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Research Paper

Solution and intuitionistic fuzzy stability of n-dimensional quadratic functional equation: direct and fixed point methods

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Abstract

In this paper, the authors established the solution in vector space and Intuitionistic Fuzzy stability of n-dimensional quadratic functional equation

$$\sum_{i=1}^{n} f\left(\sum_{j=1}^{n} x_{ij}\right) = (-n^2 + 6n - 4)\sum_{i=1}^{n} f(x_i) + (n-4) \sum_{1\le i<j\le n} f\left(x_i + x_j\right)$$

where

$$x_{ij} = \begin{cases} -x_j & if \quad i = j, \\ x_j & if \quad i \neq j, \end{cases}$$

and n is a positive integer using direct and fixed point methods.

Keywords: *fixed point, generalized Ulam - Hyers stability, Intuitionistic Fuzzy normed space, quadratic functional equation.*

1. Introduction

The problem of the stability of functional equations was originally stated by S.M.Ulam [41]. In 1941 D.H. Hyers [18] proved the stability of the linear functional equation for the case when the groups are Banach spaces. In 1950, T. Aoki discussed the Hyers-Ulam stability theorem in [2]. His result was further generalized and rediscovered by Th.M. Rassias [32] in 1978. These stability results are further generalized and excellent results was investigated by a number of authors [16, 30, 35]. The terminology generalized Ulam - Hyers stability originates from these historical backgrounds. These terminologies are also applied to the case of other functional equations. For more detailed definitions of such terminologies, one can refer to [1, 13, 19, 22, 24, 34].

The quadratic function $f(x) = cx^2$ satisfies the functional equation

$$f(x+y) + f(x-y) = 2f(x) + 2f(y) \tag{1}$$

and therefore the equation (1) is called quadratic functional equation.

The Hyers - Ulam stability theorem for the quadratic functional equation (1) was proved by F.Skof [40] for the functions $f : E_1 \to E_2$ where E_1 is a normed space and E_2 be a Banach space. The result of Skof is still true if the relevant domain E_1 is replaced by an Abelian group and it was delt by P.W.Cholewa [11]. S.Czerwik [12] proved the Hyers-Ulam-Rassias stability of the quadratic functional equation (1). This result further generalized by Th.M.Rassais [33], C.Borelli, and G.L.Forti [8].

Several other quadratic functional equations were introduced and investigated by I.S. Chang et al., [9], K.W. Jun and H.M. Kim [20], Pl.Kannappan [23], S.M. Jung [21] and references cited there in.

Recently, M. Arunkumar and S. Karthikeyan [4] introduced and investicated the solution and stability of n-dimensional additive functional equation

$$\sum_{i=1}^{n} f\left(\sum_{j=1}^{n} x_{ij}\right) = (n-2)\sum_{j=1}^{n} f\left(x_j\right) \tag{2}$$

in C^*-algebra.

Consider the following quadratic identity

$$\begin{aligned}&\left(-x_1+x_2+\cdots+x_n\right)^2+\left(x_1-x_2+\cdots+x_n\right)^2+\cdots+\left(x_1+x_2+\cdots-x_n\right)^2=\left(-n^2+6n-4\right)\left(x_1^2+x_2^2+\cdots+x_n^2\right)\\&\quad+(n-4)\left[(x_1+x_2)^2+(x_1+x_3)^2+\cdots+(x_1+x_n)^2+(x_2+x_3)^2+(x_2+x_4)^2+\cdots+\cdots+(x_{n-1}+x_n)^2\right].\end{aligned} \tag{3}$$

The above identity can be transformed into the quadratic functional equation

$$\sum_{i=1}^{n} f\left(\sum_{j=1}^{n} x_{ij}\right) = (-n^2+6n-4)\sum_{i=1}^{n} f(x_i) + (n-4)\sum_{1\le i<j\le n} f\left(x_i+x_j\right) \tag{4}$$

where

$$x_{ij} = \begin{cases} -x_j & if \quad i=j,\\ x_j & if \quad i\neq j\end{cases}$$

and n is a positive integer.

In this paper, the authors established the solution and Intuitionistic Fuzzy stability of n-dimensional quadratic functional equation (4) using direct and fixed point methods.

2. General solution

In this section, the general solution of the functional equation (4) is present.

Theorem 2.1 *Let X and Y be real vector spaces. The mapping $f : X \to Y$ satisfies the functional equation (1) for all $x, y \in X$ if and only if $f : X \to Y$ satisfies the functional equation (4) for all $x_1, \cdots, x_n \in X$.*

Proof. Let $f : X \to Y$ satisfies the functional equation (4). Replacing $(x_2, x_3, \cdots, x_n)$ by $(0, 0, \cdots, 0)$ in (4), we get

$$f(-x_1) = f(x_1), \qquad \forall \;\; x_1 \in X.$$

Therefore f is an even function. Setting $(x_3, x_4, \cdots, x_n)$ by $(0, 0, \cdots, 0)$ in (4), we obtain

$$\begin{aligned}&f\left(-x_1+x_2\right)+f\left(x_1-x_2\right)+(n-2)f(x_1+x_2)\\&\quad=\left(-n^2+6n-4\right)\left[f\left(x_1\right)+f\left(x_2\right)\right]+(n-4)\left[f(x_1+x_2)+(n-2)f(x_1)+(n-2)f(x_2)\right]\end{aligned} \tag{5}$$

for all $x_1, x_2 \in X$. Replacing (x_1, x_2) by (x, y) in (5) and using evenness of f and rearranging the functions, our result is desired.

Conversely, assume that $f : X \to Y$ satisfies (1). Setting $x = y = 0$ in (1), we get $f(0) = 0$. Let $y = 0$ in (1), we obtain $f(-x) = f(x)$ for all $x \in X$. Therefore f is an even function. Replacing y by x and $2x$ respectively in (1), we get $f(2x) = 2^2 f(x)$ and $f(3x) = 3^2 f(x)$ for all $x \in X$. In general for any positive integer a, we have $f(ax) = a^2 f(x)$ for all $x \in X$.

Multiplying both sides by 2 in equation (1) and using evenness of f, we obtain

$$f(-x_1 + x_2) + f(x_1 - x_2) = 4[f(x_1) + f(x_2)] - 2f(x_1 + x_2) \tag{6}$$

for all $x_1, x_2 \in X$. Replacing (x_1, x_2) by $(x_1, x_2 - x_3)$ in (6), we get,

$$f(x_1 + x_2 - x_3) + f(x_1 - x_2 + x_3) = 2f(x_1) + 2f(x_2 - x_3) \tag{7}$$

for all $x_1, x_2, x_3 \in X$. Again replacing (x_1, x_2) by $(x_2, x_1 - x_3)$ in (6), we arrive

$$f(x_1 + x_2 - x_3) + f(-x_1 + x_2 + x_3) = 2f(x_2) + 2f(x_1 - x_3) \tag{8}$$

for all $x_1, x_2, x_3 \in X$. Also, replacing (x_1, x_2) by $(x_1 - x_2, x_3)$ in (6), we obtain

$$f(x_1 - x_2 + x_3) + f(-x_1 + x_2 + x_3) = 2f(x_1 - x_2) + 2f(x_3) \tag{9}$$

for all $x_1, x_2, x_3 \in X$. Adding (7),(8) and (9), we have

$$\begin{aligned} &f(-x_1 + x_2 + x_3) + f(x_1 - x_2 + x_3) + f(x_1 + x_2 - x_3) \\ &\quad = (-3^2 + 6 \cdot 3 - 4)\sum_{i=1}^{3} f(x_i) + (3 - 4)\sum_{1 \le i < j \le 3} f(x_i + x_j) \end{aligned} \tag{10}$$

for all $x_1, x_2, x_3 \in X$. Similarly, one can easily verify for four variables that

$$\begin{aligned} &f(-x_1 + x_2 + x_3 + x_4) + f(x_1 - x_2 + x_3 + x_4) + f(x_1 + x_2 - x_3 + x_4) \\ &\quad + f(x_1 + x_2 + x_3 - x_4) = (-4^2 + 6 \cdot 4 - 4)\sum_{i=1}^{4} f(x_i) + (4 - 4)\sum_{1 \le i < j \le 4} f(x_i + x_j) \end{aligned} \tag{11}$$

for all $x_1, x_2, x_3, x_4 \in X$. Extending this result, for any positive integer n, we arrive (4) for all $x_1, x_2, \cdots, x_n \in X$.

3. Preliminaries of intuitionistic fuzzy normed spaces

In this section, using the idea of intuitionistic fuzzy metric spaces introduced by J.H. Park [29] and R. Saadati and J.H. Park [37, 38], we define the new notion of intuitionistic fuzzy metric spaces with the help of the notion of continuous $t-$representable (see [17]).

Lemma 3.1 *[14] Consider the set L^* and the order relation $\le_{L^*}$ defined by:*

$$L^* = \left\{(x_1, x_2) : (x_1, x_2) \in [0,1]^2 \text{ and } x_1 + x_2 \le 1\right\},$$

$$(x_1, x_2) \le_{L^*} (y_1, y_2) \Leftrightarrow x_1 \le y_1, x_2 \ge y_2, \quad \forall (x_1, x_2), (y_1, y_2) \in L^*$$

Then $(L^*, \le_{L^*})$ is a complete lattice.

Definition 3.2 *[5] An intuitionistic fuzzy set $A_{\zeta,\eta}$ in a universal set U is an object*

$$A_{\zeta,\eta} = \{(\zeta_A(u), \eta_A(u)) \,|u \in U\}$$

for all $u \in U$, $\zeta_A(u) \in [0,1]$ and $\eta_A(u) \in [0,1]$ are called the membership degree and the non-membership degree, respectively, of u in $A_{\zeta,\eta}$ and, furthermore, they satisfy $\zeta_A(u) + \eta_A(u) \le 1$.

We denote its units by $0_{L^*} = (0,1)$ and $1_{L^*} = (1,0)$. Classically, a *triangular norm* $* = T$ on $[0,1]$ is defined as an increasing, commutative, associative mapping $T : [0,1]^2 \to [0,1]$ satisfying $T(1,x) = 1 * x = x$ for all $x \in [0,1]$. A *triangular conorm* $S = \diamondsuit$ is defined as an increasing, commutative, associative mapping $S : [0,1]^2 \to [0,1]$ satisfying $S(0,x) = 0 \diamondsuit x = x$ for all $x \in [0,1]$.

Using the lattice $(L^*, \le_{L^*})$, these definitions can be straightforwardly extended.

Definition 3.3 *[5] A triangular norm ($t-$norm) on L^* is a mapping $T:(L^*)^2 \to L^*$ satisfying the following conditions:*

(i) $(\forall \in L^*)(T(x, 1_{L^*}) = x)$ *(boundary condition);*

(ii) $\left(\forall (x,y) \in (L^*)^2\right)(T(x,y) = T(y,x))$ *(commutativity);*

(iii) $\left(\forall (x,y,z) \in (L^*)^3\right)(T(x, T(y,z)) = T(T(x,y), z))$ *(associativity);*

(iv) $\left(\forall (x,x',y,y') \in (L^*)^4\right)(x \leq_{L^*} x' \text{ and } y \leq_{L^*} y' \Rightarrow T(x,y) \leq_{L^*} T(x',y'))$ *(monotonicity).*

If $(L^*, \leq_{L^*}, T)$ is an Abelian topological monoid with unit 1_{L^*}, then L^* is said to be a *continuous $t-$norm.*

Definition 3.4 *[5] A continuous $t-$norms T on L^* is said to be continuous $t-$representable if there exist a continuous $t-$norm $*$ and a continuous $t-$conorm $\diamondsuit$ on $[0,1]$ such that, for all $x = (x_1, x_2), y = (y_1, y_2) \in L^*$,*

$$T(x,y) = (x_1 * y_1, x_2 \diamondsuit y_2).$$

For example,

$$T(a,b) = (a_1 b_1,\ \min\{a_2 + b_2, 1\})$$

and

$$M(a,b) = (\min\{a_1, b_1\}, \max\{a_2, b_2\})$$

for all $a = (a_1, a_2),\ b = (b_1, b_2) \in L^*$ are continuous $t-$representable.

Now, we define a sequence T^n recursively by $T^1 = T$ and

$$T^n\left(x^{(1)}, \ldots, x^{(n+1)}\right) = T\left(T^{n-1}\left(x^{(1)}, \ldots, x^{(n)}\right), x^{(n+1)}\right), \quad \forall n \geq 2,\ x^{(i)} \in L^*.$$

Definition 3.5 *[39] A negator on L^* is any decreasing mapping $N: L^* \to L^*$ satisfying $N:(0_{L^*}) = 1_{L^*}$ and $N(1_{L^*}) = 0_{L^*}$. If $N(N(x)) = x$ for all $x \in L^*$, then N is called an involutive negator. A negator on $[0,1]$ is a decreasing mapping $N:[0,1] \to [0,1]$ satisfying $P_{\mu,\nu}(0) = 1$ and $P_{\mu,\nu}(1) = 0$. N_s denotes the standard negator on $[0,1]$ defined by*

$$N_s(x) = 1 - x, \quad \forall x \in [0,1].$$

Definition 3.6 *[39] Let μ and ν be membership and nonmembership degree of an intuitionistic fuzzy set from $X \times (0, +\infty)$ to $[0,1]$ such that $\mu_x(t) + \nu_x(t) \leq 1$ for all $x \in X$ and all $t > 0$. The triple $(X, P_{\mu,\nu}, T)$ is said to be an intuitionistic fuzzy normed space (briefly IFN-space) if X is a vector space, T is a continuous $t-$representable and $P_{\mu,\nu}$ is a mapping $X \times (0, +\infty) \to L^*$ satisfying the following conditions: for all $x, y \in X$ and $t, s > 0$,*

(IFN1) $P_{\mu,\nu}(x, 0) = 0_{L^*}$;

(IFN2) $P_{\mu,\nu}(x,t) = 1_{L^*}$ if and only if $x = 0$;

(IFN3) $P_{\mu,\nu}(\alpha x, t) = P_{\mu,\nu}\left(x, \frac{t}{|\alpha|}\right)$ for all $\alpha \neq 0$;

(IFN4) $P_{\mu,\nu}(x+y, t+s) \geq_{L^*} T(P_{\mu,\nu}(x,t), P_{\mu,\nu}(y,s))$.

In this case, $P_{\mu,\nu}$ is called an intuitionistic fuzzy norm. Here, $P_{\mu,\nu}(x,t) = (\mu_x(t), \nu_x(t))$.

Example 3.7 *[39] Let $(X, \|\cdot\|)$ be a normed space. Let $T(a,b) = (a, b,\ \min\ (a_2 + b_2, 1))$ for all $a = (a_1, a_2), b = (b_1, b_2) \in L^*$ and μ, ν be membership and non-membership degree of an intuitionistic fuzzy set defined by*

$$P_{\mu,v}(x,t) = (\mu_x(t), v_x(t)) = \left(\frac{t}{t + \|x\|}, \frac{\|x\|}{t + \|x\|}\right), \ \forall t \in R^+.$$

Then $(X, P_{\mu,\nu}, T)$ is an IFN-sapce.

Definition 3.8 *[39] A sequence $\{x_n\}$ in an IFN-space $(X, P_{\mu,\nu}, T)$ is called a Cauchy sequence if, for any $\varepsilon > 0$ and $t > 0$, there exists $n_0 \in N$ such that*

$$P_{\mu,\nu}(x_n - x_m, t) >_{L^*} (N_s(\varepsilon), \varepsilon), \quad \forall n,\ m \geq n_0,$$

where N_s is the standard negator.

Definition 3.9 *[39] The sequence $\{x_n\}$ is said to be convergent to a point $x \in X$ (denoted by $x_n \xrightarrow{P_{\mu,\nu}} x$) if $P_{\mu,\nu}(x_n - x, t) \to 1_{L^*}$ as $n \to \infty$ for every $t > 0$.*

Definition 3.10 *[39] An IFN-space $(X, P_{\mu,\nu}, T)$ is said to be complete if every Cauchy sequence in X is convergent to a point $x \in X$.*

Zhou [42] proved a stability property of the functional equation

$$f(x+y) + f(x-y) = 2f(x)$$

to prove a conjecture of Z.Ditzian about the relationship between the smoothness of a mapping and the degree of its approximation by associated Bernstein polynomials. Very recently, S. Shakeri [39] investigate the stability of Jensen type mapping of the form

$$f(x+y) - f(x-y) = 2f(y)$$

in the setting of intuitionistic fuzzy normed spaces.

4. Stability results: direct method

In this section, authors present the generalized Ulam-Hyers stability of the functional equation (4) in intuitionistic fuzzy normed spaces using direct method.

Now use the following notation for a given mapping $f : X \to Y$

$$Df(x_1, \cdots, x_n) = \sum_{i=1}^{n} f\left(\sum_{j=1}^{n} x_{ij}\right) - (-n^2 + 6n - 4)\sum_{i=1}^{n} f(x_i) - (n-4)\sum_{1 \leq i < j \leq n} f(x_i + x_j)$$

where

$$x_{ij} = \begin{cases} -x_j & if \quad i = j, \\ x_j & if \quad i \neq j, \end{cases}$$

and n is a positive integer and for all $x_1, \cdots, x_n \in X$.

Theorem 4.1 *Let $\beta \in \{-1, 1\}$. Let X be a linear space, $\left(Z, P'_{\mu,\nu}, T\right)$ be an IFN-space, $\alpha : X^n \to Z$ be a mapping with $0 < \left(\frac{d}{2^2}\right)^{\beta} < 1$,*

$$P'_{\mu,\nu}\left(\alpha\left(2^{\beta}x, 2^{\beta}x, 0, \ldots, 0\right), r\right) \geq_{L^*} P'_{\mu,\nu}\left(d^{\beta}\alpha(x, x, 0, \ldots, 0), r\right) \tag{12}$$

for all $x \in X$ and all $r > 0$, and

$$\lim_{n\to\infty} P'_{\mu,\nu}\left(\alpha\left(2^{\beta n}x_1, \ldots, 2^{\beta n}x_n\right), 2^{\beta 2n}r\right) = 1_{L^*} \tag{13}$$

for all $x_1, \ldots, x_n \in X$ and all $r > 0$. $\left(Y, P'_{\mu,\nu}, T\right)$ be an IFN-space. Suppose that a function $f : X \to Y$ satisfies the inequality

$$P_{\mu,\nu}(Df(x_1, \ldots, x_n), r) \geq_{L^*} P'_{\mu,\nu}(\alpha(x_1, \ldots, x_n), r) \tag{14}$$

for all $r > 0$ and all $x_1, \ldots, x_n \in X$. Then the limit

$$P_{\mu,\nu}\left(Q(x) - \frac{f(2^{\beta n}x)}{2^{\beta 2n}}\right) \to 1_{L^*}, as\ n \to \infty, r > 0 \tag{15}$$

exists for all $x \in X$ and the mapping $Q : X \to Y$ is a unique quadratic mapping satisfying (4) and

$$P_{\mu,\nu}(f(x) - Q(x), r) \geq_{L^*} P'_{\mu,\nu}\left(\alpha(x, x, 0, \ldots, 0), 2r|2^2 - d|\right) \tag{16}$$

for all $x \in X$ and all $r > 0$.

Proof. First assume $\beta = 1$. Replacing $(x_1, x_2, x_3, \ldots, x_n)$ by $(x, x, 0, \ldots, 0)$ in (14), we get

$$P_{\mu,\nu}\left(2f(2x) - 2^3 f(x), r\right) \geq_{L^*} P'_{\mu,\nu}\left(\alpha(x, x, 0, \ldots, 0), r\right) \tag{17}$$

for all $x \in X$ and all $r > 0$. Replacing x by $2^n x$ in (17)and using (IFN3), we obtain

$$P_{\mu,\nu}\left(\frac{f(2^{n+1}x)}{2^2} - f(2^n x), \frac{r}{2^3}\right) \geq_{L^*} P'_{\mu,\nu}\left(\alpha(2^n x, 2^n x, 0, \ldots, 0), r\right) \tag{18}$$

for all $x \in X$ and all $r > 0$. Using (12), $(IFN3)$ in (18), we arrive

$$P_{\mu,\nu}\left(\frac{f(2^{n+1}x)}{2^2} - f(2^n x), \frac{r}{2^3}\right) \geq_{L^*} P'_{\mu,\nu}\left(\alpha(x, x, 0, \ldots, 0), \frac{r}{d^n}\right) \tag{19}$$

for all $x \in X$ and all $r > 0$. It is easy to verify from (19), that

$$P_{\mu,\nu}\left(\frac{f(2^{n+1}x)}{2^{2(n+1)}} - \frac{f(2^n x)}{2^{2n}}, \frac{r}{2^3 \cdot 2^{2n}}\right) \geq_{L^*} P'_{\mu,\nu}\left(\alpha(x, x, 0, \ldots, 0), \frac{r}{d^n}\right) \tag{20}$$

holds for all $x \in X$ and all $r > 0$. Replacing r by $d^n r$ in (20), we get

$$P_{\mu,\nu}\left(\frac{f(2^{n+1}x)}{2^{2(n+1)}} - \frac{f(2^n x)}{2^{2n}}, \frac{d^n\ r}{2^3 \cdot 2^{2n}}\right) \geq_{L^*} P'_{\mu,\nu}\left(\alpha(x, x, 0, \ldots, 0), r\right) \tag{21}$$

for all $x \in X$ and all $r > 0$. It is easy to see that

$$\frac{f(2^n x)}{2^{2n}} - f(x) = \sum_{i=0}^{n-1} \frac{f(2^{i+1}x)}{2^{2(i+1)}} - \frac{f(2^i x)}{2^{2i}} \tag{22}$$

for all $x \in X$. From equations (21) and (22), we have

$$\begin{aligned} P_{\mu,\nu}\left(\frac{f(2^n x)}{2^{2n}} - f(x), \sum_{i=0}^{n-1} \frac{d^i\ r}{2^2 \cdot 2^{2i}}\right) &\geq_{L^*} T_{i=0}^{n-1}\left\{P'_{\mu,\nu}\left(\frac{f(2^{i+1}x)}{2^{2(i+1)}} - \frac{f(2^i x)}{2^{2i}}, \frac{d^i\ r}{2^3 \cdot 2^{2i}}\right)\right\} \\ &\geq_{L^*} T_{i=0}^{n-1}\left\{P'_{\mu,\nu}\left(\alpha(x, x, 0, \ldots, 0), r\right)\right\} \\ &\geq_{L^*} P'_{\mu,\nu}\left(\alpha(x, x, 0, \ldots, 0), r\right) \end{aligned} \tag{23}$$

for all $x \in X$ and all $r > 0$. Replacing x by $2^m x$ in (23) and using (12), we obtain

$$P_{\mu,\nu}\left(\frac{f(2^{n+m}x)}{2^{2(n+m)}} - \frac{f(2^m x)}{2^{2m}}, \sum_{i=0}^{n-1} \frac{d^i\ r}{2^3 \cdot 2^{2(i+m)}}\right) \geq_{L^*} P'_{\mu,\nu}\left(\alpha(x, x, 0, \ldots, 0), \frac{r}{d^m}\right) \tag{24}$$

for all $x \in X$ and all $r > 0$ and all $m, n \geq 0$. Replacing r by $d^m r$ in (24), we get

$$P_{\mu,\nu}\left(\frac{f(2^{n+m}x)}{2^{2(n+m)}} - \frac{f(2^m x)}{2^{2m}}, \sum_{i=m}^{m+n-1} \frac{d^i\ r}{2^3 \cdot 2^{2i}}\right) \geq_{L^*} P'_{\mu,\nu}\left(\alpha(x, x, 0, \ldots, 0), r\right) \tag{25}$$

for all $x \in X$ and all $r > 0$ and all $m, n \geq 0$. Using $(IFN3)$ in (25), we obtain

$$P_{\mu,\nu}\left(\frac{f(2^{n+m}x)}{2^{2(n+m)}} - \frac{f(2^m x)}{2^{2m}}, r\right) \geq_{L^*} P'_{\mu,\nu}\left(\alpha(x, x, 0, \ldots, 0), \frac{r}{\sum_{i=m}^{m+n-1} \frac{d^i}{2^3 \cdot 2^{2i}}}\right) \tag{26}$$

for all $x \in X$ and all $r > 0$ and all $m, n \geq 0$. Since $0 < d < 2^2$ and

$$\sum_{i=0}^{n} \left(\frac{d}{2^2}\right)^i < \infty.$$

Thus

$$\left\{\frac{f(2^n x)}{2^{2n}}\right\}$$

is a Cauchy sequence in $(Y, P_{\mu,\nu,}, T)$. Since $(Y, P_{\mu,\nu}, T)$ is a complete IFN-space, this sequence converges to some point $Q(x) \in Y$. So one can define the mapping $Q : X \to Y$ by

$$P_{\mu,\nu}\left(Q(x) - \frac{f(2^{\beta n}x)}{2^{\beta 2n}}\right) \to 1_{L^*}, as\ n \to \infty, r > 0$$

for all $x \in X$. Letting $m = 0$ in (26), we get

$$P_{\mu,\nu}\left(\frac{f(2^n x)}{2^{2n}} - f(x), r\right) \geq_{L^*} P'_{\mu,\nu}\left(\alpha(x, x, 0, \ldots, 0), \frac{r}{\sum_{i=0}^{n-1} \frac{d^i}{2^3 \cdot 2^{2i}}}\right) \tag{27}$$

for all $x \in X$ and all $r > 0$. Letting $n \to \infty$ in (27), we arrive

$$P_{\mu,\nu}\left(f(x) - Q(x), r\right) \geq_{L^*} P'_{\mu,\nu}\left(\alpha(x, x, 0, \ldots, 0), 2r(2^2 - d)\right)$$

for all $x \in X$ and all $r > 0$. To prove Q satisfies the (4), replacing $(x_1, \ldots, x_n)$ by $(2^n x_1, \ldots, 2^n x_n)$ in (14), respectively , we obtain

$$P_{\mu,\nu}\left(\frac{1}{2^{2n}} Df(2^n x_1, \ldots, 2^n x_n), r\right) \geq_{L^*} P'_{\mu,\nu}\left(\alpha(2^n x_1, \ldots, 2^n x_n), 2^{2n} r\right) \tag{28}$$

for all $r > 0$ and all $x_1, \ldots, x_n \in X$. Now,

$$\begin{aligned}
&P_{\mu,\nu}\left(\sum_{i=1}^{n} Q\left(\sum_{j=1}^{n} x_{ij}\right) - (-n^2 + 6n - 4)\sum_{i=1}^{n} Q(x_i) - (n-4)\sum_{1\leq i<j\leq n} Q\left(x_i + x_j\right), r\right)\\
&\geq_{L^*} T\left\{P_{\mu,\nu}\left(\sum_{i=1}^{n} Q\left(\sum_{j=1}^{n} x_{ij}\right) - \frac{1}{2^{2n}}\sum_{i=1}^{n} f\left(\sum_{j=1}^{n} 2^n x_{ij}\right), \frac{r}{4}\right),\right.\\
&\qquad P_{\mu,\nu}\left(-(-n^2 + 6n - 4)\sum_{i=1}^{n} Q(x_i) + \frac{1}{2^{2n}}(-n^2 + 6n - 4)\sum_{i=1}^{n} f(2^n x_i), \frac{r}{4}\right),\\
&\qquad P_{\mu,\nu}\left(-(n-4)\sum_{1\leq i<j\leq n} Q\left(x_i + x_j\right) + \frac{1}{2^{2n}}(n-4)\sum_{1\leq i<j\leq n} f\left(2^n(x_i + x_j)\right), \frac{r}{4}\right),\\
&\qquad P_{\mu,\nu}\left(\frac{1}{2^{2n}}\sum_{i=1}^{n} f\left(\sum_{j=1}^{n} 2^n x_{ij}\right) - \frac{1}{2^{2n}}(-n^2 + 6n - 4)\sum_{i=1}^{n} f(2^n x_i)\right.\\
&\qquad\qquad \left.\left. - \frac{1}{2^{2n}}(n-4)\sum_{1\leq i<j\leq n} f\left(2^n(x_i + x_j)\right), \frac{r}{4}\right)\right\}
\end{aligned} \tag{29}$$

for all $x_1, \ldots, x_n \in X$ and all $r > 0$. Using (28) in (29), we arrive

$$\begin{aligned}
&P_{\mu,\nu}\left(\sum_{i=1}^{n} Q\left(\sum_{j=1}^{n} x_{ij}\right) - (-n^2 + 6n - 4)\sum_{i=1}^{n} Q(x_i) - (n-4)\sum_{1\leq i<j\leq n} Q\left(x_i + x_j\right), r\right)\\
&\qquad \geq_{L^*} T\left\{1_{L^*}, 1_{L^*}, 1_{L^*}, P'_{\mu,\nu}\left(\alpha(2^n x_1, \ldots, 2^n x_n), 2^{2n} r\right)\right\}\\
&\qquad \geq_{L^*} P'_{\mu,\nu}\left(\alpha(2^n x_1, \ldots, 2^n x_n), 2^{2n} r\right)
\end{aligned} \tag{30}$$

for all $x_1, \ldots, x_n \in X$ and all $r > 0$. Letting $n \to \infty$ in (30) and using (13), we see that

$$P_{\mu,\nu}\left(\sum_{i=1}^{n} Q\left(\sum_{j=1}^{n} x_{ij}\right) - (-n^2 + 6n - 4)\sum_{i=1}^{n} Q(x_i) - (n-4)\sum_{1\leq i<j\leq n} Q\left(x_i + x_j\right), r\right) = 1_{L^*}$$

for all $x_1, \ldots, x_n \in X$ and all $r > 0$. Using $(IFN2)$ in the above inequality, it gives

$$\sum_{i=1}^{n} Q\left(\sum_{j=1}^{n} x_{ij}\right) = (-n^2 + 6n - 4)\sum_{i=1}^{n} Q(x_i) + (n-4)\sum_{1\leq i<j\leq n} Q\left(x_i + x_j\right)$$

for all $x_1, \ldots, x_n \in X$. Hence Q satisfies the quadratic functional equation (4). In order to prove $Q(x)$ is unique, we let $Q'(x)$ be another quadratic functional equation satisfying (4) and (16). Hence,

$$\begin{aligned} P_{\mu,\nu}\left(Q(x) - Q'(x), r\right) &= P_{\mu,\nu}\left(\frac{Q(2^n x)}{2^{2n}} - \frac{Q'(2^n x)}{2^{2n}}, r\right) \\ &\geq_{L^*} T\Big\{P_{\mu,\nu}\left(\frac{Q(2^n x)}{2^{2n}} - \frac{f(2^n x)}{2^{2n}}, \frac{r}{2}\right), P_{\mu,\nu}\left(\frac{f(2^n x)}{2^{2n}} - \frac{Q'(2^n x)}{2^{2n}}, \frac{r}{2}\right)\Big\} \\ &\geq_{L^*} P'_{\mu,\nu}\left(\alpha(2^n x, 2^n x, 0, \ldots, 0), \frac{2r\ 2^{2n}(2^2 - d)}{2}\right) \\ &\geq_{L^*} P'_{\mu,\nu}\left(\alpha(x, x, 0, \ldots, 0), \frac{r\ 2^{2n}(2^2 - d)}{d^n}\right) \end{aligned}$$

for all $x \in X$ and all $r > 0$. Since

$$\lim_{n\to\infty} \frac{r\ 2^{2n}(2^2 - d)}{d^n} = \infty,$$

we obtain

$$\lim_{n\to\infty} P'_{\mu,\nu}\left(\alpha(x, x, 0, \ldots, 0), \frac{r\ 2^{2n}(2^2 - d)}{d^n}\right) = 1_{L^*}.$$

Thus

$$P_{\mu,\nu}(Q(x) - Q'(x), r) = 1_{L^*}$$

for all $x \in X$ and all $r > 0$, hence $Q(x) = Q'(x)$. Therefore $Q(x)$ is unique.

For $\beta = -1$, we can prove the result by a similar method. This completes the proof of the theorem.

From Theorem (4.1), we obtain the following corollaries concerning the Hyers-Ulam-Rassias and JMRassias stabilities for the functional equation (4).

Corollary 4.2 *Suppose that a function $f : X \to Y$ satisfies the inequality*

$$P_{\mu,\nu}\left(D\ f(x_1, x_2, x_3, \cdots, x_n), r\right) \geq_{L^*} \begin{cases} P'_{\mu,\nu}\left(\epsilon \sum_{i=1}^{n} ||x_i||^s, r\right), \\ P'_{\mu,\nu}\left(\epsilon\left(\prod_{i=1}^{n} ||x_i||^s + \sum_{i=1}^{n} ||x_i||^{ns}\right), r\right), \end{cases} \tag{31}$$

for all $r > 0$ and all $x_1, \ldots, x_n \in X$, where ϵ, s are constants with $\epsilon > 0$. Then there exists a unique quadratic mapping $Q : X \to Y$ such that

$$P_{\mu,\nu}\left(f(x) - Q(x), r\right) \geq_{L^*} \begin{cases} P'_{\mu,\nu}\left(\epsilon||x||^s, 2r|2^2 - 2^s|\right), s \neq 2 \\ P'_{\mu,\nu}\left(\epsilon||x||^{2s}, 2r|2^2 - 2^{ns}|\right), s \neq \frac{2}{n} \end{cases} \tag{32}$$

for all $x \in X$ and all $r > 0$.

If we define

$$\varphi(x_1, \ldots, x_n) = \begin{cases} \epsilon \sum_{i=1}^{n} ||x_i||^s, \\ \epsilon\left\{\prod_{i=1}^{n} ||x_i||^s + \left(\epsilon \sum_{i=1}^{n} ||x_i||^{ns}\right)\right\} \end{cases}$$

then the corollary is followed from Theorem 4.1, by

$$d = \begin{cases} 2^s, \\ 2^{ns}. \end{cases}$$

5. Stability results: fixed point method

In this section, the authors presented the generalized Ulam - Hyers stability of the functional equation (4) in intuitionistic Fuzzy normed space using fixed point method.

Now we will recall the fundamental results in fixed point theory.

Theorem 5.1 *(Banach's contraction principle) Let (X, d) be a complete metric space and consider a mapping $T : X \to X$ which is strictly contractive mapping, that is*

(A1) $d(Tx, Ty) \leq Ld(x, y)$ *for some (Lipschitz constant)* $L < 1$. *Then,*
(i) The mapping T has one and only fixed point $x^* = T(x^*)$;
(ii)The fixed point for each given element x^* is globally attractive, that is

(A2) $lim_{n\to\infty} T^n x = x^*$, *for any starting point* $x \in X$;
(iii) One has the following estimation inequalities:

(A3) $d(T^n x, x^*) \leq \frac{1}{1-L} \quad d(T^n x, T^{n+1}x), \forall \quad n \geq 0, \forall \quad x \in X;$

(A4) $d(x, x^*) \leq \frac{1}{1-L} \quad d(x, x^*), \forall \quad x \in X.$

Theorem 5.2 *[27](The alternative of fixed point) Suppose that for a complete generalized metric space (X, d) and a strictly contractive mapping $T : X \to X$ with Lipschitz constant L. Then, for each given element $x \in X$, either*
$(B1)$ $$d(T^n x, T^{n+1}x) = \infty \quad \forall \quad n \geq 0,$$
or
$(B2)$ *there exists a natural number n_0 such that:*
(i) $d(T^n x, T^{n+1}x) < \infty$ *for all* $n \geq n_0$;
(ii)The sequence $(T^n x)$ is convergent to a fixed point y^* of T
(iii) y^* is the unique fixed point of T in the set $Y = \{y \in X : d(T^{n_0}x, y) < \infty\}$;
(iv) $d(y^*, y) \leq \frac{1}{1-L} \quad d(y, Ty)$ *for all* $y \in Y$.

For to prove the stability result we define the following:
δ_i is a constant such that

$$\delta_i = \begin{cases} 2 & if \quad i = 0, \\ \frac{1}{2} & if \quad i = 1 \end{cases}$$

and Ω is the set such that

$$\Omega = \{g \mid g : X \to Y, g(0) = 0\}.$$

Theorem 5.3 *Let $f : X \to Y$ be a mapping for which there exist a function $\alpha : X^n \to Z$ with the condition*

$$\lim_{k\to\infty} P'_{\mu,\nu}\left(\alpha\left(\delta_i^k x_1, \delta_i^k x_2, \cdots, \delta_i^k x_n\right), \delta_i^{2k} r\right) = 1_{L^*} \quad \forall\ x_1, x_2, \cdots, x_n \in X, r > 0 \tag{33}$$

and satisfying the functional inequality

$$P_{\mu,\nu}\left(D\ f(x_1, x_2, \cdots, x_n), r\right) \geq_{L^*} P'_{\mu,\nu}\left(\alpha(x_1, x_2, \cdots, x_n), r\right)\ \forall\ x_1, x_2, \cdots, x_n \in X, r > 0. \tag{34}$$

If there exists $L = L(i)$ such that the function

$$x \to \beta(x) = \frac{1}{2}\alpha\left(\frac{x}{2}, \frac{x}{2}, 0, \cdots, 0\right),$$

has the property

$$P'_{\mu,\nu}\left(L\frac{1}{\delta_i^2}\beta(\delta_i x), r\right) = P'_{\mu,\nu}\left(\beta(x), r\right),\ \forall\ x \in X, r > 0. \tag{35}$$

Then there exists unique quadratic function $Q : X \to Y$ satisfying the functional equation (4) and

$$P_{\mu,\nu}\left(f(x) - Q(x), r\right) \geq_{L^*} P'_{\mu,\nu}\left(\beta(x), \frac{L^{1-i}}{1-L}r\right),\ \forall\ x \in X, r > 0. \tag{36}$$

*Proof.*Let d be a general metric on Ω, such that

$$d(g, h) = inf\left\{K \in (0, \infty) | P_{\mu,\nu}\left(g(x) - h(x), r\right) \geq_{L^*} P'_{\mu,\nu}\left(\beta(x), Kr\right), x \in X, r > 0\right\}.$$

It is easy to see that (Ω, d) is complete. Define $T : \Omega \to \Omega$ by $Tg(x) = \dfrac{1}{\delta_i^2} g(\delta_i x)$, for all $x \in X$. For $g, h \in \Omega$, we have $d(g,h) \le K$

$$\begin{aligned}
\Rightarrow \quad & P_{\mu,\nu}\left(g(x) - h(x), r\right) \geq_{L^*} P'_{\mu,\nu}\left(\beta(x), Kr\right) \\
\Rightarrow \quad & P_{\mu,\nu}\left(\frac{g(\delta_i x)}{\delta_i^2} - \frac{h(\delta_i x)}{\delta_i^2}, r\right) \geq_{L^*} P'_{\mu,\nu}\left(\beta(\delta_i x), K\delta_i^2 r\right) \\
\Rightarrow \quad & P_{\mu,\nu}\left(Tg(x) - Th(x), r\right) \geq_{L^*} P'_{\mu,\nu}\left(\beta(x), KLr\right) \\
\Rightarrow \quad & d\left(Tg(x), Th(x)\right) \le KL \\
\Rightarrow \quad & d\left(Tg, Th\right) \le L d(g,h)
\end{aligned} \tag{37}$$

for all $g, h \in \Omega$. There fore T is strictly contractive mapping on Ω with Lipschitz constant L. Replacing $(x_1, x_2, x_3, \cdots, x_n)$ by $(x, x, 0, \cdots, 0)$ in (34), we get

$$P_{\mu,\nu}\left(2f(2x) - 2^3 f(x), r\right) \geq_{L^*} P'_{\mu,\nu}\left(\alpha(x, x, 0, \cdots, 0), r\right). \tag{38}$$

for all $x \in X, r > 0$. Using (IFN2) in (38), we arrive

$$P_{\mu,\nu}\left(\frac{f(2x)}{2^2} - f(x), r\right) \geq_{L^*} P'_{\mu,\nu}\left(\alpha(x, x, 0, \cdots, 0), 2^3 r\right) \tag{39}$$

for all $x \in X, r > 0$ with the help of (35) when $i = 0$, it follows from (39), we get

$$\begin{aligned}
\Rightarrow \quad & P_{\mu,\nu}\left(\frac{f(2x)}{2^2} - f(x), r\right) \geq_{L^*} P'_{\mu,\nu}\left(\beta(x), Lr\right) \\
\Rightarrow \quad & d(Tf, f) \le L = L^1 = L^{1-i}
\end{aligned} \tag{40}$$

Replacing x by $\frac{x}{2}$ in (38), we obtain

$$P_{\mu,\nu}\left(f(x) - 2^2 f\left(\frac{x}{2}\right), r\right) \geq_{L^*} P'_{\mu,\nu}\left(\alpha\left(\frac{x}{2}, \frac{x}{2}, 0, \cdots, 0\right), 2r\right) \tag{41}$$

for all $x \in X, r > 0$ with the help of (35) when $i = 1$, it follows from (41) we get

$$\begin{aligned}
\Rightarrow \quad & P_{\mu,\nu}\left(f(x) - 2^2 f\left(\frac{x}{2}\right), r\right) \geq_{L^*} P'_{\mu,\nu}\left(\beta(x), r\right) \\
\Rightarrow \quad & d(f, Tf) \le 1 = L^0 = L^{1-i}
\end{aligned} \tag{42}$$

Then from (40) and (42) we can conclude,

$d(f, Tf) \le L^{1-i} < \infty$

Now from the fixed point alternative in both cases, it follows that there exists a fixed point Q of T in Ω such that

$$\lim_{n \to \infty} P_{\mu,\nu}\left(\frac{f\left(\delta_i^n x\right)}{\delta_i^n} - Q(x), r\right) \to 1_{L^*}, \qquad \forall x \in X, r > 0. \tag{43}$$

Replacing $(x_1, x_2, \cdots, x_n)$ by $(\delta_i x_1, \delta_i x_2, \cdots, \delta_i x_n)$ in (34), we arrive

$$P_{\mu,\nu}\left(\frac{1}{\delta_i^{2n}} Df(\delta_i x_1, \delta_i x_2, \cdots, \delta_i x_n), r\right) \geq_{L^*} P'_{\mu,\nu}\left(\alpha(\delta_i x_1, \delta_i x_2, \cdots, \delta_i x_n), \delta_i^{2n} r\right) \tag{44}$$

for all $r > 0$ and all $x_1, x_2, \cdots, x_n \in X$ By proceeding the same procedure as in the Theorem 4.1, we can prove the function, $Q : X \to Y$ satisfies the functional equation (4).

By fixed point alternative, since Q is unique fixed point of T in the set

$$\Delta = \left\{f \in \Omega | d(f, Q) < \infty\right\},$$

therefore Q is a uniqe function such that

$$P_{\mu,\nu}\left(f(x) - Q(x), r\right) \geq_{L^*} P'_{\mu,\nu}\left(\beta(x), Kr\right) \tag{45}$$

for all $x \in X, r > 0$ and $K > 0$. Again using the fixed point alternative, we obtain

$$\begin{aligned} & d(f,Q) \leq \frac{1}{1-L} d(f,Tf) \\ \Rightarrow \quad & d(f,Q) \leq \frac{L^{1-i}}{1-L} \\ \Rightarrow \quad & P_{\mu,\nu}\left(f(x)-Q(x),r\right) \geq_{L^*} P'_{\mu,\nu}\left(\beta(x), \frac{L^{1-i}}{1-L} r\right), \end{aligned} \tag{46}$$

firall $x \in X$ and $r > 0$. This completes the proof of the theorem.

From Theorem 5.3, we obtain the following corollary concerning the stability for the functional equation (4).

Corollary 5.4 *Suppose that a function $f : X \to Y$ satisfies the inequality*

$$P_{\mu,\nu}\left(D\ f(x_1,x_2,\cdots,x_n),r\right) \geq \begin{cases} P'_{\mu,\nu}\left(\epsilon \sum_{i=1}^{n} ||x_i||^s, r\right), \\ P'_{\mu,\nu}\left(\epsilon \left(\prod_{i=1}^{n} ||x_i||^s + \sum_{i=1}^{n} ||x_i||^{ns}\right), r\right), \end{cases} \tag{47}$$

for all $x_1, x_2, \cdots, x_n \in X$ and $r > 0$, where ϵ, s are constants with $\epsilon > 0$. Then there exists a unique quadratic mapping $Q : X \to Y$ such that

$$P_{\mu,\nu}\left(f(x)-Q(x),r\right) \geq_{L^*} \begin{cases} P'_{\mu,\nu}\left(\epsilon||x||^s, \frac{2^{s+2}}{|2^2-2^s|} r\right), & s<2 \quad or \quad s>2; \\ P'_{\mu,\nu}\left(\epsilon||x||^{ns}, \frac{2^{ns+2}}{|2^2-2^{ns}|} r\right), & s<\frac{2}{n} \quad or \quad s>\frac{2}{n}; \end{cases} \tag{48}$$

for all $x \in X$ and all $r > 0$.

Proof. Setting

$$\alpha(x_1,x_2,\cdots,x_n) = \begin{cases} \epsilon \sum_{i=1}^{n} ||x_i||^s, \\ \epsilon \left(\prod_{i=1}^{n} ||x_i||^s + \sum_{i=1}^{n} ||x_i||^{ns}\right) \end{cases}$$

for all $x_1, x_2, \cdots, x_n \in X$. Then,

$$\begin{aligned} & P'_{\mu,\nu}\left(\alpha(\delta_i^k x_1, \delta_i^k x_2, \cdots, \delta_i^k x_n), \delta_i^{2k} r\right) \\ & = \begin{cases} P'_{\mu,\nu}\left(\epsilon \sum_{i=1}^{n} ||x||^s, (\delta_i^{2-s})^k r\right) \\ P'_{\mu,\nu}\left(\epsilon \left(\prod_{i=1}^{n} ||x_i||^s + \sum_{i=1}^{n} ||x_i||^{ns}\right), \left(\delta_i^{2-ns}\right)^k r\right) \end{cases} \\ & = \begin{cases} \to 1_{L^*} \text{ as } k \to \infty \text{ for } s<2 \text{ if } i=0 \text{ and } s>2 \text{ if } i=1, \\ \to 1_{L^*} \text{ as } k \to \infty \text{ for } s<\frac{2}{n} \text{ if } i=0 \text{ and } s>\frac{2}{n} \text{ if } i=1. \end{cases} \end{aligned}$$

Thus, (33) is holds. But we have $\beta(x) = \frac{1}{2}\alpha\left(\frac{x}{2}, \frac{x}{2}, 0, \cdots, 0\right)$ has the property

$$P'_{\mu,\nu}\left(L\frac{1}{\delta_i^2}\beta(\delta_i x), r\right) \geq_{L^*} P'_{\mu,\nu}\left(\beta(x), r\right) \ \forall\ x \in X, r > 0.$$

Hence

$$P'_{\mu,\nu}\left(\beta(x),r\right) = P'_{\mu,\nu}\left(\frac{1}{2}\alpha\left(\frac{x}{2}, \frac{x}{2}, 0, \cdots, 0\right), r\right) = \begin{cases} P'_{\mu,\nu}\left(\frac{\epsilon}{2^s}||x||^s, r\right), \\ P'_{\mu,\nu}\left(\frac{\epsilon}{2^{ns}}||x||^{ns}, r\right). \end{cases}$$

Now,

$$P'_{\mu,\nu}\left(\frac{1}{\delta_i^2}\beta(\delta_i x), r\right) = \begin{cases} P'_{\mu,\nu}\left(\frac{\epsilon}{\delta_i^2}\left(\frac{1}{2^s}\right)||\delta_i x||^s, r\right), \\ P'_{\mu,\nu}\left(\frac{\epsilon}{\delta_i^2}\left(\frac{1}{2^{ns}}\right)||\delta_i x||^{ns}, r\right) \end{cases} = \begin{cases} P'_{\mu,\nu}\left(\beta(x), \delta_i^{2-s} r\right), \\ P'_{\mu,\nu}\left(\beta(x), \delta_i^{2-ns} r\right). \end{cases}$$

Hence the inequality (35) holds either, $L = 2^{2-s}$ for $s > 2$ if $i = 0$ and $L = 2^{s-2}$ for $s < 2$ if $i = 1$ also $L = 2^{2-ns}$ for $s > \frac{2}{n}$ if $i = 0$ and $L = 2^{ns-2}$ for $s < \frac{2}{n}$ if $i = 1$.
Now from (36), we prove the following cases for conditions (i) and (ii).
Case:1 $L = 2^{2-s}$ for $s > 2$ if $i = 0$

$$P_{\mu,\nu}\left(f(x) - Q(x), r\right) \geq_{L^*} = P'_{\mu,\nu}\left(\epsilon\left(\frac{1}{2^s}\right)||x||^s, \frac{2^{2-s}}{1-2^{2-s}}r\right) = P'_{\mu,\nu}\left(\epsilon||x||^s, \frac{2^{s+2}}{2^s - 2^2}r\right).$$

Case:2 $L = 2^{s-2}$ for $s < 2$ if $i = 1$

$$P_{\mu,\nu}\left(f(x) - Q(x), r\right) \geq_{L^*} = P'_{\mu,\nu}\left(\epsilon\left(\frac{1}{2^s}\right)||x||^s, \frac{1}{1-2^{s-2}}r\right) = P'_{\mu,\nu}\left(\epsilon||x||^s, \frac{2^{s+2}}{2^2 - 2^s}r\right).$$

Case:3 $L = 2^{2-ns}$ for $s > \frac{2}{n}$ if $i = 0$

$$P_{\mu,\nu}\left(f(x) - Q(x), r\right) \geq_{L^*} = P'_{\mu,\nu}\left(\epsilon\left(\frac{1}{2^{ns}}\right)||x||^{ns}, \frac{2^{2-ns}}{1-2^{2-ns}}r\right) = P'_{\mu,\nu}\left(\epsilon||x||^{ns}, \frac{2^{ns+2}}{2^{ns} - 2^2}r\right).$$

Case:4 $L = 2^{ns-2}$ for $s < \frac{2}{n}$ if $i = 1$

$$P_{\mu,\nu}\left(f(x) - Q(x), r\right) \geq_{L^*} = P'_{\mu,\nu}\left(\epsilon\left(\frac{1}{2^{ns}}\right)||x||^{ns}, \frac{1}{1-2^{ns-2}}r\right) = P'_{\mu,\nu}\left(\epsilon||x||^{ns}, \frac{2^{ns+2}}{2^2 - 2^{4s}}r\right).$$

Hence the proof is complete.

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