below. Combining (9) with the hypothesis (H_1) yields

$$\lambda_1 a(t)U(t) \ge -\lambda_1 M_1 C_\Phi K,$$

and hence we see from (8) that

$$U'(t) - \lambda_1 M_1 C_\Phi K - M_2 C_\Phi \varphi(K) \le -a(t)\Psi_1(t) + F_1(t), \quad t > 0.$$

Therefore, we observe that $U(t)$ is a solution of (2) with $\tilde{K} = K$ which is bounded from below. This contradicts the hypothesis. Next we treat the case where

$$u(x,t) \le K, \quad (x,t) \in \overline{G} \times [0,\infty).$$

Letting $v := -u$, we find that $v \ge -K$ on $\overline{G} \times [0,\infty)$ and

$$\begin{aligned} &\frac{\partial v}{\partial t} - a(t)\Delta v + p(x,t)\varphi(v) = -f(x,t), \quad (x,t) \in G \times (0,\infty), \\ &v = -\psi_1 \quad \text{on } \partial G \times (0,\infty). \end{aligned}$$

It is easy to see that

$$V(t) := \int_G v\Phi(x)\,dx \ge -KC_\Phi.$$

Proceeding as in the case where $-K \le u(x,t)$ on $\overline{G} \times [0,\infty)$, we conclude that $V(t)$ is a solution of (3) with $\tilde{K} = K$ which is bounded from below. This contradicts the hypothesis. The proof is complete. $\square$

Theorem 2.2 *Every classical solution u of the boundary value problem* (1), (B$_2$) *is unbounded on $\overline{G} \times [0, \infty)$ if for any constant $\tilde{K} > 0$, all solutions $y(t)$ of the first-order ordinary differential inequalities*

$$y' - M_1 M_3 |\partial G| \tilde{K} - M_2 |G| \varphi(\tilde{K}) \le a(t)\Psi_2(t) + F_2(t), \ \ t > 0, \tag{10}$$

$$y' - M_1 M_3 |\partial G| \tilde{K} - M_2 |G| \varphi(\tilde{K}) \le -\big(a(t)\Psi_2(t) + F_2(t)\big), \ \ t > 0 \tag{11}$$

are not bounded from below, where

$$|G| = \int_G dx \ \ (\textit{the volume of } G),$$

$$|\partial G| = \int_{\partial G} dS \ \ (\textit{the surface area of } G),$$

$$\Psi_2(t) = \int_{\partial G} \psi_2 \, dS,$$

$$F_2(t) = \int_G f(x, t) \, dx.$$

Proof. Suppose that there is a solution u of the boundary value problem (1), (B$_2$) which is bounded on $\overline{G} \times [0, \infty)$. Then there exists a constant $K > 0$ such that

$$|u(x, t)| \le K \quad \text{on } \overline{G} \times [0, \infty),$$

or

$$-K \le u(x, t) \le K \quad \text{on } \overline{G} \times [0, \infty).$$

First we let

$$-K \le u(x, t) \quad \text{on } \overline{G} \times [0, \infty).$$

Arguing as in the proof of Theorem 2.1, we find that the inequality (5) holds. Integrating (5) over G yields

$$\frac{d}{dt} \int_G u \, dx - a(t) \int_G \Delta u \, dx - M_2 \varphi(K) \int_G dx \le \int_G f(x, t) \, dx, \quad t > 0. \tag{12}$$

From the divergence theorem and the boundary condition (B$_2$) we see that

$$\int_G \Delta u \, dx = \int_{\partial G} \frac{\partial u}{\partial \nu} \, dS = \int_{\partial G} (-\mu u + \psi_2) dS \le M_3 |\partial G| K + \Psi_2(t) \tag{13}$$

in view of

$$\int_{\partial G} (-\mu u) dS \le \int_{\partial G} K \mu \, dS \le M_3 |\partial G| K.$$

Combining (12) with (13), we obtain

$$\tilde{U}'(t) - a(t) M_3 |\partial G| K - M_2 |G| \varphi(K) \le a(t)\Psi_2(t) + F_2(t), \quad t > 0$$

and therefore

$$\tilde{U}'(t) - M_1 M_3 |\partial G| K - M_2 |G| \varphi(K) \le a(t)\Psi_2(t) + F_2(t), \quad t > 0 \tag{14}$$

in light of (H$_1$), where $\tilde{U}(t) = \int_G u \, dx$. It is readily seen that $\tilde{U}(t) \ge -K|G|$. Hence, $\tilde{U}(t)$ is a solution of (10) with $\tilde{K} = K$ which is bounded from below. This contradicts the hypothesis. The case where $u(x, t) \le K$ on $\overline{G} \times [0, \infty)$ can be handled similarly, and we conclude that (11) with $\tilde{K} = K$ has a solution which is bounded from below. This is a contradiction, and the proof is complete. □

3 Ordinary differential inequalities of first order

In this section we deal with the first-order ordinary differential inequality

$$y' - \gamma \le g(t), \quad t > 0, \tag{15}$$

and provide a sufficient condition for every solution $y(t)$ of (15) to be unbounded from below. It is assumed that γ is a nonnegative constant and $g(t)$ is a real-valued continuous function on $[0, \infty)$.

Theorem 3.1 *Every solution $y(t)$ of* (15) *is not bounded from below if*

$$\liminf_{t\to\infty} \frac{1}{t}\int_0^t g(\xi)\,d\xi = -\infty. \tag{16}$$

Proof. Suppose to the contrary that there exists a solution $y(t)$ of (15) which is bounded from below. Let $y(t) \geq -M$ on $(0,\infty)$ for some constant $M > 0$. Integrating (15) over $(0,t)$, we obtain

$$y(t) - y(0) - \gamma t \leq \int_0^t g(\xi)\,d\xi,$$

and therefore

$$-M - y(0) - \gamma t \leq \int_0^t g(\xi)\,d\xi. \tag{17}$$

Dividing (17) by t yields

$$-\frac{M + y(0)}{t} - \gamma \leq \frac{1}{t}\int_0^t g(\xi)\,d\xi. \tag{18}$$

The left hand side of (18) is bounded from below as $t \to \infty$, whereas the right hand side of (18) is not bounded from below from the hypothesis (16). This contradiction proves the theorem. □

4 Main results

We present unboundedness results for (1) by combining Theorems 2.1 and 2.2 of Section 2 and Theorem 3.1 of Section 3.

Theorem 4.1 *Every classical solution u of the boundary value problem* (1), (B_1) *is unbounded on $\overline{G} \times [0,\infty)$ if*

$$\liminf_{t\to\infty} \frac{1}{t}\int_0^t \big(-a(\xi)\Psi_1(\xi) + F_1(\xi)\big)\,d\xi = -\infty, \tag{19}$$

$$\limsup_{t\to\infty} \frac{1}{t}\int_0^t \big(-a(\xi)\Psi_1(\xi) + F_1(\xi)\big)\,d\xi = \infty. \tag{20}$$

Proof. It follows from Theorem 3.1 and the hypothesis (19) that every solution y of (2) is not bounded from below. Since

$$\begin{aligned} & \liminf_{t\to\infty} \frac{1}{t}\int_0^t \Big(-\big(-a(\xi)\Psi_1(\xi) + F_1(\xi)\big)\Big)\,d\xi \\ = & -\limsup_{t\to\infty} \frac{1}{t}\int_0^t \big(-a(\xi)\Psi_1(\xi) + F_1(\xi)\big)\,d\xi \\ = & -\infty, \end{aligned}$$

we find that every solution y of (3) is not bounded from below. The conclusion follows from Theorem 2.1. □

Theorem 4.2 *Every classical solution u of the boundary value problem* (1), (B_2) *is unbounded on $\overline{G} \times [0,\infty)$ if*

$$\liminf_{t\to\infty} \frac{1}{t}\int_0^t \big(a(\xi)\Psi_2(\xi) + F_2(\xi)\big)\,d\xi = -\infty, \tag{21}$$

$$\limsup_{t\to\infty} \frac{1}{t}\int_0^t \big(a(\xi)\Psi_2(\xi) + F_2(\xi)\big)\,d\xi = \infty. \tag{22}$$

Proof. The conclusion follows by combining Theorem 2.2 with Theorem 3.1. □

Remark. Following the proofs of Theorems 2.1 and 4.1, we conclude that if (19) is satisfied, every classical solution u of the boundary value problem (1), (B_1) is not bounded from below on $\overline{G} \times [0,\infty)$, i.e., there exists a sequence $\{(x_n, t_n)\} \subset \overline{G} \times [0,\infty)$ such that

$$\lim_{n\to\infty} u(x_n, t_n) = -\infty.$$

Analogously, if (20) holds, then every classical solution u of the boundary value problem (1), (B_1) is not bounded from above on $\overline{G} \times [0,\infty)$, i.e., there exists a sequence $\{(\tilde{x}_n, \tilde{t}_n)\} \subset \overline{G} \times [0,\infty)$ such that

$$\lim_{n\to\infty} u(\tilde{x}_n, \tilde{t}_n) = \infty.$$

The similar remarks hold in Theorem 4.2.

Corollary 4.3 *Every classical solution u of the boundary value problem* (1), (B_1) *is oscillatory in $G \times (0,\infty)$ and is neither bounded from below on $\overline{G} \times [0,\infty)$ nor bounded from above on $\overline{G} \times [0,\infty)$ if* (19) and (20) *are satisfied.*

Proof. If the function

$$\int_0^t \big(-a(\xi)\Psi_1(\xi) + F_1(\xi)\big)\, d\xi$$

is bounded from below on $\overline{G} \times [0,\infty)$ as $t \to \infty$, then the function

$$\frac{1}{t}\int_0^t \big(-a(\xi)\Psi_1(\xi) + F_1(\xi)\big)\, d\xi$$

is also bounded from below on $\overline{G} \times [0,\infty)$ as $t \to \infty$. Therefore, (19) implies that

$$\liminf_{t\to\infty} \int_0^t \big(-a(\xi)\Psi_1(\xi) + F_1(\xi)\big)\, d\xi = -\infty.$$

Analogously, it follows from (20) that

$$\limsup_{t\to\infty} \int_0^t \big(-a(\xi)\Psi_1(\xi) + F_1(\xi)\big)\, d\xi = \infty.$$

From the results of Yoshida [15], [16, Section 2.1] we see that every classical solution u of the boundary value problem (1), (B_1) is oscillatory in $G \times (0,\infty)$. The conclusion follows from Theorem 4.1 and Remark. $\square$

Similarly we obtain the analogue of Corollary 4.3.

Corollary 4.4 *Every classical solution u of the boundary value problem* (1), (B_2) *is oscillatory in $G \times (0,\infty)$ and is neither bounded from below on $\overline{G} \times [0,\infty)$ nor bounded from above on $\overline{G} \times [0,\infty)$ if* (21) and (22) *are satisfied.*

Example 4.5 We consider the parabolic equation

$$\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} + u = (\sin x)\big(2t^2 \sin t + t^2 \cos t + 2t \sin t\big), \quad (x,t) \in (0,\pi) \times (0,\infty), \tag{23}$$

and the boundary condition

$$u(0,t) = u(\pi,t) = 0, \quad t > 0. \tag{24}$$

Here $n = 1$, $G = (0,\pi)$, $a(t) = 1$, $p(x,t) = 1$, $\varphi(u) = u$, $\psi_1 = 0$ and

$$f(x,t) = (\sin x)\big(2t^2 \sin t + t^2 \cos t + 2t \sin t\big).$$

It is easy to see that $\lambda_1 = 1$, $\Phi(x) = \sin x$, $\Psi_1(t) = 0$ and

$$F_1(t) = \int_0^\pi f(x,t) \sin x\, dx = \frac{\pi}{2}\big(2t^2 \sin t + t^2 \cos t + 2t \sin t\big).$$

A simple computation shows that

$$\begin{aligned} \frac{1}{t}\int_0^t F_1(\xi)\, d\xi &= \frac{\pi}{2}\left(t \sin t - 2t \cos t + 4 \sin t + \frac{1}{t}\,(4 \cos t - 4)\right) \\ &= \frac{\sqrt{5}}{2}\pi t \sin(t + \alpha) + B(t), \end{aligned}$$

where α is some constant and $B(t)$ is a bounded function as $t \to \infty$. Therefore we observe that

$$\liminf_{t\to\infty} \frac{1}{t}\int_0^t F_1(\xi)\,d\xi = -\infty,$$
$$\limsup_{t\to\infty} \frac{1}{t}\int_0^t F_1(\xi)\,d\xi = \infty.$$

Hence, Theorem 4.1 implies that every solution u of (23), (24) is unbounded on $[0,\pi]\times[0,\infty)$. One such solution is

$$u = (\sin x)t^2 \sin t.$$

Example 4.6 We consider the problem

$$\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} + u = (\cos x + 1)\big(t^2\cos t - t^2 \sin t + 2t\cos t\big) + (\cos x)t^2\cos t, \quad (x,t)\in(0,\pi)\times(0,\infty), \tag{25}$$
$$-\frac{\partial u}{\partial x}(0,t) = \frac{\partial u}{\partial x}(\pi,t) = 0, \quad t > 0. \tag{26}$$

Here $n = 1$, $G = (0,\pi)$, $a(t) = 1$, $p(x,t) = 1$, $\varphi(u) = u$, $\mu = 0$, $\psi_2 = 0$ and

$$f(x,t) = (\cos x + 1)\big(t^2\cos t - t^2\sin t + 2t\cos t\big) + (\cos x)t^2\cos t.$$

It is easily verified that $\Psi_2(t) = 0$ and

$$F_2(t) = \int_0^\pi f(x,t)\,dx = \pi\big(t^2\cos t - t^2\sin t + 2t\cos t\big).$$

An easy calculation shows that

$$\begin{aligned}\frac{1}{t}\int_0^t F_2(\xi)\,d\xi &= \pi\left(t\sin t + t\cos t + 2\cos t - \frac{2}{t}\sin t\right)\\ &= \sqrt{2}\pi t\sin\left(t + \frac{\pi}{4}\right) + \tilde{B}(t),\end{aligned}$$

where $\tilde{B}(t)$ is a bounded function as $t \to \infty$. Hence we obtain

$$\liminf_{t\to\infty} \frac{1}{t}\int_0^t F_2(\xi)\,d\xi = -\infty,$$
$$\limsup_{t\to\infty} \frac{1}{t}\int_0^t F_2(\xi)\,d\xi = \infty.$$

It follows from Theorem 4.2 that every solution u of (25), (26) is unbounded on $[0,\pi]\times[0,\infty)$. For example,

$$u = (\cos x + 1)t^2\cos t$$

is such a solution.

5 Conclusion

We have investigated unboundedness of classical global solutions of semilinear parabolic equations with forcing terms. We consider two kinds of boundary conditions, i.e., Dirichlet type condition and Robin type condition. It is shown that every solution of the boundary value problem for a class of semilinear parabolic equations with a forcing term is unbounded on the cylindrical domain $\overline{G}\times[0,\infty)$ under some assumptions on the forcing term and the boundary function. Our key tool is to reduce the unboundedness of parabolic equations to that of ordinary differential inequalities. Two examples which illustrate our main theorems are also given.

References

[1] C. Y. Chan, Singular and unbounded matrix solutions for both time-dependent matrix and vector differential systems, J. Math. Anal. Appl. 87 (1982) 147–157.

[2] C. Y. Chan, E. C. Young, Unboundedness of solutions and comparison theorems for time-dependent quasilinear differential matrix inequalities, J. Differential Equations 14 (1973) 195–201.

[3] R. Courant, D. Hilbert, Methods of Mathematical Physics, Vol. I, Interscience, New York, 1966.

[4] D. R. Dunninger, Sturmian theorems for parabolic inequalities, Rend. Accad. Sci. Fis. Mat. Napoli 36 (1969) 406–410.

[5] S. Itô, Fundamental solutions of parabolic differential equations and boundary value problems, Japan. J. Math. 27 (1957) 55-102.

[6] S. Itô, Neumann problem for non-symmetric second order partial differential equations of elliptic type, J. Fac. Sci. Univ. Tokyo, Sec. I, 10 (1963) 20-28.

[7] J. Jaroš, T. Kusano, N. Yoshida, Oscillatory properties of solutions of superlinear-sublinear parabolic equations via Picone-type inequalities, Math. J. Toyama Univ. 24 (2001) 83–91.

[8] J. Jaroš, T. Kusano, N. Yoshida, Oscillation properties of solutions of a class of nonlinear parabolic equations, J. Comput. Appl. Math. 146 (2002) 277–284.

[9] K. Kobayashi, N. Yoshida, Unboundedness of solutions of time-dependent differential systems of parabolic type, Math. J. Toyama Univ. 25 (2002) 65–75.

[10] K. Kobayashi, N. Yoshida, Unboundedness of solutions of Timoshenko beam equations with damping and forcing terms, Int. J. Differ. Equ. 2013, Art. ID 435456, 6 pages.

[11] L. M. Kuks, Unboundedness of solutions of high-order parabolic systems in the plane and a Sturm-type comparison theorem, Differ. Equ. 14 (1978) 623–627.

[12] L. M. Kuks, A. A. Svirchevskii, Theorems in the qualitative theory of partial difference equations, Differ. Equ. 4 (1968) 200–201.

[13] T. Kusano, M. Narita, Unboundedness of solutions of parabolic differential inequalities, J. Math. Anal. Appl. 57 (1977) 68–75.

[14] A. McNabb, A note on the boundedness of solutions of linear parabolic equations, Proc. Amer. Math. Soc. 13 (1962) 262–265.

[15] N. Yoshida, Forced oscillations of solutions of parabolic equations, Bull. Austral. Math. Soc. 36 (1987) 289–294.

[16] N. Yoshida, Oscillation Theory of Partial Differential Equations, World Scientific Publishing Co. Pte. Ltd., 2008.