International Journal of Advanced Mathematical Sciences, 10 (2) (2024) 19-23

International Journal of Advanced Mathematical Sciences

Website: www.sciencepubco.com/index.php/IJAMS

Research paper

On some characterizations of classical operators of composition Hardy type

Anupama Gupta[1*]

[1] *Govt. Degree College Marheen, District Kathua, 184148, J&K, India*
[*] *Corresponding author E-mail: anugup49@gmail.com*

Abstract

In this paper, we study classical operators of composite Hardy type on L^p- space.Our focus is to discuss some operator theoretic properties of the classical operators of composite Hardy type. An attempt has been made to compute the n^{th} approximation number of classical operators of composite Hardy type.

Keywords: *Classical operators of Composite Hardy type; Composition operator; Expectation operator; Radon- Nikodym derivative*

1. Introduction

Let (X,Ω,μ) be a $\sigma-$finite measure space. For $1 \leq p < \infty$, suppose $L^p(\mu)$ is a Banach space of all real valued Lebesgue measurable functions $f : X \to \mathbb{R}$ with usual norm, $||f||_p = (\int_0^1 |f(x)|^p d\mu(x))^{1/p} < \infty$. Also, $L^2(\mu)$, is the space of square-integrable functions is a Hilbert space. For each $f \in L^p(\mu), 1 \leq p < \infty$, there exists unique $\phi^{-1}(\Omega)$ measurable function $E(f)$ such that $\int gfd\mu = \int gE(f)d\mu$ for every $\phi^{-1}(\Omega)$ measurable function g for which left integral exists and the function $E(f)$ is called conditional expectation of f with respect to the sub- algebra $\phi^{-1}(\Omega)$. Also,we have,

$$E(f)o\phi^{-1} = g \iff E(f) = g \circ \phi.$$

For wider perspective of expectation operator, one can refer to Parthasarthy [13]. Let $\phi : X \to X$ be a non-singular measurable transformation

$$(\mu(E) = 0 \Rightarrow \mu\phi^{-1}(E) = 0, \forall E \in \Omega).$$

We can define a composition transformation, for $1 \leq p < \infty$, as $C_\phi : L^p(\mu) \to L^p(\mu)$ by

$$C_\phi f = f \circ \phi \forall f \in L^p(\mu).$$

If C_ϕ is continuous, then C_ϕ is called as composition operator induced by ϕ. The composition operators are also known as substitution operators or Koopman operators. Singh [15] established that C_ϕ is a bounded operator if and only if $f_d = \frac{d\mu\phi^{-1}}{d\mu}$, the Radon-Nikodym derivative of the measure $\mu\phi^{-1}$ with respect to the measure μ, is essentially bounded. In 1990, the adjoint of a composition operator was computed by Campbell [1] on $L^2(\mu)$ which is given by

$$C_\phi^* f(x) = f_d(x)E(f \circ \phi^{-1})(x).$$

Let $\theta : X \to \mathbb{C}$ be a function such that $\theta.f \in L^p(\mu)$, for all $f \in L^p(\mu)$, then we can define a multiplication transformation $M_\theta : L^p(\mu) \to L^p(\mu)$ by $M_\theta f = \theta.f$. If M_θ is continuous,we call it a multiplication operator induced by θ. We defined Volterra operator (Hardy operator) V, as $V : L^p \to L^p$

$$(Vf)(x) = \int_0^x f(y)d\mu(y), \; \forall\, f \in L^p[0,1].$$

Thus, the Volterra operator V is an integral operator induced by the kernel $k(x,y)$ defined as

$$k(x,y) = \begin{cases} 0 & x \leq 0 \\ 1 & x > 1. \end{cases}$$

Copyright © 2018 Author. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

The adjoint of Volterra (Hardy operator) V is given by

$$V^*f(x) = \int_x^1 f(t)d\mu(t).$$

Whitley [18] established the Lyubic's conjecture [11] and generalized it to Volterra composition operators on $L^p[0,1]$. The Volterra composition operator is a composition of Volterra integral operator V and a composition operator C_ϕ defined as

$$(V_\phi f)(x) = (C_\phi Vf)(x) = (Vf)\circ\phi(x)$$

$$(V_\phi f)(x) = \int_0^{\phi(x)} f(t)d\mu(t)$$

for every $f \in L^p[0,1]$, where $\phi : [0,1] \to [0,1]$ is a measurable function. Gupta and Komal [2] defined the Volterra composite integral operators using expectation operator as given below

$$(V_\phi f)(x) = \int_0^x f(\phi(y))d\mu(y) = \int_0^x E(f_d o\phi^{-1})(y)f(y)d\mu(y).$$

For $1 \le p < \infty$, classical Hardy operator (averaging operator/Cesaro operator),

$$\mathbb{H} : L^p[0,1] \to L^p[0,1]$$

is defined by

$$\mathbb{H}(f)(x) = \frac{1}{x}\int_0^x f(y)d\mu(y).$$

The adjoint of classical Hardy operator $\mathbb{H}^*$ is given by

$$\mathbb{H}^* f(x) = \int_x^1 \frac{f(t)}{t}d\mu(t).$$

The classical Hardy operators have been extensively studied since their introduction in 1915, as they constitute simple examples of positive integral operators. For $x \in [0,1]$, the classical Hardy operator can be written as

$$\mathbb{H}f(x) = \int_0^1 f(xu)du.$$

The classical Hardy operators can also be defined using convolution product as

$$\mathbb{H}f(x) = \frac{1}{\int_0^x \chi_{[0,\infty)}(t)dt}(f * \chi_{[0,\infty)})(x); x \ge 0.$$

For $p > 1$, the first Hardy inequality

$$(\int_0^\infty (\frac{1}{x}\int_0^x |f(t)|d\mu(t))^p)^{\frac{1}{p}} \le \frac{p}{1-p}(\int_0^\infty |f(t)|^p)^{\frac{1}{p}}.$$

For detail about the history of Hardy-type inequalities upto 2007, one can refer to Kufner ([8][9]), Nikolova [12] and Krulic [7]. The concept of classical operators of composite Hardy type was introduced by Gupta [3] in 2022. The definition of classical operators of composite operator of Hardy type is motivated by Whitley [18], wherein composition operator with Volterra integral operator were composed to study Lyubic's conjecture. Gupta [3] composed composition operator C_ϕ with classical Hardy operator $\mathbb{H}$, and the classical operator of composite Hardy type is defined as

$$(\mathbb{H}_\phi f)(x) = (C_\phi \mathbb{H}f)(x) = (\mathbb{H}f)\circ\phi(x)$$

$$(\mathbb{H}_\phi f(x) = \frac{1}{x}\int_0^x f(\phi(t))d\mu(t)$$

for every $f \in L^p[0,1]$, where $\phi : [0,1] \to [0,1]$ is a measurable function. From the recent literature available in operator theory and functional analysis, we explore that multiplication operators and composition operators are very much intimately connected with classical Hardy operators. There exits a vast literature on the properties of these operators in different function spaces. For more recent studies about these operators, we refer to Singh and Manhas [16], Takagi [17], Halmos [4], Halmos and Sunder [5], Kreshaw[6], Lao and Whitley[10] and Whitley [18]. Our focus in this paper is mainly on operator theoretic properties of classical operators of composite of Hardy type.

It has been proved that classical operators of composite Hardy type from $L^p(\mu))$ into $L^p(\mu)$ is bounded. The conditions for classical operators of composite Hardy type to be a contraction are explored. In addition, the spectrum of classical operators of composite Hardy type is computed. The main purpose of this paper is to evaluate approximation number of classical operators of composite Hardy type. The introduction conclude by recalling some standard notation in operator theory. Let H be a Hilbert space and $B(H)$ be the algebra of all bounded linear operators acting on H. For a pair p,q such that $1 \le p,q < \infty$, and a continuous linear operator $T : L^p \to L^q$, the norm of T is given by

$$||Tf|| = \sup\{||Tf|| : ||f|| = 1\}.$$

A mapping $T : X \to X$ is said to be contraction if $|||Tx - Ty|| \le d|||x - y||$ for some $d < l$ and $x,y \in X$. For any natural number n, suppose $S \in B(X,Y)$, then the n^{th} approximation number of P is defined as $\alpha_n(S) = \inf\{||S - Q|| : Q \in B(X,Y)\}$, where Q is one-dimensional operator.

2. Operator theoretic properties of Classical Operators of Composite Hardy Type (CCHO)

Theorem 2.1 *Let $\phi : [0,1] \to [0,1]$ be a non-singular measurable transformation. For $1 \le p,q < \infty$, if $\mathbb{H}_\phi : L^p[0,1] \to L^q[0,1]$ is a linear transformation, then $\mathbb{H}_\phi$ is continuous and hence,a bounded operator.*

Proof. We have make use of closed graph theorem in order to prove that $\mathbb{H}_\phi$ is bounded operator. Let $\{f_n\}, \{\mathbb{H}_\phi f_n\} \in L^p[0,1]$ such that

$$f_n \to f, \mathbb{H}_\phi f_n \to g$$

for some $f,g \in L^p[0,1]$. That is,

$$\frac{1}{x}\int_0^x f_n o\phi(y)dy = g(x).$$

Then, there exists a dominated subsequence $\{f_{n_k}\}$ of $\{f_n\}$ such that

$$f_{n_k}(x) \to f(x), \mathbb{H}_\phi f_{n_k}(x) \to g(x)$$

for $\mu-$ almost all $x \in [0,1]$ Then, we have

$$\frac{1}{x}\int_0^x f_{n_k} o\phi(y)dy \to g(x)$$

for $\mu-$ almost all $x \in [0,1]$. Also, $|f_{n_k}| \le h$ for some $h \in L^p[0,1]$. By using Lebesgue dominated convergence theorem and non-singularity of ϕ, we have

$$\frac{1}{x}\int_0^x f_{n_k} o\phi(y)dy \to \frac{1}{x}\int_0^x f o\phi(y)dy$$

for $\mu-$ almost all $x \in [0,1]$. Thus, we get

$$\mathbb{H}_\phi f(x) = \frac{1}{x}\int_0^x f o\phi(y)dy = g(x).$$

This shows that graph of $\mathbb{H}_\phi$ is closed and therefore, by using closed graph theorem,$\mathbb{H}_\phi$ is continuous and hence, bounded. □

In the following example, we have discussed bounded classical operators of composite Hardy type H_ϕ.

Example 2.2 Let $X = [0,1]$. Define $\phi : [0,1] \to [0,1]$ by $\phi(x) = \sqrt{x}, \forall x \in [0,1]$. Then $\phi^{-1}(x) = x^2$ and $f_d(x) = 2x$. Suppose $f(x) = x^2, \forall x \in [0,1]$. Then $\forall x \in [0,1]$,we have

$$\begin{aligned} H_\phi f(x) &= \frac{1}{x}\int_0^x f(\phi(y))d\mu(y) \\ &= \frac{1}{x}(\int_0^x f(\sqrt{y})d\mu(y) \\ &= \frac{1}{x}\int_0^x y d\mu(y) \\ &= \frac{x}{2}. \end{aligned}$$

Hence, $\mathbb{H}_\phi$ is a bounded operator.

In the next result, we explore the condition for classical operators of composite Hardy Type (CCHO) to be a contraction on $L^p(\mu)$. In the the proof the next theorem, we have make use of the result of Gupta[3] stated as follows:

For $1 < p,q < \infty$, $\frac{1}{p} + \frac{1}{q} = 1$, let $\mathbb{H}_\phi \in B(L^p(\mu))$. The $\mathbb{H}_\phi$ is a bounded operator and

$$||\mathbb{H}_\phi f||_p \le \alpha \frac{p}{1-p} ||f-g||_p.$$

Theorem 2.3 *For $1 < p,q < \infty$, $\frac{1}{p} + \frac{1}{q} = 1$ and q is the conjugate exponent of p. Let $\mathbb{H}_\phi \in B(L^p(\mu))$. Then $\mathbb{H}_\phi$ is a contraction if*

$$f_d = \frac{d\mu\phi^{-1}}{d\mu} \le \alpha < 1.$$

Proof. For $f,g \in L^p(\mu)$, by using bounded condition from Gupta [3], we have

$$\begin{aligned} ||\mathbb{H}_\phi f - \mathbb{H}_\phi g|| &\le |||\mathbb{H}_\phi|||| f-g||_p \\ &\le \alpha \frac{p}{1-p}||f-g||_p \\ &\le \alpha(-q)||f-g||_p. \end{aligned}$$

Here we have used the given conditions, $\frac{1}{p} + \frac{1}{q} = 1 \Rightarrow \frac{p}{1-p} = -q$. and $f_d \le \alpha < 1$.

Thus, we have

$$||\mathbb{H}_\phi f - \mathbb{H}_\phi g|| \le \alpha(-q)||f-g||_p.$$

Also, $1 < q < \infty$ implies that $1 < q$ and $-q < 1$

$$\frac{||\mathbb{H}_\phi(f-g)||}{||f-g||} \le \alpha(-q).$$

Hence, we get the required result

$$||\mathbb{H}_\phi f||_p \le \alpha.$$

Therefore, $\mathbb{H}_\phi$ is a contraction if $f_d \le \alpha < 1$. □

In the next result, consider a mapping $T_a \in B(L^2[0,1])$ defined as $T_a f(x) = f(x+a)$ for some fixed a.
Theorem 2.4 *Let* $\mathbb{H}_\phi \in B(L^2[0,1])$. *Then* $\mathbb{H}_\phi$ *commutes with* T_a.

Proof. Given $f \in L^2[0,1]$ and $x \in [0,1]$ we have

$$\begin{aligned} \mathbb{H}_\phi T_a f(x) &= \mathbb{H}_\phi(T_a f(x)) \\ &= \mathbb{H}_\phi(f(x+a)) \\ &= \frac{1}{x+a}\int_0^{x+a} f(\phi(y))d\mu(y) \\ &= T_a \mathbb{H}_\phi f(x) \end{aligned}$$

Again, we have

$$T_a H_\phi f(x) = H_\phi T_a f(x).$$

Hence, the result follows. □

Theorem 2.5 *Let* $\mathbb{H}_\phi \in B(L^p[0,1])$. *Then* $C_\phi f(x) = 1$ *for all* $x \in [0,1]$ *if and only if* $\mathbb{H}_\phi f(x) = 1$ *for all* $x \in [0,1]$.

Proof. Firstly, suppose $\mathbb{H}_\phi(x) = 1$ for all $x \in [0,1]$. Then, we have

$$\frac{1}{x}\int_0^x fo\phi(y)dy = 1 \Longrightarrow \int_0^x fo\phi(y)dy = x.$$

On differentiation, we have $f(\phi(x)) = 1$, hence, we get $C_\phi f(x) = 1$ for all $x \in [0,1]$. Conversely, suppose

$$C_\phi f(x) = 1 \forall x \in [0,1] \Longrightarrow f(\phi(x)) = 1.$$

That is,

$$\int_0^x fo\phi(y)dy = x \Longrightarrow \frac{1}{x}\int_0^x fo\phi(y)dy = 1.$$

Hence, we have

$$\mathbb{H}_\phi f(x) = 1$$

for all $x \in [0,1]$. □

In the next theorem, we have evaluated the spectrum of $\mathbb{H}_\phi$ on L^p- space.
Theorem 2.6 *Let* $\mathbb{H}_\phi \in B(L^p[0,1])$. *Suppose* $f(0) = 0$, *then the spectrum of* $\mathbb{H}_\phi$ *is zero,* $\sigma(\mathbb{H}_\phi) = \{0\}$.

Proof. If possible, suppose $\sigma(\mathbb{H}_\phi) \neq \{0\}$.

$$\mathbb{H}_\phi f(x) = \lambda f(x) \Longrightarrow \frac{1}{x}\int_0^x fo\phi(y)dy = \lambda f(x).$$

That is,

$$\frac{1}{x}\int_0^{\phi(x)} f(y)dy = \lambda f(x).$$

On differentiation with respect to x, we have

$$\frac{1}{x} fo\phi(x)\phi'(x) + (-\frac{1}{x^2})\int_0^{\phi(x)} f(y)dy = \lambda f'(x).$$

Moreover, using given condition $f(0) = 0$, we have $f'(0) = 0$. Continue in this way, we get $f''(0) = 0, f'''(0) = 0, ...f^n = 0$. This shows that f is a zero function, which is a contradiction. Hence,

$$\sigma(\mathbb{H}_\phi) = \{0\}.$$

□

In the next result, we have to evaluate nth approximation number of classical composite operator of Hardy type. Consider an one-dimensional opearator A on L^1- space defined as

$$Af(x) = \frac{1}{2x}\int_0^1 fo\phi(y)dy.$$

Theorem 2.7 *Let* $\mathbb{H}_\phi \in B(L^1[0,1])$. *Suppose* $f_d < 1$. *Then, for* $n \leq 2$,

$$\sigma_n(\mathbb{H}_\phi) < \frac{1}{2}.$$

Proof. Let $f,g \in L^1[0,1]$, we have

$$\begin{aligned}
||\mathbb{H}_\phi f - Af|| &= \sup_{0\leq x\leq 1}\{|\frac{1}{x}\int_0^x fo\phi(y)dy - \frac{1}{2x}\int_0^1 fo\phi(y)dy|\} \\
&= \sup_{0\leq x\leq 1}\{|\frac{1}{2x}\int_0^x fo\phi(y)dy - \frac{1}{2x}\int_x^1 fo\phi(y)dy|\} \\
&\leq \sup_{0\leq x\leq 1}\{\frac{1}{2|x|}\int_0^x |fo\phi(y)|dy + \frac{1}{2|x|}\int_x^1 |fo\phi(y)|dy\} \\
&= \sup_{0\leq x\leq 1}\{\frac{1}{2|x|}\int_0^1 |fo\phi(y)|dy \\
&= \sup_{0\leq x\leq 1}\{\frac{1}{2|x|}\int_0^1 |\, f_a f(y)|dy \\
&< \sup_{0\leq x\leq 1}\{\frac{1}{2|x|}\int_0^1 |f(y)|dy \\
&= \frac{1}{2|x|}||f||.
\end{aligned}$$

Thus, $x \in [0,1],\ |x| < 1$ we have

$$||\mathbb{H}_\phi - A|| < \frac{1}{2}.$$

Hence,

$$\sigma_n(\mathbb{H}_\phi) \leq \frac{1}{2}.$$

□

3. Conclusion

In this paper,we have studied classical operators of composite Hardy type. The characterizations for classical operators of composite Hardy type to be contraction has been explored and the spectrum of classical operators of composite Hardy type is computed. The problem of computing n^{th} approximation number of classical operators of composite Hardy type is addressed. The study may open new horizons to explore more properties of classical operators of composite Hardy type. One can investigate different applications of classical operators of composite Hardy type.

References

[1] Campbell, J. T. and Jamison, J. E. (1991), The Analysis of Composition Operators on Lp and the Hopf Decomposition, *J. Math. Anal. Appl.*, 159; 520-531.
[2] Gupta, A. and Komal, B. S. (2011),Volterra Composition Operators, *Int. J. Contemp. Math Sciences*, 6(7); 345-351.
[3] Gupta, A. (2022), Classical operators of composite Hardy type, *Mathematical Forum*, 29; 42-52.
[4] Halmos, P. R. (1982), A Hilbert space problem Book, *Springer-Verlag, New York*.
[5] Halmos, P. R. and Sunder, V. S. (1978), Bounded integral operators on L^2 spaces, *Springer- Verlag,New York*.
[6] Kershaw, D. (1999), Operator norms of powers of the Volterra operator, *Journal of Integral Equations and Applications*, 11(3); 351-362.
[7] Krulic Himmelreich, K. (2022), Some new inequalities involving the generalized Hardy operator, *Mathematica Pannonica New series*, 28(2); 127-132.
[8] Kufner, A., Maligranda, L. and Persson, L.E. (2007), The Hardy inequalities. About its history and some related results, *Vydavatelski Servis Publishing House, Pilsen*.
[9] Kufner, Persson, L.E. and Samko, N.(2017) weighted inequalities of Hardy type, 2nd ed., *World Scientific Publishing Co. New Jersey.*
[10] Lao, N.and Whitley, R. (1997), Norms of powers of the Volterra operator, *Integral Equation and Operator Theory*, 27; 419-425.
[11] Lyubic, Yu. I. (1984), Composition of integration and substitution, Linear and complex Analysis, Problem Book, *Springer Lect. Notes in Maths.*, 1043, Berlin; 249-250.
[12] Nikolova, L., Person, L.E. and Samko, N.(2019), Some new inequalities involving Hardy operator, *Mathematische Nachrichten*, 2020; 376-385
[13] Parthasarathy, K. R. (1977), Introduction to probability and measure, Macmillan Limited.
[14] Shapiro, J.H. (1993), Composition Operators and Classical Function Theory, *Springer-verlag, New York.*
[15] Singh, R. K. (1972), Composition operators, Thesis University of New Hampshire.
[16] Singh, R. K. and Manhas, J. S. (1993), Composition operators on function spaces, North Holland, Mathematics studies 179, *Elsevier sciences publishers Amsterdam, New York*.
[17] Takagi, H., Yokouchi, K. (1999), Multiplication and composition operators between two Lp-spaces. *Contemp. Math.,* 232; 321 -338.
[18] Whitley, R. (1987), The spectrum of a Volterra composition operator, *Integral equation and Operator theory,* 10; 146-149.