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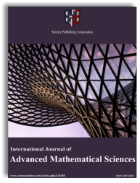

On a Subclass of Multivalent Functions with Bounded Positive Real Part

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Abstract

In the present paper, by introducing a new subclass of multivalent functions with respect to (j, k) - symmetric points, we have obtained the integral representations and conditions for starlikeness using differential subordination.

Keywords: *multivalent functions; (j,k) - symmetric points; Differential subordination.*

1. Introduction, Definitions And Preliminaries

Let $\mathscr{H}$ be the class of functions analytic in the open unit disc $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$. Let $\mathscr{H}(a,m)$ be the subclass of $\mathscr{H}$ consisting of functions of the form $f(z) = z + a_{m+1}z^{m+1} + a_{m+2}z^{m+2} + \cdots$.
Let $\mathscr{A}_p$ be the class of functions $f(z)$, of the form

$$f(z) = z^p + \sum_{n=p+1}^{\infty} a_n z^n \tag{1}$$

which are analytic in the unit disc $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$. And let $\mathscr{A}$=$\mathscr{A}_1$.
We denote by $\mathscr{S}^*, \mathscr{C}, \mathscr{K}$ and $\mathscr{C}^*$ the familiar subclasses of $\mathscr{A}$ consisting of functions which are respectively starlike, convex, close-to-convex and quasi-convex in $\mathbb{U}$.
Let $\mathscr{S}$ be the subclass of $\mathscr{A}$ consisting of all functions which are univalent in $\mathbb{U}$. Also, let $\mathscr{P}$ denote the class of functions of the form

$$p(z) = 1 + \sum_{n=1}^{\infty} c_n z^n$$

which are analytic and convex in $\mathbb{U}$ and satisfy the condition

$$\Re(p(z)) > 0, \ (z \in \mathbb{U}).$$

Let $f(z)$ and $g(z)$ be analytic in $\mathbb{U}$. Then we say that the function $f(z)$ is subordinate to $g(z)$ in $\mathbb{U}$, if there exists an analytic function $w(z)$ in $\mathbb{U}$ such that $|w(z)| < |z|$ and $f(z) = g(w(z))$, denoted by $f(z) \prec g(z)$. If $g(z)$ is univalent in $\mathbb{U}$, then the subordination is equivalent to $f(0) = g(0)$ and $f(\mathbb{U}) \subset g(\mathbb{U})$.
Motivated by the concept introduced by Sakaguchi in [8], recently several subclasses of analytic functions with respect to k-symmetric points were introduced and studied by various authors (see [1], [2], [9], [10] and [12]). Parvatham in ([7]) introduced and investigated $K_n(\alpha, h)$ - so called class of α starlike functions with respect to n symmetric points.
Let k be a positive integer and $j = 0, 1, 2, \ldots (k-1)$. A domain D is said to be (j, k)-fold symmetric if a rotation of D about the origin through an angle $2\pi j/k$ carries D onto itself. A function $f \in \mathscr{A}$ is said to be (j, k)-symmetrical if for each $z \in \mathbb{U}$

$$f(\varepsilon z) = \varepsilon^j f(z), \tag{2}$$

where $\varepsilon = \exp(2\pi i/k)$. The family of (j, k)-symmetrical functions will be denoted by $\mathscr{F}_k^j$. For every function f defined on a symmetrical subset $\mathbb{U}$ of $\mathbb{C}$, there exits a unique sequence of (j, k)-symmetrical functions $f_{j,k}(z), j = 0, 1, \ldots, k-1$ such that

$$f = \sum_{j=0}^{k-1} f_{j,k}.$$

Also let $f_{j,k}(z)$ be defined by the following equality

$$f_{j,k}(z)=\frac{1}{k}\sum_{\nu=0}^{k-1}\frac{f(\varepsilon^{\nu}z)}{\varepsilon^{\nu pj}},\quad (f\in\mathscr{A}_p;\, k=1,2,\ldots;\, j=0,1,2,\ldots(k-1)). \tag{3}$$

where, ν is an integer.
This decomposition is a generalization of the well known fact that each function defined on a symmetrical subset $\mathbb{U}$ of $\mathbb{C}$ can be uniquely represented as the sum of an even function and an odd functions (see Theorem 1 of [5]). It is obvious that $f_{j,k}(z)$ is a linear operator from $\mathbb{U}$ into $\mathbb{U}$. The notion of (j,k)-symmetrical functions was first introduced and studied by P. Liczberski and J. Polubiński in [5].
The following identities directly follow from (3):

$$\begin{aligned} f_{j,k}(\varepsilon^{\nu}z)&=\varepsilon^{\nu pj}f_{j,k}(z)\\ f'_{j,k}(\varepsilon^{\nu}z)&=\varepsilon^{\nu pj-\nu}f'_{j,k}(z)\\ f''_{j,k}(\varepsilon^{\nu}z)&=\varepsilon^{\nu pj-2\nu}f''_{j,k}(z)\end{aligned} \tag{4}$$

In [4], Karthikeyan et.al., investigated the class

$$\mathscr{S}^{p}_{j,k}(b;\alpha,\beta)=\left\{f\in\mathscr{A}_p:\alpha<Re\left\{1+\frac{1}{b}\left(\frac{zf^{(m+1)}(z)}{f^{(m)}_{j,k}(z)}-p+m\right)\right\}<\beta,0\le\alpha<1<\beta\right\}.$$

Motivated by the above concept, in this paper, we introduce and investigate a new subclass of multivalent functions with respect to symmetric points. We now define the following:

Definition 1.1. The function $f\in\mathscr{A}_p$ and $\frac{f(z)f'(z)}{z}\neq 0$ in $\mathbb{U}$ is said to be in the class $\mathscr{S}_p^{j,k}(\gamma;\lambda,\alpha,\beta)$ of p - valently functions of complex order $\gamma\neq 0$ if and only if it satisfies the condition

$$\alpha<\Re\left\{1+\frac{1}{\gamma}\left(\frac{(1-\lambda)zf'(z)+\lambda z(zf'(z))'}{(1-\lambda)f_{j,k}(z)+\lambda zf'_{j,k}(z)}-p\right)\right\}<\beta,\qquad (z\in\mathbb{U}), \tag{5}$$

where, $0\le\alpha<1<\beta$, $0\le\lambda\le 1$ and $f_{j,k}(z)\neq 0$ is defined by the equality (3). Similarly, we say that a function $f\in\mathscr{A}_p$ is in the class $\mathscr{C}_p^{j,k}(\gamma;\lambda,\alpha,\beta)$ if and only if

$$zf'\in\mathscr{S}_p^{j,k}(\gamma;\lambda,\alpha,\beta).$$

Remark 1.1. If $\lambda=0$, $j=k=p=1$ and $\alpha\geq 0$, then $f(z)$ reduces to the well-known class of starlike functions of complex order. Similarly, if we let $\lambda=1$, $j=k=p=1$ and $\alpha\geq 0$, then $f(z)$ reduces to the well-known class convex functions of complex order.

We observe that for a given α and β $(0\le\alpha<1<\beta)$, $f\in\mathscr{S}_p^{j,k}(\gamma;\lambda,\alpha,\beta)$ satisfies each of the following subordination equations

$$1+\frac{1}{\gamma}\left(\frac{(1-\lambda)zf'(z)+\lambda z(zf'(z))'}{(1-\lambda)f_{j,k}(z)+\lambda zf'_{j,k}(z)}-p\right)\prec\frac{1+(1-2\alpha)z}{1-z}$$

and

$$1+\frac{1}{\gamma}\left(\frac{(1-\lambda)zf'(z)+\lambda z(zf'(z))'}{(1-\lambda)f_{j,k}(z)+\lambda zf'_{j,k}(z)}-p\right)\prec\frac{1+(1-2\beta)z}{1-z}.$$

Both superordinate functions in the above expressions maps the unit disc onto right half plane, so it is obvious that the above expression is mapped on to a plane having real part greater than α but less than β.
Kuroki and Owa [3], defined an analytic function $p:\mathbb{U}\to\mathbb{C}$ by

$$p(z)=1+\frac{\beta-\alpha}{\pi}i\log\left(\frac{1-e^{2\pi i\frac{(1-\alpha)}{(\beta-\alpha)}}z}{1-z}\right).$$

The above function p maps $\mathbb{U}$ onto a convex domain $\Lambda=\{w:\alpha<Re\{w\}<\beta\}$, conformally. Using this fact and the definition of subordination, we can obtain the following:
Let $f\in\mathscr{A}_p$ and $0\le\alpha<1<\beta$. Then $f\in\mathscr{S}_p^{j,k}(\gamma;\lambda,\alpha,\beta)$ if and only if

$$1+\frac{1}{\gamma}\left(\frac{(1-\lambda)zf'(z)+\lambda z(zf'(z))'}{(1-\lambda)f_{j,k}(z)+\lambda zf'_{j,k}(z)}-p\right)\prec p(z),$$

and $p(z)$ is of the form

$$p(z)=1+\sum_{n=1}^{\infty}c_nz^n$$

and $c_n=\left(\frac{\beta-\alpha}{n\pi}\right)i\left(1-e^{2n\pi i\frac{(1-\alpha)}{(\beta-\alpha)}}\right)$.

Lemma 1.1. *[6] Let the functions q be univalent in the open unit disc $\mathbb{U}$ and θ and ϕ be analytic in a domain D containing $q(\mathbb{U})$ with $\phi(w) \neq 0$ when $w \in q(\mathbb{U})$. Set $Q(z) = zq'(z)\phi(q(z))$, $h(z) = \theta(q(z)) + Q(z)$. Suppose that*

1. *Q is starlike univalent in $\mathbb{U}$ and*
2. *$\Re\left(\frac{zh'(z)}{Q(z)}\right) > 0$ for $z \in \mathbb{U}$.*

If

$$\theta(p(z)) + zp'(z)\phi(p(z)) \prec \theta(q(z)) + zq'(z)\phi(q(z)),$$

then $p(z) \prec q(z)$ and q is the best dominant.

2. Main Results

In this section, we prove the integral representation of the function class $\mathscr{S}_p^{j,k}(\gamma; \lambda, \alpha, \beta)$.

Theorem 2.1. *Let $f \in \mathscr{S}_p^{j,k}(\gamma; \lambda, \alpha, \beta)$ with $0 \leq \alpha < 1 < \beta$ and $0 < \lambda \leq 1$. Then we have*

$$f_{j,k}(z) = \frac{1}{\lambda} z^{1-\frac{1}{\lambda}} \int_0^z \exp\left\{\frac{\gamma}{k}\sum_{\nu=0}^{k-1}\int_0^u \frac{1}{\zeta}\left[\frac{\beta-\alpha}{\pi} i \log\left(\frac{1-e^{2\pi i\frac{(1-\alpha)}{(\beta-\alpha)}} w(\varepsilon^\nu \zeta)}{1-w(\varepsilon^\nu \zeta)}\right)\right] d\zeta\right\} u^{\frac{1}{\lambda}+p-2} du \tag{6}$$

where $f_{j,k}(z)$ defined by (3), $w(z)$ is analytic in $\mathbb{U}$ with $w(0) = 0$ and $|w(z)| < 1$.

Proof. Let $f \in \mathscr{S}_p^{j,k}(\gamma; \lambda, \alpha, \beta)$ with $0 \leq \alpha < 1 < \beta$ and $0 < \lambda \leq 1$. Then we have

$$1 + \frac{1}{\gamma}\left(\frac{(1-\lambda) zf'(z) + \lambda z (zf'(z))'}{(1-\lambda) f_{j,k}(z) + \lambda z f'_{j,k}(z)} - p\right) = 1 + \frac{\beta-\alpha}{\pi} i \log\left(\frac{1-e^{2\pi i\frac{(1-\alpha)}{(\beta-\alpha)}} w(z)}{1-w(z)}\right), \tag{7}$$

where $w(z)$ is analytic in $\mathbb{U}$ with $w(0) = 0$ and $|w(z)| < 1$. Substituting z by $\varepsilon^\nu z$ in (7), we have

$$1 + \frac{1}{\gamma}\left(\frac{(1-\lambda) \varepsilon^\nu zf'(\varepsilon^\nu z) + \lambda \varepsilon^\nu z (\varepsilon^\nu zf'(\varepsilon^\nu z))'}{(1-\lambda) f_{j,k}(\varepsilon^\nu z) + \lambda \varepsilon^\nu z f'_{j,k}(\varepsilon^\nu z)} - p\right) = 1 + \frac{\beta-\alpha}{\pi} i \log\left(\frac{1-e^{2\pi i\frac{(1-\alpha)}{(\beta-\alpha)}} w(\varepsilon^\nu z)}{1-w(\varepsilon^\nu z)}\right). \tag{8}$$

Using the identities (4), we have

$$1 + \frac{1}{\gamma}\left(\frac{(1-\lambda) \varepsilon^\nu zf'(\varepsilon^\nu z) + \lambda \varepsilon^\nu z (\varepsilon^\nu zf'(\varepsilon^\nu z))'}{(1-\lambda) \varepsilon^{\nu pj} f_{j,k}(z) + \lambda \varepsilon^\nu z \varepsilon^{\nu pj-\nu} f'_{j,k}(z)} - p\right) = 1 + \frac{\beta-\alpha}{\pi} i \log\left(\frac{1-e^{2\pi i\frac{(1-\alpha)}{(\beta-\alpha)}} w(\varepsilon^\nu z)}{1-w(\varepsilon^\nu z)}\right). \tag{9}$$

On simplifying, we get

$$\frac{1}{\gamma}\left(\frac{(1-\lambda) \varepsilon^{\nu-\nu pj} zf'(\varepsilon^\nu z) + \lambda \varepsilon^{2\nu-\nu pj} z (zf'(\varepsilon^\nu z))'}{(1-\lambda) f_{j,k}(z) + \lambda z f'_{j,k}(z)} - p\right) = \frac{\beta-\alpha}{\pi} i \log\left(\frac{1-e^{2\pi i\frac{(1-\alpha)}{(\beta-\alpha)}} w(\varepsilon^\nu z)}{1-w(\varepsilon^\nu z)}\right). \tag{10}$$

Let $\nu = 0, 1, 2, \ldots, (k-1)$ in (10) respectively and summing them, we get

$$\frac{1}{\gamma}\left(\frac{(1-\lambda) zf'_{j,k}(z) + \lambda z \left(zf'_{j,k}(z)\right)'}{(1-\lambda) f_{j,k}(z) + \lambda z f'_{j,k}(z)} - p\right) = \frac{1}{k}\sum_{\nu=0}^{k-1} \frac{\beta-\alpha}{\pi} i \log\left(\frac{1-e^{2\pi i\frac{(1-\alpha)}{(\beta-\alpha)}} w(\varepsilon^\nu z)}{1-w(\varepsilon^\nu z)}\right). \tag{11}$$

From this equality , we get

$$\frac{(1-\lambda) f'_{j,k}(z) + \lambda \left(zf'_{j,k}(z)\right)'}{(1-\lambda) f_{j,k}(z) + \lambda z f'_{j,k}(z)} - \frac{p}{z} = \frac{\gamma}{kz}\sum_{\nu=0}^{k-1} \frac{\beta-\alpha}{\pi} i \log\left(\frac{1-e^{2\pi i\frac{(1-\alpha)}{(\beta-\alpha)}} w(\varepsilon^\nu z)}{1-w(\varepsilon^\nu z)}\right). \tag{12}$$

Integrating, we get

$$\log\left(\frac{(1-\lambda) f_{j,k}(z) + \lambda z f'_{j,k}(z)}{z^p}\right) = \frac{\gamma}{k}\sum_{\nu=0}^{k-1}\int_0^z \frac{1}{t}\left(\frac{\beta-\alpha}{\pi} i \log\left(\frac{1-e^{2\pi i\frac{(1-\alpha)}{(\beta-\alpha)}} w(\varepsilon^\nu t)}{1-w(\varepsilon^\nu t)}\right)\right) dt. \tag{13}$$

Simplifying (13), we have

$$(1-\lambda) f_{j,k}(z) + \lambda z f'_{j,k}(z) = z^p \exp\left\{\frac{\gamma}{k}\sum_{\nu=0}^{k-1}\int_0^z \frac{1}{t}\left(\frac{\beta-\alpha}{\pi} i \log\left(\frac{1-e^{2\pi i\frac{(1-\alpha)}{(\beta-\alpha)}} w(\varepsilon^\nu t)}{1-w(\varepsilon^\nu t)}\right)\right) dt\right\}. \tag{14}$$

A simple computation in (14), gives the required conclusion of this theorem. □

Theorem 2.2. *Let* $f \in \mathscr{S}_p^{j,k}(\gamma;\lambda,\alpha,\beta)$ *with* $0 \le \alpha < 1 < \beta$ *and* $0 < \lambda \le 1$. *Then we have*

$$f(z) = \frac{1}{\lambda} z^{1-\frac{1}{\lambda}} \int_0^z \int_0^u \exp\left\{ \frac{\gamma}{k} \sum_{\nu=0}^{k-1} \int_0^\eta \frac{1}{\zeta} \left[\frac{\beta-\alpha}{\pi} i \log \left(\frac{1-e^{2\pi i \frac{(1-\alpha)}{(\beta-\alpha)}} w(\varepsilon^\nu \zeta)}{1-w(\varepsilon^\nu \zeta)} \right) \right] d\zeta \right\} \times \left[p + \frac{\gamma(\beta-\alpha)}{\pi} i \log \left(\frac{1-e^{2\pi i \frac{(1-\alpha)}{(\beta-\alpha)}} w(z)}{1-w(z)} \right) \right] u^{\frac{1}{\lambda}+p-3} d\eta du \tag{15}$$

where $w(z)$ *is analytic in* $\mathbb{U}$ *with* $w(0)=0$ *and* $|w(z)|<1$.

Proof. From (7), we have

$$(1-\lambda) z f'(z) + \lambda z \left(z f'(z) \right)' = \left((1-\lambda) f_{j,k}(z) + \lambda z f'_{j,k}(z) \right) \times \left[p + \frac{\gamma(\beta-\alpha)}{\pi} i \log \left(\frac{1-e^{2\pi i \frac{(1-\alpha)}{(\beta-\alpha)}} w(z)}{1-w(z)} \right) \right]. \tag{16}$$

From (14) and (16), we have

$$(1-\lambda) f'(z) + \lambda \left(z f'(z) \right)' = z^{p-1} \exp\left\{ \frac{\gamma}{k} \sum_{\nu=0}^{k-1} \int_0^z \frac{1}{t} \left(\frac{\beta-\alpha}{\pi} i \log \left(\frac{1-e^{2\pi i \frac{(1-\alpha)}{(\beta-\alpha)}} w(\varepsilon^\nu t)}{1-w(\varepsilon^\nu t)} \right) \right) dt \right\} \times \left[p + \frac{\gamma(\beta-\alpha)}{\pi} i \log \left(\frac{1-e^{2\pi i \frac{(1-\alpha)}{(\beta-\alpha)}} w(z)}{1-w(z)} \right) \right]. \tag{17}$$

On simplifying and integrating the above equality (17), we get (15). □

If we put $\lambda = 1$, $j = k = 1$ in Definition 1.1 and Theorem 2.1, we get the following corollary:

Corollary 2.3. *If* $f \in \mathscr{A}_p$ *satisfies the analytic condition*

$$\alpha < \Re\left\{ 1 + \frac{1}{\gamma} \left(1 + \frac{z f''(z)}{f'(z)} - p \right) \right\} < \beta,$$

then the integral representation of $f(z)$ *is given by*

$$f(z) = \int_0^z t^{p-1} \exp\left\{ \gamma \sum_{\nu=0}^{k-1} \int_0^t \frac{1}{\zeta} \left(\frac{\beta-\alpha}{\pi} i \log \left(\frac{1-e^{2\pi i \frac{(1-\alpha)}{(\beta-\alpha)}} w(\varepsilon^\nu \zeta)}{1-w(\varepsilon^\nu \zeta)} \right) \right) d\zeta \right\} dt.$$

Remark 2.1. If we put $\lambda = 1$, $j = k = 1$ in (5) then this result is reduced into the Corollary 2.5 in [4].

Remark 2.2. If we put $\lambda = 0$, $j = k = 1$ in (14), then

$$f(z) = z^p \exp\left\{ \gamma \sum_{\nu=0}^{k-1} \int_0^z \frac{1}{t} \left(\frac{\beta-\alpha}{\pi} i \log \left(\frac{1-e^{2\pi i \frac{(1-\alpha)}{(\beta-\alpha)}} w(\varepsilon^\nu t)}{1-w(\varepsilon^\nu t)} \right) \right) dt \right\}.$$

Take $p = 1$, this result was proved by K.Kuroki and S.Owa [3].

Theorem 2.4. *Let the function* $h(z)$ *analytic in* $\mathbb{U}$ *be defined by*

$$h(z) = \delta + (\delta+\kappa) \frac{\beta-\alpha}{\pi} i \log \left(\frac{1-e^{2\pi i \frac{(1-\alpha)}{(\beta-\alpha)}} z}{1-z} \right) + \kappa \left(\frac{\beta-\alpha}{\pi} \right) i \frac{z \left(1-e^{2\pi i \frac{(1-\alpha)}{(\beta-\alpha)}} \right)}{(1-z)\left(1-e^{2\pi i \frac{(1-\alpha)}{(\beta-\alpha)}} z \right)} - \kappa \left(\frac{\beta-\alpha}{\pi} \right)^2 \left[\log \left(\frac{1-e^{2\pi i \frac{(1-\alpha)}{(\beta-\alpha)}} z}{1-z} \right) \right]^2 \tag{18}$$

where $\kappa > 0$, $\kappa + \delta > 0$. *If* $f \in \mathscr{A}$ *with* $\frac{f_{j,k}(z)}{z} \neq 0$ *satisfies the condition*

$$\delta + \frac{(\delta+\kappa)}{\gamma} \left[\frac{F(z)}{F_{j,k}(z)} - 1 \right] + \frac{\kappa}{\gamma^2} \left[\frac{F(z)}{F_{j,k}(z)} - 1 \right]^2 + \frac{\kappa}{\gamma} \left[\frac{z F'(z)}{F_{j,k}(z)} - \frac{z F(z) F'_{j,k}(z)}{(F_{j,k}(z))^2} \right] \prec h(z), \tag{19}$$

where

$$F(z) = (1-\lambda) z f'(z) + \lambda z \left(z f'(z) \right)' \quad \text{and} \quad F_{j,k}(z) = (1-\lambda) f_{j,k}(z) + \lambda z f'_{j,k}(z) \tag{20}$$

then $f \in \mathscr{S}_1^{j,k}(\gamma;\lambda,\alpha,\beta)$.

Proof. Let the function $p(z)$ be defined by

$$p(z) = 1 + \frac{1}{\gamma}\left(\frac{F(z)}{F_{j,k}(z)} - 1\right) \qquad (z \in \mathbb{U}; z \neq 0; f \in \mathscr{A}), \tag{21}$$

where $p(z) = 1 + c_1 z + c_2 z^2 + \cdots \in \mathscr{P}$, $F(z)$ and $F_{j,k}(z)$ defined by (20). On simplification, we get

$$zp'(z) = \frac{1}{\gamma}\left[\frac{zF'(z)}{F_{j,k}(z)} - \frac{zF(z)F'_{j,k}(z)}{(F_{j,k}(z))^2}\right].$$

Thus by (19), we have

$$\kappa z p'(z) + \kappa p^2(z) + (\delta - \kappa)p(z) \prec h(z). \tag{22}$$

Let

$$g(z) = 1 + \frac{\beta - \alpha}{\pi} i \log\left(\frac{1 - e^{2\pi i \frac{(1-\alpha)}{(\beta-\alpha)}} w(z)}{1 - w(z)}\right). \tag{23}$$

Set

$$\theta(w) = \kappa w^2 + (\delta - \kappa)w \qquad \text{and} \qquad \phi(w) = \kappa,$$

it can be easily verified that θ is analytic in $\mathbb{C}$, ϕ is analytic in $\mathbb{C}$ with $\phi(0) \neq 0$ in the w-plane. Also, let

$$Q(z) = zg'(z)\phi(g(z)) = \kappa z g'(z)$$

and

$$h(z) = \theta(g(z)) + Q(z) = \kappa(g(z))^2 + (\delta - \kappa)g(z) + \kappa z g'(z).$$

Since $g(z)$ is convex univalent in $\mathbb{U}$ provided $\alpha \geq 0$, it gives that $Q(z)$ is starlike univalent in $\mathbb{U}$. In view of the result proved in [3], $g(z)$ given by (23) is starlike for $\alpha \geq 0$, we have

$$\Re\left(\frac{zh'(z)}{Q(z)}\right) = \Re\left\{\kappa\left(\frac{g(z)}{zg'(z)}(g(z) - 1) + 1\right) + \delta\frac{g(z)}{zg'(z)}\right\} > 0.$$

By the application of Lemma 1.1, we get the required assertion of this theorem. □

If we put $\lambda = 0$, $\gamma = 1$ in Theorem 2.4, we get the following corollary:

Corollary 2.5. *Let the function $h(z)$ be defined as in (18). If $f \in \mathscr{A}$ with $\frac{f_{j,k}(z)}{z} \neq 0$ satisfies the condition*

$$\kappa\left\{\frac{z^2 f''(z)}{f_{j,k}(z)} - \frac{z^2 f'(z) f'_{j,k}(z)}{(f_{j,k}(z))^2} + \frac{z^2 (f'(z))^2}{(f_{j,k}(z))^2}\right\} + \delta\left(\frac{zf'(z)}{f_{j,k}(z)}\right) \prec h(z),$$

then

$$\frac{zf'(z)}{f_{j,k}(z)} \prec 1 + \frac{\beta - \alpha}{\pi} i \log\left(\frac{1 - e^{2\pi i \frac{(1-\alpha)}{(\beta-\alpha)}} z}{1 - z}\right).$$

Remark 2.3. If we take $j = k = 1$ in the corollary 2.5, then this result was analogous to the result obtained by Xu et al. in [11].

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