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Research Paper

Construction of generalized atomic decompositions in Banach spaces

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Abstract

G-atomic decompositions for Banach spaces with respect to a model space of sequences have been introduced and studied as a generalization of atomic decompositions. Examples and counter example have been provided to show its existence. It has been proved that an associated Banach space for G-atomic decomposition always has a complemented subspace. The notion of a representation system is introduced and exhibits its relation with G-atomic decomposition. Also It has been observed that G-atomic decompositions are exactly compressions of Schauder decompositions for a larger Banach space. We give a characterization for finite G-atomic decomposition in terms of finite-dimensional expansion of identity.

Keywords: *complemented coefficient spaces, finite-dimensional expansion of identity, G-atomic decomposition, representation system.*

1. Introduction

Frames are main tools for use in signal and image processing, compression, sampling theory, optics, filter banks, signal detection etc. In order to have many more uses of frames, several notions generalizing the concept of frames have been introduced and studied, namely; Banach frames [13], pseudo frames [15], oblique frames [7], frames of subspaces (fusion frames) [4, 5], G-frames [20] etc.

Coifman and Weiss [9] introduced a concept, similar to that of frames, called atomic decompositions for function spaces. Later, the concept of frames in Hilbert spaces was extended to Banach spaces by Feichtinger and Grochenig [11] who introduced the concept of atomic decompositions in Banach spaces. This concept was further generalized by Grochenig [13] who introduced the notion of Banach frames for Banach spaces. Frazier and Jawerth [12] had constructed wavelet atomic decompositions for Besov spaces, which they called "ϕ-transform". Feichtinger [10] constructed Gabor atomic decomposition for the modulation spaces. Christensen [6] in 1996, studied atomic decomposition via group representation, while Christensen and Heil [8], in 1997, discussed stability of atomic decompositions for Banach spaces under small perturbations. Casazza, Han and Larson [3] relate an atomic decomposition with several forms of the approximation property in Banach space theory and with Banach frames. Banach frames and atomic decompositions were further studied in [1, 2].

In this article we generalize the classical construction of Pelczynski [16]. In fact, we introduce the notion of G-atomic decomposition for Banach spaces. It has been proved that an associated Banach space, for G-atomic decomposition always has a complemented subspace. We define representation system and exhibits its relation with

G-atomic decomposition. Also, it has been observed that G-atomic decompositions are exactly compressions of Schauder decompositions for a larger Banach space. We give a characterization for finite G-atomic decomposition in terms of finite-dimensional expansion of identity.

2. Preliminaries

Throughout this paper, E will denote a Banach space over the scalar field $\mathbb{K}$ ($\mathbb{R}$ or $\mathbb{C}$), E^* the conjugate space of E, and $L(E,F)$ will denote the Banach space of all continuous linear operators from E into F.

A sequence $\{x_n\}$ in E is said to be *complete* if $[x_n] = E$ and a sequence $\{f_n\}$ in E^* is said to be *total* over E if $\{x \in E : f_n(x) = 0,\ n \in \mathbb{N}\} = \{0\}$. In the case where $F = E$, we write $L(E) = L(E,E)$. A sequence $\{v_n\} \subset L(E)$ is said to be total on E if $v_n(x) = 0$, for all $n \in \mathbb{N}$, implies $x = 0$.

By a Banach sequence space (often called a BK-space) we mean a Banach space of scalar sequences, indexed by $\mathbb{N}$, for which the coordinate functionals are continuous. We say that the space is a Schauder sequence space if, in addition, the unit vectors (e_i) given by $(e_i)_j = \delta_{ij}$(where δ_{ij} is the Kronecker delta) form a basis for it.

Definition 2.1 ([11])**.** *Let E be a Banach space and let E_d be an associated Banach space of scalar-valued sequences indexed by $\mathbb{N}$. Let $\{x_n\}$ be a sequence in E and let $\{f_n\}$ be a sequence in E^*. Then, the pair $(\{f_n\},\{x_n\})$ is called an atomic decomposition for E with respect to E_d, if*

(a) $\{f_n(x)\} \in E_d$, *for all* $x \in E$

(b) *there exist constants* A, B *with* $0 < A \leq B < \infty$ *such that*

$$A\|x\|_E \leq \|\{f_n(x)\}\|_{E_d} \leq B\|x\|_E, \quad x \in E$$

(c) $x = \sum_{n=1}^{\infty} f_n(x)x_n$, *for all* $x \in E$.

The positive constants A, B are called atomic bounds for the atomic decomposition $(\{f_n\},\{x_n\})$.

Definition 2.2 ([18])**.** *A sequence $\{G_n\}$ of subspaces of E is a decomposition of E if for each $x \in E$ there exists a unique sequence $\{x_n\}$ in E such that*

$$x = \sum_{n=1}^{\infty} x_n, \quad x_n \in G_n \text{ for each } n,$$

the convergence being in the norm topology of E. Uniqueness implies the existence of projections (not necessarily continuous) v_n from E onto G_n such that $v_iv_j = \delta_{ij}v_j$, where δ_{ij} is the Kronecker delta. If each v_n is continuous, then decomposition is called a Schauder decomposition.

Let $(\{f_n\},\{x_n\})$ be an atomic decomposition for E with respect to a Banach sequence space $\mathcal{Z}$. There is a natural procedure that allows us to replace $\mathcal{Z}$ by a Schauder sequence space so that $(\{f_n\},\{x_n\})$ is also, an atomic decomposition of E with respect to E_d (see [3, Theorem 2.6]).

3. Main results

The theory of spaces of sequences of scalars admits a natural generalization to a vector sequence spaces. If $\Phi = \{G_n\}$ is a sequence of Banach spaces, a sequence space X_Φ associated with $\{G_n\}$ is a linear subspace of $\prod_{n=1}^{\infty} G_n$ (the collection of all sequences $\{y_n\}$ with $y_n \in G_n$, $n = 1, 2, \ldots$, endowed with product topology). The coordinate operators $P_n : X_\Phi \to G_n$ are defined by $P_n(\{y_i\}) = y_n$, $n = 1, 2, \ldots$. Then X_Φ is called a generalized BK-space induced by $\{G_n\}$ if X_Φ is a Banach space and P_n is a continuous operator on X_Φ, for every $n \in \mathbb{N}$. The scalar BK-spaces containing all unit vectors e_n are generalized by the spaces X_Φ containing all canonical subspaces

$$F_n = \{0\} \times \ldots \times \{0\} \times \underset{\substack{\downarrow \\ n\text{th place}}}{G_n} \times \{0\} \times \ldots \quad (G_n \neq \{0\}, n = 1, 2 \ldots).$$

These F_n's are closed linear subspaces of X_Φ. We refer to the space X_Φ as a model space.

The following is the example of such type of a model space.

Let $\Phi = \{G_n\}$ be a sequence of closed linear subspaces of a Banach space E. Consider the linear space X_Φ of the system Φ, that is, the space of all element sequences $y = \{y_n\}_{n=1}^{\infty}$ for which the series $\sum_{n=1}^{\infty} y_n$ is convergent equipped with the norm

$$\|y\|_{X_\Phi} = \sup_{n\geq 1} \left\| \sum_{k=1}^{n} y_k \right\|_E, \quad y_n \in G_n \ (n = 1, 2 \ldots).$$

The space X_Φ is complete with respect to this norm and the system $\{F_n\}$ defined by above is a Schauder decomposition of X_Φ. Clearly, any model space X_Φ can be obtained by the method described above, indeed, if X is a model space of the sequence of subspaces $\Phi = \{F_n\}$ then $X_\Phi = X$.

We begin with the following generalization of Atomic decomposition.

Definition 3.1. *Let $\Phi = \{G_n\}$ be a sequence of non-trivial subspaces of a Banach space E and $\{v_n : v_n \in L(E, G_n)\}$ be a sequence of linear operators (not necessarily projections). Let X_Φ be a model space associated with E. Then we say $(\{G_n\}, \{v_n\})$ is G-atomic decomposition for E with respect to X_Φ if*

(a) *$\{v_n(x)\} \in X_\Phi$, for all $x \in E$*

(b) *there exist constants A, B with $0 < A \leq B < \infty$ such that*

$$A\|x\|_E \leq \|\{v_n(x)\}\|_{X_\Phi} \leq B\|x\|_E, \quad x \in E$$

(c) *$x = \sum_{n=1}^{\infty} v_n(x)$, for all $x \in E$.*

The positive constants A and B, respectively, are called lower and upper atomic bounds for the G-atomic decomposition $(\{G_n\}, \{v_n\})$.

Next, we have following lemma on the line of [19, p. 189], which used in the subsequent work.

Lemma 3.2. *Let $\{G_n\}$ be a sequence of subspaces of E and $\{v_n\} \subset L(E, G_n)$ be a sequence of operators, $\forall\, n \in \mathbb{N}$. If $\{v_n\}$ is total over E, then $X = \{\{v_n(x)\} : x \in E\}$ is a Banach space with norm $\|\{v_n(x)\}\|_X = \|x\|_E, x \in E$.*

Regarding existence of G-atomic decompositions, we have the following examples. The modified sequence $\{G_n\}$ used below was constructed in [14].

Example 3.3. *Consider the Banach space*

$$E = \ell^\infty(\chi) = \{\{x_n\} : x_n \in \chi;\ \sup_{1\leq n<\infty} \|x_n\|_\chi < \infty\}$$

equipped with the norm $\|\{x_n\}\|_E = \sup_{1\leq n<\infty} \|x_n\|_\chi$, $\{x_n\} \in E$, where $(\chi, \|\cdot\|)$ is a Banach space.

Define a sequence $\{G_n\}$ of subspaces of E by

$$G_{2n-1} = \{\delta_{2n-1}^x + 2^n \delta_{2n}^x : x \in \chi\}$$
$$G_{2n} = \{\delta_{2n}^x : x \in \chi\}$$

where

$$\delta_n^x = (0, 0, \ldots, 0, \underset{\substack{\downarrow \\ nth\ place}}{x}, 0, \ldots) \quad \text{for all } n \in \mathbb{N} \text{ and } x \in \chi.$$

Define operators $v_n : \ell^\infty(\chi) \to \ell^\infty(\chi)$ by

$$v_{2n-1}(x) = \delta_{2n-1}^{x_{2n-1}} + 2^n \delta_{2n}^{x_{2n-1}}$$
$$v_{2n}(x) = 2^n \delta_{2n}^{(\frac{1}{2^n} x_{2n} - x_{2n-1})} \quad \text{for all } x = \{x_n\} \in E \text{ and } n \in \mathbb{N}.$$

Then, by Lemma 3.2, there exists an associated model space $X = \{v_n(x) : x \in E\}$ with norm given by

$$\|\{v_n(x)\}\|_X = \|x\|_E, \quad x \in E.$$

Also

$$\sum_{n=1}^{\infty} v_n(x) = x, \quad x \in E.$$

Therefore $(\{G_n\}, \{v_n\})$ *is* G*-atomic decomposition of* E *with respect to model space* X.

Example 3.4. *Let* $E = c_0$ *and* $\{e_n\}$ *be the unit vector basis in* c_0. *Write*

$$G_n = [x_n] \quad \text{and} \quad v_n(x) = f_n(x)x_n, \quad n \in \mathbb{N},$$

where $\{x_n\} \subset E$ *and* $\{f_n\} \subset E^*$ *are given by*

$$x_{2n-1} = 2^{1-n} e_{2n-1} - e_{2n}, \quad x_{2n} = e_{2n} \quad (n \in \mathbb{N})$$
$$f_{2n-1} = 2^{n-1} h_{2n-1}, \quad f_{2n} = 2^{n-1} h_{2n-1} + h_{2n} \quad (n \in \mathbb{N}),$$

$\{h_n\}$ *being the sequence of coordinate functionals to* $\{e_n\}$. *Then, it can be easily prove that there exist an associated model space* $X = \{v_n(x) : x \in E\}$, *such that* $(\{G_n\}, \{v_n\})$ *is* G*-atomic decomposition for* E *with respect to* X.

Example 3.5. *Let* E *be a Banach space defined as*

$$E = \ell^2(\chi) = \left\{ \{x_n\} : x_n \in \chi; \sum_{n=1}^{\infty} \|x_n\|_{\chi}^2 < \infty \right\},$$

where $(\chi, \|\cdot\|)$ *is a Banach space, equipped with the norm given by*

$$\|\{x_n\}\|_E = \left(\sum_{n=1}^{\infty} \|x_n\|_{\chi}^2 \right)^{\frac{1}{2}}.$$

Define for $n \in \mathbb{N}$, $G_n = \{\delta_1^x + \delta_{n+1}^x : x \in \chi\}$ *and* $v_n(x) = \delta_1^{x_{n+1}} + \delta_{n+1}^{x_{n+1}}$, $x = \{x_n\} \in E$, *where* $\delta_n^x = (0, 0, \ldots, 0, \underset{nth\ place}{\underset{\downarrow}{x}}, 0, \ldots)$ *for all* $n \in \mathbb{N}$ *and* $x \in \chi$. *But, since for any* $0 \neq x \in \chi$, $\delta_1^x = (x, 0, 0, \ldots) \in E$ *is such that* $v_n(\delta_1^x) = 0$, *for all* $n \in \mathbb{N}$, *there exist no associated model space* X *such that* $(\{G_n\}, \{v_n\})$ *is a* G*-atomic decomposition for* E *with respect to* X.

Remark 3.6. *Any Banach space* E *admits the trivial* G*-atomic decomposition* $\{G_n\}$, *where* $G_1 = E$ *and* $G_n \neq \{0\}$ $(n = 2, 3, \ldots)$ *are arbitrary with operators* $v_1 = I_E$, $v_n = 0$ $(n = 2, 3, \ldots)$.

Theorem 3.7. *If* $(\{G_n\}, \{v_n\})$ *is a* G*-atomic decomposition for* E *with respect to* X_Φ, *then there exist a complemented coefficient subspace* G *of* X_Φ *and an isomorphism* T *from* E *into* X_Φ *such that* $X_\Phi = T(E) \oplus G$.

Proof. Let $(\{G_n\}, \{v_n\})$ be a G-atomic decomposition of E with respect to X_Φ
where

$$X_\Phi = \left\{ \{y_n\} \subset E \,\middle|\, \sum_{n=1}^{\infty} y_n \text{ converges}; y_n \in G_n \ (n = 1, 2, \ldots) \right\} \tag{1}$$

equipped with norm $\|\{y_n\}\|_{X_\Phi} = \sup\limits_{1 \leq n < \infty} \left\| \sum\limits_{i=1}^{n} y_i \right\|$.

Then the mapping $T : E \to X_\Phi$ defined by

$$T(x) = \{v_n(x)\}, \quad x \in E$$

is an isomorphism from E into X_Φ. Since $\sum\limits_{n=1}^{\infty} v_n(x)$ converges to x by (1) and

$$\begin{aligned} \|x\|_E &= \left\| \sum_{i=1}^{\infty} v_i(x) \right\| \leq \sup_{1 \leq n < \infty} \left\| \sum_{i=1}^{n} v_i(x) \right\| \\ &= \|\{v_n(x)\}\|_{X_\Phi} \leq B\|x\|_E, \quad x \in E, \end{aligned}$$

where

$$B = \sup_{1\le n<\infty} \|S_n\| < \infty \quad \text{and} \quad S_n(x) = \sum_{i=1}^{n} v_i(x).$$

Now, define $S : X_\Phi \to E$ by

$$S(\{x_n\}) = \sum_{i=1}^{\infty} x_i, \quad \{x_n\} \in X_\Phi,\ n \in \mathbb{N}.$$

Then S is a bounded linear operator from X_Φ to E. Put $G = \ker S$. Then

$$G = \left\{ \{x_n\} \subset E \,\middle|\, x_n \in G_n (n = 1, 2, \ldots), \sum_{i=1}^{\infty} x_i = 0 \right\},$$

is a closed subspace of X_Φ.

Furthermore, if $\{v_n(x)\} \in G$ for some $x \in E$, then

$$0 = S(\{v_n(x)\}) = \sum_{n=1}^{\infty} v_n(x) = x.$$

So

$$T(E) \cap G = \{0\}.$$

Now, let $\{x_n\} \in X_\Phi$ be arbitrary such that $x = \sum\limits_{i=1}^{\infty} x_i$. Then $\{v_n(x)\} \in T(E)$ such that

$$\begin{aligned}\sum_{i=1}^{\infty}(x_i - v_i(x)) &= \sum_{i=1}^{\infty} x_i - \sum_{i=1}^{\infty} v_i(x)\\ &= x - x = 0.\end{aligned}$$

Therefore $\{x_n - v_n(x)\} \in G$ such that

$$\{x_n\} = \{v_n(x)\} + \{x_n^{(0)}\}, \quad \text{where } \{v_n(x)\} \in T(E) \text{ and } \{x_n^{(0)}\} = \{x_n - v_n(x)\} \in G.$$

Hence, we have $X_\Phi = T(E) \oplus G$. □

Definition 3.8. *A system $\Phi = \{G_n\}$ of closed linear subspaces of a Banach space E, with $G_n \neq \{0\}$ $(n = 1, 2, \ldots)$ is called a representation system of E with respect to model space X_Φ if for every $x \in E$, there exists a sequence $\{x_n\} \subset E$ with $x_n \in G_n$ $(n = 1, 2, \ldots)$ such that $x = \sum\limits_{n=1}^{\infty} x_n$ and $G = \left\{ \{x_n\} \subset E \,\middle|\, \sum\limits_{n=1}^{\infty} x_n = 0 \right\}$ is a complemented coefficient subspace of X_Φ.*

Theorem 3.9. *Let E be a Banach space and X_Φ be an associated Banach space indexed by $\mathbb{N}$. Then $\Phi = \{G_n\}$ is a representation system if and only if $(\{G_n\}, \{v_n\})$ is a G-atomic decomposition with respect to X_Φ.*

Proof. **Necessity.** Let $\Phi = \{G_n\}$ be a representation system of E then for $\{x_n\} \subset E$ with $x_n \in G_n$, we have $x = \sum\limits_{n=1}^{\infty} x_n$, $n \in \mathbb{N}$. Let $G = \left\{ \{x_n\} \subset E \,\middle|\, \sum\limits_{n=1}^{\infty} x_n = 0 \right\}$ be a complemented coefficient subspace of X_Φ then $X_\Phi = G \oplus F$ and F is complemented to G. Define $S : X_\Phi \to E$ by

$$S(\{x_n\}) = \sum_{n=1}^{\infty} x_n, \quad \{x_n\} \in X_\Phi, n \in \mathbb{N}.$$

As in Theorem 3.7, T is an isomorphism from E into X_Φ, then $S|_F$ is an isomorphism from F onto E. Indeed, if $S(\{x_n\}) = 0$ for some $\{x_n\} \in F$, then $\sum\limits_{n=1}^{\infty} x_n = 0$. Hence $\{x_n\} \in G \cap F = \{0\}$, which proves that $S|_F$ is one to one. Also, if $y \in E$ then, since Φ is a representation system, there exists a sequence $\{y_n\} \in X_\Phi$ such that $y = \sum_{n=1}^{\infty} y_n = S(\{y_n\})$, write

$$\{y_n\} = \{x_n^{(0)}\} + \{x_n\}, \quad \text{where } \{x_n^{(0)}\} \in G,\ \{x_n\} \in F.$$

Then $y = S(\{x_n^{(0)}\}) + S(\{x_n\}) = S(\{x_n\})$, which proves that $S|_F$ is onto.

Now let $x \in E$ be an arbitrary element and let $\{v_n(x)\} = (S|_F)^{-1}(x) \in F$ then

$$\{x_n\} = \{v_n(x)\} + \{x_n^{(0)}\}, \quad \{x_n^{(0)}\} \in G.$$

So, we have

$$S(\{x_n\}) = S(\{v_n(x)\}).$$

Therefore

$$x = S(\{v_n(x)\}) = \sum_{n=1}^{\infty} v_n(x).$$

Since, $F \subset X_\Phi$, we have $v_n(x) \in G_n$, $x \in E$, $n \in \mathbb{N}$.

$$x = S(\{v_n(x)\}) = \sum_{n=1}^{\infty} v_n(x), \quad x \in E,$$

and each v_n is linear on E and satisfies

$$\begin{aligned}
\|v_n(x)\| &\leq 2 \sup_{1\leq k<\infty} \left\| \sum_{i=1}^{k} v_i(x) \right\| \\
&= 2\|\{v_n(x)\}\| \\
&\leq 2\|(S|_F)^{-1}\|\|x\| \quad (x \in E, n = 1, 2, \ldots).
\end{aligned}$$

Also, by the the principle of uniform boundedness,

$$\begin{aligned}
\|x\|_E &= \left\| \sum_{n=1}^{\infty} v_n(x) \right\| \\
&\leq \sup_{1\leq n<\infty} \left\| \sum_{i=1}^{n} v_i(x) \right\| \\
&= \|\{v_n(x)\}\|_{X_\Phi} \leq B\|x\|_E,
\end{aligned}$$

where $B = \sup_{1\leq n<\infty} \|S_n\| < \infty$ and $S_n(x) = \sum_{i=1}^{n} v_i(x)$. Therefore $(\{G_n\}, \{v_n\})$ is a G-atomic decomposition of E with respect to X_Φ.

Sufficiency, follows with the argument of the proof of Theorem 3.7. □

In the following result, we show that an G-atomic decomposition for a Banach space produces another G-atomic decomposition for the space.

Theorem 3.10. *If $(\{G_n\}, \{v_n\})$ is a G-atomic decomposition for E with respect to X_Φ, then there exists a projection P of X_Φ onto $T(E)$ along G such that $(\{T^{-1}P(F_n)\}, \{v_n\})$ is an G-atomic decomposition for E with respect to X_Φ, where $\{F_n\}$ is the Schauder decomposition of X_Φ.*

Proof. Let P be a projection of X_Φ onto $T(E)$ along G. Then

$$P(\{x_n\}) = \left\{ v_n\left(\sum_{i=1}^{\infty} x_i \right) \right\}, \quad \{x_n\} \in X_\Phi. \tag{2}$$

Since for every $\{v_n(x)\} \in T(E)$, we have

$$P(\{v_n(x)\}) = \{v_n(x)\} = \left\{ v_n \left(\sum_{i=1}^{\infty} v_i(x) \right) \right\}$$

and since for every $\{x_n\} \in G$ we have

$$P(\{x_n\}) = 0 = \{v_n(0)\} = \left\{ v_n \left(\sum_{i=1}^{\infty} x_i \right) \right\}.$$

By (2) we have, in particular, for any $\{\delta_{nk} x_n\} \in F_k$, $k = 1, 2, \dots$

$$P(\{\delta_{nk} x_n\}) = \left\{ v_n \left(\sum_{i=1}^{\infty} \delta_{ik} x_i \right) \right\} = \{v_n(x_k)\} = T(x_k)$$

where

$$\delta_{ik} = \begin{cases} 1 & \text{if } i = k \\ 0 & \text{if } i \neq k. \end{cases}$$

Since T is invertible, then $x_k = T^{-1}(P(\{\delta_{nk} x_n\}))$, $x_k \in G_k$, $k = 1, 2, \dots$. Therefore $G_k = T^{-1} P(F_k)$, $k = 1, 2, \dots$.
Hence $(\{T^{-1} P(F_n)\}, \{v_n\})$ is a G-atomic decomposition for E with respect to model space X_Φ. □

In next theorem we want to classify G-atomic decomposition in terms of bases of subspaces (Schauder decomposition) for Banach space.

Theorem 3.11. *If D is any Banach space with Schauder decomposition $\{F_n\}$ and an isomorphism T from E into D and a projection P from D onto $T(E)$ such that $G_n = T^{-1} P(F_n)$. Then, there exists an associated sequence of operators $\{v_n\}$ such that $(\{G_n\}, \{v_n\})$ is a G-atomic decomposition for E.*

Proof. Since $\{F_n\}$ is a Schauder decomposition for D. Assume $\{u_n\} \subset L(D, D)$ is an associated sequence of coordinate projection to $\{F_n\}$. Then, for $y \in P(D)$

$$y = P(y) = P\left(\sum_{n=1}^{\infty} u_n(y) \right) = \sum_{n=1}^{\infty} P(u_n(y)). \tag{3}$$

Since $Pu_n|_{P(D)} \in L(P(D), P(F_n))$, $n = 1, 2 \dots$. Therefore $(\{P(F_n)\}, \{Pu_n\})$ is G-atomic decomposition of $P(D) = T(E)$ by (3).
Now, T is an isomorphism from E onto $T(E)$ and $G_n = T^{-1} P(F_n)$. Put $Pu_n = v_n$, $n = 1, 2, \dots$.
It follows that $(\{G_n\}, \{v_n\})$ is G-atomic decomposition for E. □

In the next theorem we generalize Theorem 2.6 [3], which is a classical construction of Pelczynski [16].

Theorem 3.12. *Let E be a Banach space. Then the following are equivalent:*

(i) *There is a Banach space of scalar valued sequences X_Φ, so that $(\{G_n\}, \{u_n\})$ satisfies* Definition 3.1 (*i.e. is an G-atomic decomposition for E*).

(ii) *There is a Banach space D with a Schauder decomposition $\psi = \{\mathcal{A}_n\}$ so that $E \subset D$ and there is a bounded linear projection $P : D \to E$ with $P\{\mathcal{A}_n\} = G_n$, for all $n \in \mathbb{N}$.*

Proof. (i)⇒(ii) This implication is obvious with $D = E$ and $\{\mathcal{A}_n\} = \{G_n\}$ and $P = I_E$ in Theorem 3.10.

(ii)⇒(i) This follows with the argument of the above proof of Theorem 3.11. □

Now we proceed to examine the general relationship between finite-dimensional G-atomic decompositions and approximation property in Banach space theory.
We recall the following definition:

Definition 3.13 ([17]). *A sequence of non zero finite rank operators $\{v_i\}$ from a Banach space E into itself is called a finite dimensional expansion of the identity of E, if*

$$x = \sum_{i=1}^{\infty} v_i(x), \quad x \in E.$$

In view of above definition, we prove the following result.

Theorem 3.14. *A Banach space E has a finite-dimensional G-atomic decomposition $(\{G_n\}, \{v_n\})$ (i.e, such that dim $G_n < \infty$ for all $n = 1, 2, \ldots$) if and only if E admits a finite-dimensional expansion $\{v_n\}$ of the identity of E.*

Proof. Let $(\{G_n\}, \{v_n\})$ be finite dimensional G-atomic decomposition for E. Then $\dim G_n < \infty$ for all $n = 1, 2, \ldots$ and $x = \sum\limits_{n=1}^{\infty} v_n(x)$, $x \in E$.

Therefore, an associated sequence of operators $\{v_n\}$ for $\{G_n\}$ is a finite-dimension expansion of E.

Conversely, if $\{v_n\}$ is a finite-dimensional expansion of the identity of E, then

$$\{G_n\} = \{v_n(E)\}$$

i.e. $(\{G_n\}, \{v_n\})$ is a finite-dimensional G-atomic atomic decomposition of E. □

Remark 3.15. *With the help of above result we can classified finite G-atomic decompositions in terms of several forms of the approximation property for Banach spaces.*

References

[1] D. Carando and S. Lassalle, Duality, reflexivity and atomic decompositions in Banach spaces, *Studia Math.*, 191(1) (2009), 67–80.

[2] P.G. Casazza, O. Christensen and D.T. Stoeva, Frame expansions in separable Banach spaces, J. Math. Anal. Appl., 307 (2005), 710–723.

[3] P.G. Casazza, D. Han and D.R. Larson, Frames for Banach spaces, *Contem. Math.*, 247 (1999), 149–181.

[4] P.G. Casazza and G. Kutyniok, Frames of subspaces, in *Wavelets, Frames and Operator Theory* (College Park, MD, 2003), *Contemp. Math.*, 345, *Amer. Math. Soc.*, Providence, RI, 2004, 87–113.

[5] P.G. Casazza, G. Kutyniok and S. Li, Fusion frames and distributed processing, *Appl. Comput. Harmon. Anal.*, 25 (2008), 114-132.

[6] O. Christensen, Atomic decomposition via projective group representations, *Rocky Mountain J. Math.*, 26(4) (1996), 1289–1312.

[7] O. Christensen and Y. C. Eldar, Oblique dual frames with shift-invariant spaces, *Appl. Compt. Harmon. Anal.*, 17(1) (2004), 48–68.

[8] O. Christensen and C. Heil, Perturbations of Banach frames and atomic decompositions, *Math. Nachr.*, 185 (1997), 33–47.

[9] R.R. Coifman and G. Weiss, Extensions of Hardy spaces and their use in analysis, *Bull. Amer. Math. Soc.*, 83 (1977), 569–645.

[10] H.G. Feichtinger, Atomic characterizations of modulations spaces through Gabor-type representation, *Rocky Mountain J. Math.*, 19(1) (1989), 113–125.

[11] H.G. Feichtinger and K.H. Gröchenig, A unified approach to atomic decompositions via integrable group representations, Function spaces and Applications, *Lecture Notes in Mathematics*, 1302 (1988), 52–73.

[12] M. Frazier and B. Jawerth, Decompositions of Besov spaces, *Indiana Univ. Math. J.*, 34 (1985), 777–799.

[13] K.H. Gröchenig, Describing functions: Atomic decompositions versus frames, *Monatsh. fur Mathematik*, 112(3) (1991), 1–41.

[14] S.K. Kaushik and Varinder Kumar, A note on fusion Banach frames, *Archivum mathematicum(BRNO)*, Tomus 46 (2010), 203–209.

[15] S. Li and H. Ogawa, Pseudo frames for subspaces with applications, *J. Fourier Anal. Appl.*, 10(4) (2004), 409–431.

[16] A. Pelczynski, Any separable Banach space with the bounded approximation property is a complemented subspace of a Banach space with a basis, *Studia Math.*, 40(1971), 239–242.

[17] A. Pelczynski and P. Wojtaszczyk, Banach spaces with finite dimensional expansion of identity and universal bases of finite dimensional subspaces, *Studia Math.*, 40 (1971), 91–108.

[18] B.L. Sanders, On the existence of Schauder decomposition in Banach spaces, *Proc. Amer. Math. Soc.*, 16(1965), 987–990.

[19] I. Singer, Bases in Banach spaces. II, Springer (Berlin, 1981).

[20] W. Sun, G-frames and g-Riesz bases, *J. Math. Anal. Appl.*, 322(1) (2006), 437–452.