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On axially symmetric solutions of the Navier-Stokes equations

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Abstract

In the present paper, the Navier-Stokes equations are studied in several axially symmetric cases. In them incompressible viscous fluids rotate about their axes and can change their shape. In the considered cases, three exact solutions to the Navier-Stokes equations are found. The first of these solutions describes rotating viscous fluids that are gradually cooling. The second of them describes nonstationary rotations with axial motions of viscous fluids. The third of the obtained solutions to the Navier-Stokes equations concerns rotating viscous fluids with stationary velocities. It is used to describe the observable phenomenon of differential rotation of the visible surfaces of stars and giant gas planets.

Keywords: *Axial rotation of fluids, differential rotation of stars and gas giants, incompressible viscous fluids, Navier-Stokes equations, solutions to Navier-Stokes equations.*

1 Introduction

Consider the Navier-Stokes equations describing a homogeneous incompressible viscous fluid. They can be represented in the form [1-3]

$$\frac{\partial \mathbf{v}}{\partial t}+v_1\frac{\partial \mathbf{v}}{\partial x}+v_2\frac{\partial \mathbf{v}}{\partial y}+v_3\frac{\partial \mathbf{v}}{\partial z}=-\frac{1}{\theta}\,\mathrm{grad}\;p+\mathbf{f}+\nu\Delta\mathbf{v}\,, \tag{1}$$

$$\mathrm{div}\,\mathbf{v}=0\,, \tag{2}$$

where $\mathbf{v}=\mathbf{v}(t,x,y,z)$ and $p=p(t,x,y,z)$ are the vector of velocity and pressure, respectively, v_1, v_2, v_3 are the projections of the vector $\mathbf{v}$ onto the orthogonal axes $x,\ y,\ z,$ t is time, $\mathbf{f}=\mathbf{f}(t,x,y,z)$ is the force per unit mass in the considered fluid, θ is its density, and ν is its kinematic viscosity.

The Navier-Stokes equations are basic equations of fluid mechanics and extensive studies are devoted to them [4]. However, because of substantial nonlinearity of these equations, only a small number of exact solutions to them were found [5-10]. Our aim is to obtain some new exact solutions to the Navier-Stokes equations that could have interesting applications.

Further we will study the case in which the force $\mathbf{f}$ is potential. Then for its potential Φ we have the equality

$$\mathbf{f}=-\mathrm{grad}\;\Phi\,. \tag{3}$$

In this case, equation (1) can be rewritten as

$$\frac{\partial \mathbf{v}}{\partial t}+v_1\frac{\partial \mathbf{v}}{\partial x}+v_2\frac{\partial \mathbf{v}}{\partial y}+v_3\frac{\partial \mathbf{v}}{\partial z}=\mathrm{grad}\;q+\nu\Delta\mathbf{v},\quad q=-p/\theta-\Phi\,. \tag{4}$$

Let us study axially symmetric solutions to the Navier-Stokes equations. For this purpose, we will seek the components v_1, v_2, v_3 of the vector function $\mathbf{v}$ and the function q in the following form:

$$\begin{gathered} v_1=-\alpha y+\beta x,\quad v_2=\alpha x+\beta y,\quad v_3=\gamma,\quad \alpha=\alpha(t,\rho,z), \\ \beta=\beta(t,\rho,z),\quad \gamma=\gamma(t,\rho,z),\quad q=q(t,\rho,z),\quad \rho=\sqrt{x^2+y^2}\,. \end{gathered} \tag{5}$$

Here the function α presents the angular velocities of points of a fluid rotating about the axis z and the functions β and γ describe changing its shape.

Substituting expressions (5) into equation (2), we find

$$\rho\beta_\rho + 2\beta + \gamma_z = 0\,, \tag{6}$$

where $\beta_\rho \equiv \partial\beta/\partial\rho,\ \gamma_z \equiv \partial\gamma/\partial z$.

Using expressions (5), we obtain after calculations

$$\begin{aligned}
&v_1\frac{\partial v_1}{\partial x} + v_2\frac{\partial v_1}{\partial y} + v_3\frac{\partial v_1}{\partial z} = -(\rho\beta\alpha_\rho + \gamma\alpha_z + 2\alpha\beta)y + (\rho\beta\beta_\rho + \gamma\beta_z + \beta^2 - \alpha^2)x,\\
&v_1\frac{\partial v_2}{\partial x} + v_2\frac{\partial v_2}{\partial y} + v_3\frac{\partial v_2}{\partial z} = (\rho\beta\alpha_\rho + \gamma\alpha_z + 2\alpha\beta)x + (\rho\beta\beta_\rho + \gamma\beta_z + \beta^2 - \alpha^2)y,\\
&v_1\frac{\partial v_3}{\partial x} + v_2\frac{\partial v_3}{\partial y} + v_3\frac{\partial v_3}{\partial z} = \rho\beta\gamma_\rho + \gamma\gamma_z.
\end{aligned} \tag{7}$$

For the components of the Laplacian $\Delta\mathbf{v}$, we find

$$\begin{aligned}
&\Delta v_1 = -(\alpha_{\rho\rho} + 3\alpha_\rho/\rho + \alpha_{zz})y + (\beta_{\rho\rho} + 3\beta_\rho/\rho + \beta_{zz})x,\\
&\Delta v_2 = (\alpha_{\rho\rho} + 3\alpha_\rho/\rho + \alpha_{zz})x + (\beta_{\rho\rho} + 3\beta_\rho/\rho + \beta_{zz})y,\\
&\Delta v_3 = \gamma_{\rho\rho} + \gamma_\rho/\rho + \gamma_{zz},
\end{aligned} \tag{8}$$

where $\alpha_{\rho\rho} \equiv \partial^2\alpha/\partial\rho^2,\ \alpha_{zz} \equiv \partial^2\alpha/\partial z^2$.

Let us now substitute formulas (5), (7), and (8) into the Navier-Stokes equations (4). Then we come to the following three nonlinear partial differential equations:

$$\alpha_t + \beta(\rho\alpha_\rho + 2\alpha) + \gamma\alpha_z - \nu(\alpha_{\rho\rho} + 3\alpha_\rho/\rho + \alpha_{zz}) = 0, \quad \alpha_t \equiv \partial\alpha/\partial t, \tag{9}$$

$$\beta_t + \beta(\rho\beta_\rho + \beta) + \gamma\beta_z - \alpha^2 - \nu(\beta_{\rho\rho} + 3\beta_\rho/\rho + \beta_{zz}) = q_\rho/\rho, \tag{10}$$

$$\gamma_t + \rho\beta\gamma_\rho + \gamma\gamma_z - \nu(\gamma_{\rho\rho} + \gamma_\rho/\rho + \gamma_{zz}) = q_z\,. \tag{11}$$

In the second section, we will investigate the obtained four partial differential equations (6) and (9)-(11) for the unknown functions α, β, γ and q. In the third section, we will consider rotating viscous fluids that are gradually cooling and obtain a class of exact solutions to the Navier-Stokes equations. In the fourth section, we will find their exact solution for nonstationary rotations with axial motions of viscous fluids. In the fifth section, an exact solution to the Navier-Stokes equations will be found which corresponds to a rotating viscous fluid with stationary velocities. This solution can be applied to describe the observable phenomenon of differential rotation of the visible surfaces of stars and gas giant planets in which their angular velocity decreases with increased latitude [11-14].

2 Investigation of the Navier-Stokes equations in the case of axial symmetry

Consider the obtained equations (9)-(11). First, let us eliminate the function q in them. For this purpose, differentiating equations (10) and (11) with respect to z and ρ, respectively, and using the evident equality $\partial q_\rho/\partial z = \partial q_z/\partial\rho$, we find

$$\begin{aligned}
&\frac{\partial}{\partial z}\left[\beta_t + \beta(\rho\beta_\rho + \beta) + \gamma\beta_z - \alpha^2 - \nu(\beta_{\rho\rho} + 3\beta_\rho/\rho + \beta_{zz})\right]\\
&= \frac{1}{\rho}\frac{\partial}{\partial\rho}\left[\gamma_t + \rho\beta\gamma_\rho + \gamma\gamma_z - \nu(\gamma_{\rho\rho} + \gamma_\rho/\rho + \gamma_{zz})\right].
\end{aligned} \tag{12}$$

After identity transformations, this equation can be represented in the form

$$\begin{aligned}
&\beta_{tz} + \beta_z(\rho\beta_\rho + 2\beta) + \rho\beta\beta_{\rho z} + \gamma_z\beta_z + \gamma\beta_{zz} - 2\alpha\alpha_z - \nu(\beta_{\rho\rho z} + 3\beta_{\rho z}/\rho + \beta_{zzz})\\
&= (\gamma_\rho/\rho)_t + (\rho\beta_\rho + 2\beta)(\gamma_\rho/\rho) + \rho\beta(\gamma_\rho/\rho)_\rho + \gamma_z(\gamma_\rho/\rho) + \gamma(\gamma_\rho/\rho)_z\\
&- \nu[(\gamma_\rho/\rho)_{\rho\rho} + (3/\rho)(\gamma_\rho/\rho)_\rho + (\gamma_\rho/\rho)_{zz}].
\end{aligned} \tag{13}$$

Let us eliminate γ_z by using equality (6): $\gamma_z = -\rho\beta_\rho - 2\beta$. Then equation (13) acquires the following form:

$$\begin{aligned} &2\alpha\alpha_z = (\beta_z - \gamma_\rho / \rho)_t + \rho\beta(\beta_z - \gamma_\rho / \rho)_\rho + \gamma(\beta_z - \gamma_\rho / \rho)_z \\ &- \nu[(\beta_z - \gamma_\rho / \rho)_{\rho\rho} + (3/\rho)(\beta_z - \gamma_\rho / \rho)_\rho + (\beta_z - \gamma_\rho / \rho)_{zz}]. \end{aligned} \tag{14}$$

The obtained equations (6), (9), and (14) can be rewritten as

$$\begin{aligned} &\gamma_z = -\rho\beta_\rho - 2\beta, \\ &-2\alpha\beta = \alpha_t + \rho\beta\alpha_\rho + \gamma\alpha_z - \nu(\alpha_{\rho\rho} + 3\alpha_\rho / \rho + \alpha_{zz}), \\ &2\alpha\alpha_z = \varphi_t + \rho\beta\varphi_\rho + \gamma\varphi_z - \nu(\varphi_{\rho\rho} + 3\varphi_\rho / \rho + \varphi_{zz}), \quad \varphi = \beta_z - \gamma_\rho / \rho. \end{aligned} \tag{15}$$

In order to determine the function $q = -p/\theta - \Phi$ and hence the pressure p, let us turn to equations (10) and (11). From them we have

$$\begin{aligned} &q_\rho = K(t,\rho,z), \quad q_z = L(t,\rho,z), \\ &K = \rho[\beta_t + \beta(\rho\beta_\rho + \beta) + \gamma\beta_z - \alpha^2 - \nu(\beta_{\rho\rho} + 3\beta_\rho / \rho + \beta_{zz})], \\ &L = \gamma_t + \rho\beta\gamma_\rho + \gamma\gamma_z - \nu(\gamma_{\rho\rho} + \gamma_\rho / \rho + \gamma_{zz}). \end{aligned} \tag{16}$$

From (16) we derive the following equality which is equivalent to equality (12):

$$\partial K / \partial z = \partial L / \partial \rho . \tag{17}$$

This gives that in any singly connected region the expression $Kd\rho + Ldz$ is a total differential and the function q can be determined as follows:

$$q = \int_{(\rho_0, z_0)}^{(\rho, z)} (Kd\rho + Ldz) + q_0(t), \tag{18}$$

where the integral is taken along an arbitrary line connecting a fixed point (ρ_0, z_0) and point (ρ, z) and $q_0(t)$ is some function which gives values of q at the point (ρ_0, z_0).

Thus, the problem under consideration consists in finding solutions to the system of three equations (15) which will be further examined.

3 Rotation of a cooling viscous fluid

Let us apply the obtained system of equations (15) to a rotating axially symmetric viscous fluid that is gradually cooling. In this case, the following equation of heat conduction should be added to equations (15) [2]:

$$\frac{\partial T}{\partial t} + \mathbf{v}\nabla T = \chi\Delta T + \frac{\nu}{2c_p}\sum_{i,j=1}^{3}\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right)^2 , \tag{19}$$

where T is temperature, $x_1 = x,\ x_2 = y,\ x_3 = z$, x, y, z are orthogonal spatial coordinates, $\chi = \kappa / \theta c_p$, κ is heat conductivity, θ is mass density, and c_p is specific heat at constant pressure.

This equation has the following particular solution describing a gradually cooling fluid:

$$v_j = V_j(x,y,z)\exp(-k_0 t), \quad T = T_0(x,y,z)\exp(-2k_0 t), \quad k_0 = \text{const} > 0 , \tag{20}$$

where V_j and T_0 are some functions of the spatial coordinates x, y, z which are related by the equation

$$-2k_0 T_0 + \mathbf{v}\nabla T_0 = \chi\Delta T_0 + \frac{\nu}{2c_p}\sum_{i,j=1}^{3}\left(\frac{\partial V_i}{\partial x_j} + \frac{\partial V_j}{\partial x_i}\right)^2 . \tag{21}$$

Consider a cooling axially symmetric fluid with velocities v_1, v_2, v_3 that are described by formulas (5). From (5) and (20) we obtain

$$\alpha = A(\rho,z)\exp(-k_0 t), \quad \beta = B(\rho,z)\exp(-k_0 t), \quad \gamma = C(\rho,z)\exp(-k_0 t) , \tag{22}$$

where $A(\rho,z)$, $B(\rho,z)$, and $C(\rho,z)$ are some functions.

Substituting formulas (22) into equations (15), we obtain

$$C_z = -\rho B_\rho - 2B , \tag{23}$$

$$-2AB=\rho BA_\rho+CA_z,\quad k_0A+\nu(A_{\rho\rho}+3A_\rho/\rho+A_{zz})=0, \tag{24}$$

$$2AA_z=\rho BD_\rho+CD_z,\quad k_0D+\nu(D_{\rho\rho}+3D_\rho/\rho+D_{zz})=0,\quad D=B_z-C_\rho/\rho. \tag{25}$$

Let us seek a solution to equations (23)-(25) in the following form:

$$D\equiv B_z-C_\rho/\rho=k_1A,\quad B=-(1/k_1)A_z,\quad C=(1/k_1)(\rho A_\rho+2A),\quad k_1=\text{const}. \tag{26}$$

Then equation (23) and the first equations in (24) and (25) are identically satisfied and the second equations in (24) and (25) become coinciding.

As to the second equation in (24), it gives

$$k^2A+A_{\rho\rho}+3A_\rho/\rho+A_{zz}=0,\quad k=\sqrt{k_0/\nu}. \tag{27}$$

Consider now equalities (26). From them we obtain

$$D=k_1A=-(1/k_1)(A_{\rho\rho}+3A_\rho/\rho+A_{zz}). \tag{28}$$

Comparing equations (27) and (28), we find

$$|k_1|=k=\sqrt{k_0/\nu} \tag{29}$$

and the two equations become coinciding.

Let us turn to equation (27) and represent the function A in the form

$$A=f(r,z),\quad r=\sqrt{\rho^2+z^2}=\sqrt{x^2+y^2+z^2}. \tag{30}$$

Then we find

$$\begin{aligned}&A_\rho=(f_r/r)\rho,\quad A_z=(f_r/r)z+f_z,\quad f_r\equiv\partial f/\partial r,\quad f_z\equiv\partial f/\partial z,\\ &A_{\rho\rho}=(f_r/r)_r\rho^2/r+f_r/r,\quad A_{zz}=(f_r/r)_r z^2/r+f_r/r+f_{zz}\end{aligned} \tag{31}$$

and equation (27) can be rewritten as

$$k^2f+f_{rr}+4f_r/r+f_{zz}=0. \tag{32}$$

Let us seek particular solutions to equation (32) in the form

$$f=\sin(az+b)F(r),\quad a,b=\text{const}. \tag{33}$$

Then from (32) we derive

$$F''(r)+4F'(r)/r+(k^2-a^2)F(r)=0. \tag{34}$$

Let us put

$$\xi=r\sqrt{k^2-a^2},\quad F(r)=\xi^{-\frac{3}{2}}G(\xi). \tag{35}$$

Then equation (34) acquires the form

$$\xi^2G''(\xi)+\xi G'(\xi)+(\xi^2-9/4)G(\xi)=0. \tag{36}$$

As is well known, this equation has the following solution with no singularity at $r=0$:

$$G=G_0J_{\frac{3}{2}}(\xi),\quad G_0=\text{const},\quad J_{\frac{3}{2}}(\xi)=\sqrt{\frac{2}{\pi\xi}}\left(\frac{\sin\xi}{\xi}-\cos\xi\right). \tag{37}$$

From here on, $J_s(x)$ denotes the Bessel functions of the first kind which satisfy the equation $x^2J_s''+xJ_s'+(x^2-s^2)J_s=0$.

From formulas (33), (35), and (37) we obtain the following solutions to equation (32) having no singularity at $r=0$:

$$\begin{aligned}f=&\frac{1}{r^2}\int_0^k\frac{P_1(a)\sin(az)+Q_1(a)\cos(az)}{k^2-a^2}\left(\frac{\sin(r\sqrt{k^2-a^2})}{r\sqrt{k^2-a^2}}-\cos(r\sqrt{k^2-a^2})\right)da\\ &+\frac{1}{r^2}\int_k^\infty\frac{P_2(a)\sin(az)+Q_2(a)\cos(az)}{a^2-k^2}\left(\frac{\sinh(r\sqrt{a^2-k^2})}{r\sqrt{a^2-k^2}}-\cosh(r\sqrt{a^2-k^2})\right)da,\end{aligned} \tag{38}$$

where $P_1(a), Q_1(a)$ and $P_2(a), Q_2(a)$ are arbitrary absolutely integrable functions of the argument a in the domains $0\le a\le k$ and $k\le a<\infty$, respectively.

Formulas (22), (26), (30), and (38) give a class of exact solutions to the Navier-Stokes equations describing cooling axially symmetric fluids.

Consider now the Navier-Stokes equation (4) in the case of a cooling fluid that is not axially symmetric. As is well known, after excluding the function q in this equation by applying the operator rot, it acquires the following form [2]:

$$\frac{\partial}{\partial t}\,\mathrm{rot}\,\mathbf{v} = \mathrm{rot}[\mathbf{v}\times\mathrm{rot}\,\mathbf{v}] + \nu\Delta\,\mathrm{rot}\,\mathbf{v}\,. \tag{39}$$

Using formula (20) for the vector $\mathbf{v}$, from this equation we obtain

$$\mathrm{rot}[\mathbf{V}\times\mathrm{rot}\,\mathbf{V}] = 0, \quad k_0\mathrm{rot}\,\mathbf{V} + \nu\Delta\,\mathrm{rot}\,\mathbf{V} = 0\,, \tag{40}$$

where $\mathbf{V}$ is the vector with the components V_1, V_2, V_3.

Examine the following case:

$$\mathrm{rot}\,\mathbf{V} = k_1\mathbf{V}, \quad k_1 = \mathrm{const} \neq 0 \tag{41}$$

and use the well-known equalities

$$\mathrm{div}\,\mathrm{rot}\,\mathbf{V} = 0, \quad \mathrm{rot}\,\mathrm{rot}\,\mathbf{V} = \mathrm{grad}\,\mathrm{div}\,\mathbf{V} - \Delta\mathbf{V}\,. \tag{42}$$

From (41) and (42) we find

$$\mathrm{div}\,\mathbf{V} = (1/k_1)\mathrm{div}\,\mathrm{rot}\,\mathbf{V} = 0, \quad \mathrm{rot}\,\mathrm{rot}\,\mathbf{V} = -\Delta\mathbf{V} = k_1\mathrm{rot}\,\mathbf{V} = k_1^2\mathbf{V}\,. \tag{43}$$

Therefore, from equality (41) we derive

$$\mathrm{div}\,\mathbf{V} = 0, \quad \Delta\mathbf{V} + k_1^2\mathbf{V} = 0\,. \tag{44}$$

It should be noted that the first equality in (44) is equation (2) for incompressible fluids.

Substitute now equality (41) into equations (40). Then the first of them is identically satisfied and from the second equation in (40), we obtain

$$(k_0/\nu)\mathbf{V} + \Delta\mathbf{V} = 0\,. \tag{45}$$

It is evident that equation (45) coincides with the second equality in (44) in the case

$$k_1^2 = k_0/\nu\,. \tag{46}$$

Therefore, we come to the following result: Vector functions $\mathbf{V}$ fulfilling the differential equation (41), where the parameter k_1 is determined by formula (46), satisfy the differential equations (40) under consideration and equation (2) for incompressible fluids. Thus, solving the differential equation (41) of the first order, we can obtain a class of solutions to the Navier-Stokes equations.

It should be noted that in the axially symmetric case, the differential equation (41) for the vector $\mathbf{V}$ acquires the form of equations (26), taking into account formulas (5) and (22). As to equations (26), their exact solutions are given by formulas (30) and (38).

Further, we will consider two other cases in which exact solutions to the Navier-Stokes equations can be found.

4 Nonstationary rotation with axial motion of a viscous fluid

Consider the system of equations (15), describing incompressible axially symmetric viscous fluids, in the following case:

$$\alpha = \alpha(t,\rho), \quad \beta = 0, \quad \gamma = \gamma(t,\rho)\,. \tag{47}$$

Then from equations (15) we obtain

$$\alpha_t - \nu(\alpha_{\rho\rho} + 3\alpha_\rho/\rho) = 0\,, \tag{48}$$

$$\varphi_t - \nu(\varphi_{\rho\rho} + 3\varphi_\rho/\rho) = 0, \quad \varphi = -\gamma_\rho/\rho. \tag{49}$$

Let us seek particular solutions to equation (48) in the form

$$\alpha = \exp(-bt)h(\rho), \quad b = \mathrm{const} > 0\,, \tag{50}$$

where $h(\rho)$ is some differentiable function.

Then equation (48) gives

$$h''(\rho) + 3h'(\rho)/\rho + (b/\nu)h(\rho) = 0\,. \tag{51}$$

Let us put

$$\varsigma = \sqrt{b/\nu}\,\rho, \quad h(\rho) = H(\varsigma)/\varsigma\,. \tag{52}$$

Then equation (51) acquires the form

$$\varsigma^2 H''(\varsigma) + \varsigma H'(\varsigma) + (\varsigma^2 - 1)H(\varsigma) = 0\,. \tag{53}$$

As is well known, this equation has the following solution with no singularity at $\rho = 0$:

$$H = H_0 J_1(\varsigma), \quad H_0 = \mathrm{const}\,. \tag{54}$$

It should be noted that the Bessel function $J_1(\varsigma)$ is zero at $\varsigma = 0$.
Using (50), (52), and (54), we find the following solutions to equation (48) having no singularity at $\rho = 0$:

$$\alpha = \frac{1}{\rho}\int_0^\infty M(b)\exp(-bt)\frac{J_1(\rho\sqrt{b/\nu})}{\sqrt{b/\nu}}db\,, \tag{55}$$

where $M(b)$ is an arbitrary absolutely integrable function.
Since equations (48) and (49) are of the same form, from (49) we analogously derive

$$\varphi \equiv -\frac{\gamma_\rho}{\rho} = \frac{1}{\rho}\int_0^\infty N(b)\exp(-bt)\frac{J_1(\rho\sqrt{b/\nu})}{\sqrt{b/\nu}}db\,, \tag{56}$$

where $N(b)$ is an arbitrary absolutely integrable function.
As is well known, $J_0'(x) = -J_1(x)$ and $J_0(0) = 1$. Therefore, from (56) we obtain the following expression for the function $\gamma(t,\rho)$:

$$\gamma = \int_0^\infty N(b)\exp(-bt)\frac{J_0(\rho\sqrt{b/\nu})-1}{b/\nu}db + \gamma_0(t)\,, \tag{57}$$

where $\gamma_0(t)$ is some differentiable function which can be arbitrary.
The obtained formulas (55) and (57) give a class of exact solutions of form (47) to the differential equations (15) under consideration.

5 Differential rotation of stars and gas giants

Examine equations (15) in the following case:

$$\alpha = \alpha(t,\rho), \quad \beta = \beta(t), \quad \gamma = z\lambda(t)\,, \tag{58}$$

where α, β, λ are some differentiable functions. In this case, the stress tensor components σ_{xz} and σ_{yz} are zero.
Substituting expressions (58) into equations (15) and taking into account that $\varphi \equiv \beta_z - \gamma_\rho/\rho = 0$, we obtain

$$\lambda(t) = -2\beta(t)\,, \tag{59}$$

$$\beta(t)(\rho\alpha_\rho + 2\alpha) = \nu(\alpha_{\rho\rho} + 3\alpha_\rho/\rho) - \alpha_t\,. \tag{60}$$

Consider now the following stationary case:

$$\beta = \beta_0 = \text{const}, \quad \lambda = -2\beta_0, \quad \alpha = \alpha(\rho)\,. \tag{61}$$

In this case, from equation (60) we obtain

$$\beta_0(\rho\alpha' + 2\alpha) = \nu(\alpha'' + 3\alpha'/\rho), \quad \alpha = \alpha(\rho), \quad \beta_0 = \text{const}\,. \tag{62}$$

Equation (62) can be represented as follows:

$$\beta_0\rho(\rho\alpha' + 2\alpha) = \nu(\rho\alpha' + 2\alpha)'\,. \tag{63}$$

This gives

$$u'/u = \beta_0\rho/\nu, \quad u = \rho\alpha' + 2\alpha\,. \tag{64}$$

Therefore,

$$u = \rho\alpha' + 2\alpha = u_0\exp(\beta_0\rho^2/2\nu), \quad u_0 = \text{const}\,. \tag{65}$$

This equation has the following solution nonsingular at $\rho = 0$:

$$\alpha = (a_0/\rho^2)[\exp(\beta_0\rho^2/2\nu) - 1], \quad a_0 = \nu u_0/\beta_0 = \text{const}\,. \tag{66}$$

As is seen from formulas (5), the value α is the angular velocity of a rotating viscous fluid. Therefore, in the obtained solution (66) the angular velocity depends on the distance $\rho = \sqrt{x^2 + y^2}$ from the axis *z*.
Thus, the considered rotation about the axis *z* of a viscous fluid can differ from that of a solid. This phenomenon is called differential rotation. As is well known, the phenomenon of differential rotation is observed at the visible surfaces of stars and gas giant planets [11-14].
As follows from (66), when $\beta_0 > 0$, the function $|\alpha(\rho)|$ is increasing and when $\beta_0 < 0$ it is decreasing. When $\rho = 0$, formula (66) gives

$$\alpha(0) = a_0\beta_0 / 2\nu = u_0 / 2. \quad (67)$$

Let us apply formula (66) for the angular velocity of a rotating fluid to the surfaces of stars and gas giant planets. It is well known that typically at their surfaces the angular velocity $|\alpha(\rho)|$ decreases with increased latitude [11-14] and hence it is an increasing function of the argument ρ . Therefore, as a rule, at the surfaces of stars and gas giants the value of β_0 should be positive. Below we will give an explanation for this property of rotating surfaces of stars and gas planets.

Using formulas (5), (58), (61), and (66), we come to the following solution to the Navier-Stokes equations:

$$\begin{gathered} v_1 = -\alpha(\rho)y + \beta_0 x, \quad v_2 = \alpha(\rho)x + \beta_0 y, \quad v_3 = -2\beta_0 z, \\ \alpha(\rho) = (a_0 / \rho^2)[\exp(\beta_0\rho^2 / 2\nu) - 1], \quad a_0, \beta_0 = \text{const}, \quad \rho = \sqrt{x^2 + y^2}, \end{gathered} \quad (68)$$

where v_1, v_2, v_3 are the projections of the vector of velocity **v** onto the axes *x, y, z*, respectively.

Consider now the gas giant planets of the solar system and apply formulas (68) to their surfaces. As is well known, the gas giant planets have systems of rings in their equatorial planes. The velocities of particles of the rings of a gas planet are substantially larger than those of particles of the planet surface. That is why particles leaving the rings and falling on the planet should accelerate its surface mainly in the equatorial region. In this case, the value of the parameter β_0 in formulas (68) is positive and formulas (68) describe a differential rotation of the gas planet.

It should be stressed that up to now, there is no explanation for the phenomenon of differential rotation of the visible surfaces of stars and gas giant planets that is based only on their intrinsic forces. Therefore, the idea of the influence on their differential rotation of external forces looks quite attractive. Relying on this idea, we can suppose that not only gas giant planets but also the Sun and many other stars having differential rotations are surrounded by rings in their equatorial regions. This hypothesis allows one to give the following interpretation of differential rotations of the surfaces of stars and gas giant planets: The differential rotations could be regarded as results of accelerating actions of continuous streams of particles leaving the rings of stars and gas giants and falling on their surfaces.

It should be noted that formulas (68) describe not only a differential rotation of the surface of a viscous fluid but also a change of its shape. However, for a star or a gas giant planet, this change should proceed very slowly since, as follows from (68), for it the value of β_0 is $\sim \nu / R^2$, where *R* is its radius. For example, consider the Sun. At its surface, the value of ν is $\sim 2 \cdot 10^3 \text{ cm}^2/\text{sec}$ and the radius *R* of the Sun is $\approx 7 \cdot 10^{10}$ cm. Therefore, in this case, $\beta_0 \sim \nu / R^2 \sim 4 \cdot 10^{-19} \text{ sec}^{-1} \approx 1.26 \cdot 10^{-11} \text{ year}^{-1}$. That is why substantial changes of the Sun dimensions that could be caused by formulas (68) might take place only after billions of years.

6 Conclusion

In the paper, we have considered the Navier-Stokes equations for incompressible viscous fluids in the case of axially symmetry and reduced them to the system of nonlinear partial differential equations (15) for components of the vector of velocity **v**. After studying them, we found three exact solutions to the Navier-Stokes equations. One of them obtained in section 3 describes the case in which a rotating fluid is gradually cooling. In section 4, we found an exact solution corresponding to nonstationary rotations with axial motions of viscous fluids. In section 5, an exact solution to the Navier-Stokes equations was obtained which could describe a differential rotation of a viscous fluid in which points of its surface have different angular velocities. This solution was applied to explain the well-known phenomenon of differential rotation of the visible surfaces of stars and gas giant planets. It was shown that streams of particles leaving the rings of gas giant planets and falling on their surfaces could play an important role to maintain their differential rotation. This led us to the hypothesis that not only gas giant planets but also the Sun and many other stars having substantial differential rotations could be surrounded by systems of rings in their equatorial planes. Because of their great luminosity, it is very difficult to observe such rings. However, it could be supposed that the hypothetical rings of the Sun maintaining its differential rotation will be detected in the future.

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