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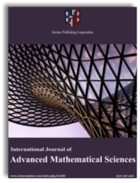

Stability Of A Generalized Alternate 3 Dimensional Additive Functional Equation In Various Banach Spaces

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Abstract

In this paper, the generalized Ulam – Gavruta stability of a generalized alternate 3 dimensional additive functional equation in various Banach spaces using Hyers method.

Keywords: *Additive functional equations; Hyers-Ulam stability; Hyers-Ulam-Rassias stability; Gavruta stability, Banach space; Generalized 2 Banach space; Fuzzy Banach Space; Random Banach space.*

1. Introduction

Over the past eight decades, there has been much discussion about the stability of functional equations. S.M. Ulam [43] presented the number of significant unresolved issues in a general address he made in front of a Mathematical Colloquium at the University of Wisconsin in 1940. One of these, the Stability Problem, is the starting point of a new line of inquiry.
D.H. Hyers [25] presented the first result about the stability of functional equations in 1941. In the case where the groups are Banach spaces, he has fully addressed Ulam's query. The following theorem was proven by him.
Theorem 1.1: [25] *Let X,Y be Banach spaces and let $f : X \to Y$ be a mapping satisfying*

$$\|f(x+y)-f(x)-f(y)\| \leq \varepsilon \tag{1}$$

for all $x,y \in X$. Then the limit

$$a(x) = \lim_{n\to\infty} \frac{f(2^n x)}{2^n} \tag{2}$$

exists for all $x \in X$ and $a : X \to Y$ is the unique additive mapping satisfying

$$\|f(x)-a(x)\| \leq \varepsilon \tag{3}$$

for all $x \in X$. Moreover, if $f(tx)$ is continuous in t for each fixed $x \in X$, then the function a is linear.
This pioneer result can be expressed as " for any pair of Banach spaces Cauchy functional equation is stable". The approach that Hyers came up with and the additive function that he generates will be referred to as a direct method. The most significant and effective technique for examining the stability of different functional equations is this one. Hyers-Ulam stability of functional equations is the name given to this stability result.
In 1951, T. Aoki [2] and in 1978 Th.M. Rassias [36] generalized the Hyers theorem in Banach spaces for approximately linear transformation, by lowering the condition for the Cauchy difference for sum of powers of norms. The following Hyers-Ulam-Aoki-Rassias theorem for the "sum" was demonstrated by both of them.
Theorem 1.2: [2, 36] *Let X and Y be two Banach spaces. Let $\theta \in [0,\infty)$ and $p \in [0,1)$. If a function $f : X \to Y$ satisfies the inequality*

$$\|f(x+y)-f(x)-f(y)\| \leq \theta\left(||x||^p + ||y||^p\right) \tag{4}$$

for all $x,y \in X$, then there exists a unique additive mapping $T : X \to Y$ such that

$$\|f(x)-T(x)\| \leq \frac{2\theta}{2-2^p}||x||^p \tag{5}$$

for all $x \in X$. Moreover, if $f(tx)$ is continuous in t for each fixed $x \in X$, then the function T is linear.
J.M. Rassias [32] established the Ulam-Gavruta-Rassias theorem in 1982 by substituting the product of powers of norms in the place of sum of powers of norms.
Theorem 1.3: [32] *Let $f : E \to E'$ be a mapping from a normed vector space E into a Banach space E' subject to the inequality*

$$\|f(x+y) - f(x) - f(y)\| \le \varepsilon \|x\|^p \|y\|^p \tag{6}$$

for all $x, y \in E$, where ε and p are constants with $\varepsilon > 0$ and $0 \le p < \frac{1}{2}$. Then the limit

$$L(x) = \lim_{n\to\infty} \frac{f(2^n x)}{2^n} \tag{7}$$

exists for all $x \in E$ and $L : E \to E'$ is the unique additive mapping which satisfies

$$\|f(x) - L(x)\| \le \frac{\varepsilon}{2 - 2^{2p}} \|x\|^{2p} \tag{8}$$

for all $x \in E$. If $p < 0$, then the inequality (6) holds for $x, y \neq 0$ and (8) for $x \neq 0$.
If $p > \frac{1}{2}$ the inequality (6) holds for $x, y \in E$ and the limit

$$A(x) = \lim_{n\to\infty} 2^n f\left(\frac{x}{2^n}\right) \tag{9}$$

exists for all $x \in E$ and $A : E \to E'$ is the unique additive mapping which satisfies

$$\|f(x) - A(x)\| \le \frac{\varepsilon}{2^{2p} - 2} \|x\|^{2p} \tag{10}$$

for all $x \in E$. If, in addition $f : E \to E'$ is a mapping such that the transformation $t \to f(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in E$, then L is $\mathbb{R}-$ linear mapping.
In 1994, P. Gavruta [22] achieved a generalization of all the stability results mentioned above and proved the following theorem.
Theorem 1.4: [22] *Let E be a abelian group, F be a Banach space and let $\phi : E \times E \to [0, \infty)$ be a function satisfying*

$$\Phi(x, y) = \sum_{k=0}^{+\infty} \frac{1}{2^{k+1}} \phi\left(2^k x, 2^k y\right) < +\infty \tag{11}$$

for all $x, y \in E$. If a function $f : E \to F$ satisfies the functional inequality

$$\|f(x+y) - f(x) - f(y)\| \le \phi(x, y) \tag{12}$$

for all $x, y \in E$. Then there exists a unique additive mapping $T : E \to F$ which satisfies

$$\|f(x) - T(x)\| \le \Phi(x, y) \tag{13}$$

for all $x \in E$. If moreover $f(tx)$ is continuous in t for fixed $x \in E$, then T is linear.
This stability theorem is called Gavruta stabiliy via Hyers method of functional equations.
In 2008, K. Ravi et. al., [35] identified a specific instance of Gavruta's theorem for the summation of the sum and product of two $p-$ norms established by J. M. Rassias. Thus, it is known as the J. M. Rassias stability of functional equations.
Theorem 1.5 : [35] *Let $(E, \perp)$ denote an orthogonality normed space with norm $\|.\|_E$ and $(F, \|.\|_E)$ is a Banach space and $f : E \to F$ be a mapping which satisfying the inequality*

$$\left\|f(mx+y) + f(mx-y) - 2f(x+y) - 2f(x-y) - 2(m^2-2)f(x) + 2f(y)\right\|_F \le \varepsilon \left\{\|x\|_E^p \|y\|_E^p + \left(\|x\|_E^{2p} + \|y\|_E^{2p}\right)\right\} \tag{14}$$

for all $x, y \in E$ with $x \perp y$, where ε and p are constants with $\varepsilon, p > 0$ and either $m > 1; p < 1$ or $m < 1; p > 1$ with $m \neq 0; m \neq \pm 1; m \neq \pm\sqrt{2}$ and $-1 \neq |m|^{p-1} < 1$. Then the limit

$$Q(x) = \lim_{n\to\infty} \frac{f(m^n x)}{m^{2n}} \tag{15}$$

exists for all $x \in E$ and $Q : E \to F$ is the unique orthogonally Euler-Lagrange quadratic mapping such that

$$\|f(x) - Q(x)\|_F \le \frac{\varepsilon}{2|m^2 - m^{2p}|} \|x\|_E^{2p} \tag{16}$$

for all $x \in E$.
The famous Cauchy Additive functional equation is

$$T(\nu_1 + \nu_w) = T(\nu_1) + T(\nu_2). \tag{17}$$

The Hyers type stability of (17) in various settings were investigated in [1, 2, 20, 25, 26, 28, 29, 37, 38]. Also, several other types of additive functional equations in various normed spaces were discussed in [3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 17, 33, 34] and references cited there in.
In this paper, the generalized Ulam – Gavruta stability of a generalized alternate 3 dimensional additive functional equation

$$\begin{aligned}
&T(\mu_1\nu_1 + \mu_2\nu_2 + \mu_3\nu_3) + T(\mu_1\nu_1 + \mu_2\nu_2 - \mu_3\nu_3) + T(\mu_1\nu_1 - \mu_2\nu_2 + \mu_3\nu_3) + T(-\mu_1\nu_1 + \mu_2\nu_2 + \mu_3\nu_3) + T(\mu_1\nu_1 - \mu_2\nu_2 - \mu_3\nu_3)\\
&+ T(-\mu_1\nu_1 + \mu_2\nu_2 - \mu_3\nu_3) + T(-\mu_1\nu_1 - \mu_2\nu_2 + \mu_3\nu_3) + T(\mu_2\nu_1 + \mu_3\nu_2 + \mu_1\nu_3) + T(\mu_2\nu_1 - \mu_3\nu_2 + \mu_1\nu_3) + T(-\mu_2\nu_1 + \mu_3\nu_2 + \mu_1\nu_3)\\
&+ T(\mu_2\nu_1 + \mu_3\nu_2 - \mu_1\nu_3) + T(-\mu_2\nu_1 - \mu_3\nu_2 + \mu_1\nu_3) + T(\mu_2\nu_1 - \mu_3\nu_2 - \mu_1\nu_3) + T(-\mu_2\nu_1 + \mu_3\nu_2 - \mu_1\nu_3) + T(\mu_3\nu_1 + \mu_1\nu_2 + \mu_2\nu_3)\\
&+ T(-\mu_3\nu_1 + \mu_1\nu_2 + \mu_2\nu_3) + T(\mu_3\nu_1 + \mu_1\nu_2 - \mu_2\nu_3) + T(\mu_3\nu_1 - \mu_1\nu_2 + \mu_2\nu_3) + T(-\mu_3\nu_1 + \mu_1\nu_2 - \mu_2\nu_3) + T(-\mu_3\nu_1 - \mu_1\nu_2 + \mu_2\nu_3)\\
&+ T(\mu_3\nu_1 - \mu_1\nu_2 - \mu_2\nu_3) = (\mu_1 + \mu_2 + \mu_3)\{T(\nu_1) + T(\nu_2) + T(\nu_3)\}
\end{aligned} \tag{18}$$

where μ_1, μ_2, μ_3 are non zero constants in various Banach spaces using Hyers method.

2. Solution of The Functional Equation (18)

In order to prove the solution of the functional equation, let us consider **A** and **B** be vector spaces.

Theorem 2.1: *If $T : \mathbf{A} \to \mathbf{B}$ satisfying the functional equation (17) if and only if $T : \mathbf{A} \to \mathbf{B}$ satisfying the functional equation (18) for all $v_1, v_2, v_3 \in \mathbf{A}$.*

Proof. Given $T : \mathbf{A} \to \mathbf{B}$ satisfying the functional equation (17). Replacing (v_1, v_2) by $(0,0)$; $(v_1, -v_1)$; (v_1, v_1); $(v_1, 2v_1)$ in (17) and for any positive integer t, we get

$$T(0) = 0;\ \ T(-v_1) = -T(v_1);\ \ T(2v_1) = 2T(v_1);\ \ T(3v_1) = 3T(v_1);\ \ T(t\,v_1) = t\,T(v_1);\quad \text{for all } v_1 \in \mathbf{A}. \tag{19}$$

Again replacing (v_1, v_2) by $(v_1, v_2 + v_3)$ in (17) and using (17), (19), we obtain

$$T(v_1 + v_2 + v_3) = T(v_1) + T(v_2) + T(v_3);\quad \text{for all } v_1, v_2, v_3 \in \mathbf{A}. \tag{20}$$

Substituting (v_1, v_2, v_3) by $(\mu_1 v_1, \mu_2 v_2, \mu_3 v_3)$; $(\mu_1 v_1, \mu_2 v_2, -\mu_3 v_3)$; $(\mu_1 v_1, -\mu_2 v_2, \mu_3 v_3)$; $(-\mu_1 v_1, \mu_2 v_2, \mu_3 v_3)$; $(\mu_1 v_1, -\mu_2 v_2, -\mu_3 v_3)$; $(-\mu_1 v_1, \mu_2 v_2, -\mu_3 v_3)$; $(-\mu_1 v_1, -\mu_2 v_2, \mu_3 v_3)$ in (20) and using (19), we attain

$$T(\mu_1 v_1 + \mu_2 v_2 + \mu_3 v_3) = \mu_1 T(v_1) + \mu_2 T(v_2) + \mu_3 T(v_3) \tag{21}$$
$$T(\mu_1 v_1 + \mu_2 v_2 - \mu_3 v_3) = \mu_1 T(v_1) + \mu_2 T(v_2) - \mu_3 T(v_3) \tag{22}$$
$$T(\mu_1 v_1 - \mu_2 v_2 + \mu_3 v_3) = \mu_1 T(v_1) - \mu_2 T(v_2) + \mu_3 T(v_3) \tag{23}$$
$$T(-\mu_1 v_1 + \mu_2 v_2 + \mu_3 v_3) = -\mu_1 T(v_1) + \mu_2 T(v_2) + \mu_3 T(v_3) \tag{24}$$
$$T(\mu_1 v_1 - \mu_2 v_2 - \mu_3 v_3) = \mu_1 T(v_1) - \mu_2 T(v_2) - \mu_3 T(v_3) \tag{25}$$
$$T(-\mu_1 v_1 + \mu_2 v_2 - \mu_3 v_3) = -\mu_1 T(v_1) + \mu_2 T(v_2) - \mu_3 T(v_3) \tag{26}$$
$$T(-\mu_1 v_1 - \mu_2 v_2 + \mu_3 v_3) = -\mu_1 T(v_1) - \mu_2 T(v_2) + \mu_3 T(v_3);\quad \text{for all } v_1, v_2, v_3 \in \mathbf{A}. \tag{27}$$

Again substituting (v_1, v_2, v_3) by $(\mu_2 v_1, \mu_3 v_2, \mu_1 v_3)$; $(\mu_2 v_1, -\mu_3 v_2, \mu_1 v_3)$; $(-\mu_2 v_1, \mu_3 v_2, \mu_1 v_3)$; $(\mu_2 v_1, \mu_3 v_2, -\mu_1 v_3)$; $(-\mu_2 v_1, -\mu_3 v_2, \mu_1 v_3)$; $(\mu_2 v_1, -\mu_3 v_2, -\mu_1 v_3)$; $(-\mu_2 v_1, \mu_3 v_2, -\mu_1 v_3)$ in (20) and using (19), we land

$$T(\mu_2 v_1 + \mu_3 v_2 + \mu_1 v_3) = \mu_2 T(v_1) + \mu_3 T(v_2) + \mu_1 T(v_3) \tag{28}$$
$$T(\mu_2 v_1 - \mu_3 v_2 + \mu_1 v_3) = \mu_2 T(v_1) - \mu_3 T(v_2) + \mu_1 T(v_3) \tag{29}$$
$$T(-\mu_2 v_1 + \mu_3 v_2 + \mu_1 v_3) = -\mu_2 T(v_1) + \mu_3 T(v_2) + \mu_1 T(v_3) \tag{30}$$
$$T(\mu_2 v_1 + \mu_3 v_2 - \mu_1 v_3) = \mu_2 T(v_1) + \mu_3 T(v_2) - \mu_1 T(v_3) \tag{31}$$
$$T(-\mu_2 v_1 - \mu_3 v_2 + \mu_1 v_3) = -\mu_2 T(v_1) - \mu_3 T(v_2) + \mu_1 T(v_3) \tag{32}$$
$$T(\mu_2 v_1 - \mu_3 v_2 - \mu_1 v_3) = \mu_2 T(v_1) - \mu_3 T(v_2) - \mu_1 T(v_3) \tag{33}$$
$$T(-\mu_2 v_1 + \mu_3 v_2 - \mu_1 v_3) = -\mu_2 T(v_1) + \mu_3 T(v_2) - \mu_1 T(v_3);\quad \text{for all } v_1, v_2, v_3 \in \mathbf{A}. \tag{34}$$

Finally substituting (v_1, v_2, v_3) by $(\mu_3 v_1, \mu_1 v_2, \mu_2 v_3)$; $(-\mu_3 v_1, \mu_1 v_2, \mu_2 v_3)$; $(\mu_3 v_1, \mu_1 v_2, -\mu_2 v_3)$; $(\mu_3 v_1, -\mu_1 v_2, \mu_2 v_3)$; $(-\mu_3 v_1, \mu_1 v_2, -\mu_2 v_3)$; $(-\mu_3 v_1, -\mu_1 v_2, \mu_2 v_3)$; $(\mu_3 v_1, -\mu_1 v_2, -\mu_2 v_3)$ in (20) and using (19), we reach

$$T(\mu_3 v_1 + \mu_1 v_2 + \mu_2 v_3) = \mu_3 T(v_1) + \mu_1 T(v_2) + \mu_2 T(v_3) \tag{35}$$
$$T(-\mu_3 v_1 + \mu_1 v_2 + \mu_2 v_3) = -\mu_3 T(v_1) + \mu_1 T(v_2) + \mu_2 T(v_3) \tag{36}$$
$$T(\mu_3 v_1 + \mu_1 v_2 - \mu_2 v_3) = \mu_3 T(v_1) + \mu_1 T(v_2) - \mu_2 T(v_3) \tag{37}$$
$$T(\mu_3 v_1 - \mu_1 v_2 + \mu_2 v_3) = \mu_3 T(v_1) - \mu_1 T(v_2) + \mu_2 T(v_3) \tag{38}$$
$$T(-\mu_3 v_1 + \mu_1 v_2 - \mu_2 v_3) = -\mu_3 T(v_1) + \mu_1 T(v_2) - \mu_2 T(v_3) \tag{39}$$
$$T(-\mu_3 v_1 - \mu_1 v_2 + \mu_2 v_3)24 = -\mu_3 T(v_1) - \mu_1 T(v_2) + \mu_2 T(v_3) \tag{40}$$
$$T(\mu_3 v_1 - \mu_1 v_2 - \mu_2 v_3) = \mu_3 T(v_1) - \mu_1 T(v_2) - \mu_2 T(v_3);\quad \text{for all } v_1, v_2, v_3 \in \mathbf{A}. \tag{41}$$

Adding all the equations from (21) to (41), we arrive (18) as desired.

Conversely, given $T : \mathbf{A} \to \mathbf{B}$ satisfying the functional equation (18). Replacing (v_1, v_2, v_3) by $(0,0,0)$ and $(v_1, 0, 0)$ in (18), we get

$$T(0) = 0 \tag{42}$$

Again replacing (v_1, v_2, v_3) by $(v_1, 0, 0)$ and (μ_1, μ_2, μ_3) by (μ_1, μ_1, μ_1) in (18), we obtain

$$4T(\mu_1 v_1) + 3T(-\mu_1 v_1) = \mu_1 T(v_1);\quad \text{for all } v_1 \in \mathbf{A}. \tag{43}$$

Substituting v_1 by $-v_1$ in (43), we have

$$4T(-\mu_1 v_1) + 3T(\mu_1 v_1) = \mu_1 T(-v_1);\quad \text{for all } v_1 \in \mathbf{A}. \tag{44}$$

Subtracting (44) from (43), we attain

$$T(\mu_1 v_1) - T(-\mu_1 v_1) = \mu_1 T(v_1) - \mu_1 T(-v_1) \Rightarrow T(\mu_1 v_1) = \mu_1 T(v_1); T(-\mu_1 v_1) = \mu_1 T(-v_1);\quad \text{for all } v_1 \in \mathbf{A}. \tag{45}$$

Using (45) in (43), we land

$$4\mu_1 T(v_1) + 3\mu_1 T(-v_1) = \mu_1 T(v_1) \Rightarrow 3\mu_1 T(-v_1) = -3\mu_1 T(v_1) \Rightarrow T(-v_1) = -T(v_1);\quad \text{for all } v_1 \in \mathbf{A}. \tag{46}$$

Substituting ν_3 by 0 in (18) and using (46), we reach

$$T(\mu_1\nu_1+\mu_2\nu_2)+T(\mu_2\nu_1+\mu_3\nu_2)+T(\mu_3\nu_1+\mu_1\nu_2)=(\mu_1+\mu_2+\mu_3)\{T(\nu_1)+T(\nu_2)\}; \quad \text{for all } \nu_1,\nu_2\in\mathbf{A}. \tag{47}$$

Again substituting (μ_1,μ_2,μ_3) by (μ_1,μ_1,μ_1) in (47) and using (45), we achieve

$$3\mu_1 T(\nu_1+\nu_2)=3\mu_1\{T(\nu_1)+T(\nu_2)\}; \quad \text{for all } \nu_1,\nu_2\in\mathbf{A} \tag{48}$$

which gives (17) as desired.
In order to stability results, let us take our functional equation as

$$\begin{aligned} T\Big(\mu_1\nu_1,\mu_2\nu_2,\mu_3\nu_3\Big) &= T(\mu_1\nu_1+\mu_2\nu_2+\mu_3\nu_3)+T(\mu_1\nu_1+\mu_2\nu_2-\mu_3\nu_3)+T(\mu_1\nu_1-\mu_2\nu_2+\mu_3\nu_3)+T(-\mu_1\nu_1+\mu_2\nu_2+\mu_3\nu_3)\\ &+T(\mu_1\nu_1-\mu_2\nu_2-\mu_3\nu_3)+T(-\mu_1\nu_1+\mu_2\nu_2-\mu_3\nu_3)+T(-\mu_1\nu_1-\mu_2\nu_2+\mu_3\nu_3)+T(\mu_2\nu_1+\mu_3\nu_2+\mu_1\nu_3)\\ &+T(\mu_1\nu_1-\mu_2\nu_2-\mu_3\nu_3)+T(\mu_2\nu_1-\mu_3\nu_2+\mu_1\nu_3)+T(-\mu_2\nu_1+\mu_3\nu_2+\mu_1\nu_3)+T(\mu_2\nu_1+\mu_3\nu_2-\mu_1\nu_3)\\ &+T(-\mu_2\nu_1-\mu_3\nu_2+\mu_1\nu_3)+T(\mu_2\nu_1-\mu_3\nu_2-\mu_1\nu_3)+T(-\mu_2\nu_1+\mu_3\nu_2-\mu_1\nu_3)+T(\mu_3\nu_1+\mu_1\nu_2+\mu_2\nu_3)\\ &+T(-\mu_3\nu_1+\mu_1\nu_2+\mu_2\nu_3)+T(\mu_3\nu_1+\mu_1\nu_2-\mu_2\nu_3)+T(\mu_3\nu_1-\mu_1\nu_2+\mu_2\nu_3)+T(-\mu_3\nu_1+\mu_1\nu_2-\mu_2\nu_3)\\ &+T(-\mu_3\nu_1-\mu_1\nu_2+\mu_2\nu_3)+T(\mu_3\nu_1-\mu_1\nu_2-\mu_2\nu_3)-[(\mu_1+\mu_2+\mu_3)\{T(\nu_1)+T(\nu_2)+T(\nu_3)\}]. \end{aligned}$$

3. Stability Results : Banach Space

In this section, we provide the generalized Ulam – Gavruta stability of the functional equation (18) in Banach Space. For that let us assume $\mathscr{A}_1$ be a normed space and $\mathscr{A}_2$ be a Banach space.
Now, we present some basic definitions related to normed spaces as in [42].
Definition 3.1 *Let X be linear space. A function $||.,.|| : X\times X\to[0,\infty)$ is called a normed space if it satisfies to following :*

(N1) $||x|| = 0$ *if and only if* $x = 0$;
(N2) $||x,y|| = ||y,x||$ *for all* $x,y\in X$;
(N3) $||\lambda x,y|| = |\lambda|||x,y||$ *for all* $x,y\in X$;
(N4) $||x+y,z|| \le ||x,z||+||y,z||$ *for all* $x,y,z\in X$.

Definition 3.2 *A sequence $\{x_n\}$ in a normed space X is called convergent if there exist $x\in X$ such that $\lim_{n\to\infty}||x_n-x|| = 0$ then $\lim_{n\to\infty}x_n = x$.*
Definition 3.3 *A sequence $\{x_n\}$ in a normed space X is called Cauchy sequence if there exist$\{x_m\}$ such that $\lim_{n\to\infty}||x_n-x_m|| = 0$.*
Definition 3.4 *A normed space X is called Banach space if every Cauchy sequence is convergent.*
Theorem 3.5 *Assume a function $T:\mathscr{A}_1\to\mathscr{A}_2$ satisfies the inequality*

$$\left\|T\Big(\mu_1\nu_1,\mu_2\nu_2,\mu_3\nu_3\Big)\right\| \le \mathscr{T}(\nu_1,\nu_2,\nu_3); \quad \textit{for all } \nu_1,\nu_2,\nu_3\in\mathscr{A}_1 \tag{49}$$

where $\mathscr{T}:\mathscr{A}_1^3\to(0,\infty]$ be a function satisfying

$$\lim_{a\to\infty}\frac{\mathscr{T}(\mu_4^{ak}\nu_1,\mu_4^{ak}\nu_2,\mu_4^{ak}\nu_3)}{\mu_4^{ak}}=0; \textit{ with } \mu_4=(\mu_1+\mu_2+\mu_3); \textit{ and } k=\pm1; \quad \textit{for all } \nu_1,\nu_2,\nu_3\in\mathscr{A}_1. \tag{50}$$

Then there exists a unique additive mapping $D:\mathscr{A}_1\to\mathscr{A}_2$ which satisfies (18) and the inequality

$$\|D(\nu_1)-T(\nu_1)\| \le \frac{1}{3\mu_4}\sum_{b=\frac{k-1}{2}}^{\infty}\frac{\mathscr{T}(\mu_4^b\nu_1,\mu_4^b\nu_1,\mu_4^b\nu_1)}{\mu_4^b}; \quad \textit{where} \quad D(\nu_1)=\lim_{a\to\infty}\frac{T(\mu_4^{ak}\nu_1)}{\mu_4^{ak}}; \quad \textit{for all } \nu_1\in\mathscr{A}_1. \tag{51}$$

Proof. Replacing (ν_1,ν_2,ν_3) by (ν_1,ν_1,ν_1) in (49) and using oddness of T, we have

$$\|3T([\mu_1+\mu_2+\mu_3]\nu_1)-3[(\mu_1+\mu_2+\mu_3)]T(\nu_1)\| \le \mathscr{T}(\nu_1,\nu_1,\nu_1) \Rightarrow \|3T(\mu_4\nu_1)-3\mu_4T(\nu_1)\| \le \mathscr{T}(\nu_1,\nu_1,\nu_1); \quad \text{for all } \nu_1\in\mathscr{A}_1. \tag{52}$$

It follows from (52) that

$$\left\|\frac{T(\mu_4\nu_1)}{\mu_4}-T(\nu_1)\right\| \le \frac{\mathscr{T}(\nu_1,\nu_1,\nu_1)}{3\mu_4}; \quad \text{for all } \nu_1\in\mathscr{A}_1. \tag{53}$$

For any positive integer a, (53) can be generalized as

$$\left\|\frac{T(\mu_4^a\nu_1)}{\mu_4^a}-T(\nu_1)\right\| \le \frac{1}{3\mu_4}\sum_{b=0}^{a-1}\frac{\mathscr{T}(\mu_4^b\nu_1,\mu_4^b\nu_1,\mu_4^b\nu_1)}{\mu_4^b}; \quad \text{for all } \nu_1\in\mathscr{A}_1. \tag{54}$$

Setting ν_1 by $\mu_4^c\nu_1$ in (54) and using (54), we arrive

$$\left\|\frac{T(\mu_4^a\mu_4^c\nu_1)}{\mu_4^a\mu_4^c}-\frac{T(\mu_4^c\nu_1)}{\mu_4^c}\right\| = \frac{1}{\mu_4^c}\left\|\frac{T(\mu_4^a\nu_1)}{\mu_4^a}-T(\mu_4^c\nu_1)\right\| \le \frac{1}{3\mu_4}\sum_{b=0}^{a-1}\frac{\mathscr{T}(\mu_4^b\mu_4^c\nu_1,\mu_4^b\mu_4^c\nu_1,\mu_4^b\mu_4^c\nu_1)}{\mu_4^b\mu_4^c} \to 0 \textit{ as } c\to\infty; \forall\, \nu_1\in\mathscr{A}_1. \tag{55}$$

Thus the sequence $\left\{\frac{T(\mu_4^a\nu_1)}{\mu_4^a}\right\}$ is a Cauchy sequence and it converges to $D(\nu_1)\in\mathscr{A}_2$. So, we define as

$$D(\nu_1)=\lim_{a\to\infty}\frac{T(\mu_4^a\nu_1)}{\mu_4^a}; \quad \text{for all } \nu_1\in\mathscr{A}_1. \tag{56}$$

Letting $a \to \infty$ in (54) and using (56), we reach

$$\|D(\nu_1) - T(\nu_1)\| \leq \frac{1}{3\mu_4} \sum_{b=0}^{\infty} \frac{\mathscr{T}(\mu_4^b \nu_1, \mu_4^b \nu_1, \mu_4^b \nu_1)}{\mu_4^b}; \quad \text{for all } \nu_1 \in \mathscr{A}_1. \tag{57}$$

Therefore (51) holds for $k = 1$. In order to show the existence of $D(\nu_1)$ satisfies (18), changing (ν_1, ν_2, ν_3) by $(\mu_4^a \nu_1, \mu_4^a \nu_2, \mu_4^a \nu_3)$ in (49), we achieve

$$\left\| \frac{1}{\mu_4^a} T\left(\mu_4^a \mu_1 \nu_1, \mu_4^a \mu_2 \nu_2, \mu_4^a \mu_3 \nu_3\right) \right\| \leq \frac{1}{\mu_4^a} \mathscr{T}(\mu_4^a \nu_1, \mu_4^a \nu_2, \mu_4^a \nu_3); \quad \text{for all } \nu_1, \nu_2, \nu_3 \in \mathscr{A}_1. \tag{58}$$

Letting $a \to \infty$ in (58) and using (50), (56), we obtain that $D(\nu_1)$ satisfies (18) for all $\nu_1, \nu_2, \nu_3 \in \mathscr{A}_1$. To prove the existence of $D(\nu_1)$ is unique, assume $D'(\nu_1)$ be a another mapping satisfies (18), (56), (57). Now,

$$\begin{aligned}
\|D(\nu_1) - D'(\nu_1)\| &\leq \frac{1}{\mu_4^c} \|D(\mu_4^c \nu_1) - T(\mu_4^c \nu_1) + T(\mu_4^c \nu_1) - D'(\mu_4^c \nu_1)\| \\
&\leq \frac{1}{\mu_4^c} \{\|D(\mu_4^c \nu_1) - T(\mu_4^c \nu_1)\| \, \|T(\mu_4^c \nu_1) - D'(\mu_4^c \nu_1)\|\} \\
&\leq \frac{2}{3\mu_4} \sum_{b=0}^{\infty} \frac{\mathscr{T}(\mu_4^{b+c} \nu_1, \mu_4^{b+c} \nu_1, \mu_4^{b+c} \nu_1)}{\mu_4^{b+c}} \to 0 \; as \; c \to \infty; \; \forall \; \nu_1 \in \mathscr{A}_1.
\end{aligned}$$

Thus $D(\nu_1)$ is unique. Therefore the theorem holds for $k = 1$.
Replacing ν_1 by $\frac{\nu_1}{\mu_4}$ in (52), we have

$$\left\| 3T(\nu_1) - 3\mu_4 T\left(\frac{\nu_1}{\mu_4}\right) \right\| \leq \mathscr{T}\left(\frac{\nu_1}{\mu_4}, \frac{\nu_1}{\mu_4}, \frac{\nu_1}{\mu_4}\right); \quad \text{for all } \nu_1 \in \mathscr{A}_1. \tag{59}$$

It follows from (59) that

$$\left\| T(\nu_1) - \mu_4 T\left(\frac{\nu_1}{\mu_4}\right) \right\| \leq \frac{1}{3} \mathscr{T}\left(\frac{\nu_1}{\mu_4}, \frac{\nu_1}{\mu_4}, \frac{\nu_1}{\mu_4}\right); \quad \text{for all } \nu_1 \in \mathscr{A}_1. \tag{60}$$

For any positive integer a, (60) can be generalized as

$$\left\| T(\nu_1) - \mu_4^a T\left(\frac{\nu_1}{\mu_4^a}\right) \right\| \leq \frac{1}{3\mu_4} \sum_{b=1}^{a} \mu_4^a \mathscr{T}\left(\frac{\nu_1}{\mu_4^a}, \frac{\nu_1}{\mu_4^a}, \frac{\nu_1}{\mu_4^a}\right); \quad \text{for all } \nu_1 \in \mathscr{A}_1. \tag{61}$$

The rest of the proof is similar lines to that of $k = 1$. So the theorem is true for $k = -1$. Hence the proof is complete.
From the above theorem, we have the following corollary regarding some stabilities of the functional equation (18).
Corollary 3.2 *Assume a function $T : \mathscr{A}_1 \to \mathscr{A}_2$ satisfies the inequality*

$$\left\| T\left(\mu_1 \nu_1, \mu_2 \nu_2, \mu_3 \nu_3\right) \right\| \leq \begin{cases} C; \\ C(|\nu_1|^E + |\nu_2|^E + |\nu_3|^E); \\ C(|\nu_1|^{E_1} + |\nu_2|^{E_2} + |\nu_3|^{E_3}); \\ C(|\nu_1|^E |\nu_2|^E |\nu_3|^E); \\ C(|\nu_1|^{E_1} |\nu_2|^{E_2} |\nu_3|^{E_3}); \\ C(|\nu_1|^{3E} + |\nu_2|^{3E} + |\nu_3|^{3E} + (|\nu_1|^E |\nu_2|^E |\nu_3|^E)); \end{cases} \quad \textit{for all } \nu_1, \nu_2, \nu_3 \in \mathscr{A}_1 \tag{62}$$

where $C > 0$ and $E \neq 0$. Then there exists a unique additive mapping $D : \mathscr{A}_1 \to \mathscr{A}_2$ which satisfies (18) and the inequality

$$\|D(\nu_1) - T(\nu_1)\| \leq \begin{cases} \left\{ \frac{C}{3|\mu_4 - 1|} \right\}; & \\ \left\{ \frac{C\,|\nu_1|^E}{|\mu_4 - \mu_4^E|} \right\}; & E \neq 1 \\ \left\{ \frac{C\,|\nu_1|^{E_1}}{3|\mu_4 - \mu_4^{E_1}|} + \frac{C\,|\nu_1|^{E_2}}{3|\mu_4 - \mu_4^{E_2}|} + \frac{C\,|\nu_1|^{E_3}}{3|\mu_4 - \mu_4^{E_3}|} \right\}; & E_1, E_2, E_3 \neq 1 \\ \left\{ \frac{C\,|\nu_1|^{3E}}{3|\mu_4 - \mu_4^{3E}|} \right\}; & 3E \neq 1 \\ \left\{ \frac{C\,|\nu_1|^{E_1+E_2+E_3}}{3|\mu_4 - \mu_4^{E_1+E_2+E_3}|} \right\}; & E_1 + E_2 + E_3 \neq 1 \\ \left\{ \frac{4\,C\,|\nu_1|^E}{3|\mu_4 - \mu_4^E|} \right\}; & 3E \neq 1 \end{cases} ; \quad \textit{for all } \nu_1 \in \mathscr{A}_1. \tag{63}$$

4. Stability Results : Generalized 2 - Banach Space

In this section, we provide the generalized Ulam – Gavruta stability of the functional equation (18) in Banach Space. For that let us assume $\mathscr{A}_1$ be a generalized 2- normed space and $\mathscr{A}_2$ be a generalized 2-Banach space. Now, we present some basic definitions related to Generalized 2-normed spaces given in [9, 10, 21].
Definition 4.1 *Let X be linear space. A function $N(.,.) : X \times X \to [0, \infty)$ is called a generalized 2-normed space if it satisfies to following :*

(G2N1) $N(x, y) = 0$ if and only if x and y are linearly independent vectors;
(G2N2) $N(x, y) = N(y, x)$ for all $x, y \in X$;
(G2N3) $N(\lambda x, y) = |\lambda| N(x, y)$ for all $x, y \in X$ and $X = \varphi, \varphi$ is a real or complex field;

(G2N4) $N(x+y,z) \leq N(x,z)+N(y,z)$ *for all* $x,y,z \in X$.

The generalized 2-normed space is denoted by $(X,N(.,.))$.

Definition 4.2 *A sequence* $\{x_n\}$ *in a generalized 2-normed space* $(X,N(.,.))$ *is called convergent if there exist* $x \in X$ *such that* $\lim_{n\to\infty} N(x_n - x,y) = 0$ *then* $\lim_{n\to\infty} N(x_n,y) = N(x,y)$ *for all* $y \in X$.

Definition 4.3 *A sequence* $\{x_n\}$ *in a generalized 2-normed space* $(X,N(.,.))$ *is called Cauchy sequence if there exist two linearly independent elements* y *and* z *in* X *such that* $\{N(x_n,y)\}$ *and* $\{N(x_n,z)\}$ *are real Cauchy sequences.*

Definition 4.4 *A generalized 2-normed space* $(X,N(.,.))$ *is called generalized 2-Banach space if every Cauchy sequence is convergent.*

Theorem 4.5 *Assume a function* $T : \mathscr{A}_1 \to \mathscr{A}_2$ *satisfies the inequality*

$$N\left(T\left(\mu_1\nu_1,\mu_2\nu_2,\mu_3\nu_3\right),\boldsymbol{C}\right) \leq \mathscr{T}((\nu_1,\nu_2,\nu_3),\boldsymbol{C}); \quad \textit{for all } \nu_1,\nu_2,\nu_3 \in \mathscr{A}_1 \textit{ and all } \boldsymbol{C} \in \mathscr{A}_1 \tag{64}$$

where $\mathscr{T} : \mathscr{A}_1^3 \to (0,\infty]$ *be a function satisfying*

$$\lim_{a\to\infty} \frac{\mathscr{T}((\mu_4^{ak}\nu_1,\mu_4^{ak}\nu_2,\mu_4^{ak}\nu_3),\boldsymbol{C})}{\mu_4^{ak}} = 0; \textit{ with } \mu_4 = (\mu_1+\mu_2+\mu_3); \textit{ and } k=\pm 1; \quad \textit{for all } \nu_1,\nu_2,\nu_3 \in \mathscr{A}_1 \textit{ and all } \boldsymbol{C} \in \mathscr{A}_1. \tag{65}$$

Then there exists a unique additive mapping $D : \mathscr{A}_1 \to \mathscr{A}_2$ *which satisfies (18) and the inequality*

$$N(D(\nu_1)-T(\nu_1),\boldsymbol{C}) \leq \frac{1}{3\mu_4}\sum_{b=\frac{k-1}{2}}^{\infty} \frac{\mathscr{T}((\mu_4^b\nu_1,\mu_4^b\nu_1,\mu_4^b\nu_1),\boldsymbol{C})}{\mu_4^b}; \textit{ where } N(D(\nu_1),\boldsymbol{C}) = \lim_{a\to\infty} N\left(\frac{T(\mu_4^{ak}\nu_1)}{\mu_4^{ak}},\boldsymbol{C}\right); \textit{ for all } \nu_1 \in \mathscr{A}_1 \textit{ and all } \boldsymbol{C} \in \mathscr{A}_1. \tag{66}$$

Proof. Replacing (ν_1,ν_2,ν_3) by (ν_1,ν_1,ν_1) in (64) and using oddness of T, we have

$$N(3T(\mu_4\nu_1)-3\mu_4T(\nu_1),\mathbf{C}) \leq \mathscr{T}((\nu_1,\nu_1,\nu_1),\mathbf{C}); \quad \text{for all } \nu_1 \in \mathscr{A}_1 \text{ and all } \mathbf{C} \in \mathscr{A}_1. \tag{67}$$

It follows from (67) that

$$N\left(\frac{T(\mu_4\nu_1)}{\mu_4} - T(\nu_1),\mathbf{C}\right) \leq \frac{\mathscr{T}((\nu_1,\nu_1,\nu_1),\mathbf{C})}{3\mu_4}; \quad \text{for all } \nu_1 \in \mathscr{A}_1 \text{ and all } \mathbf{C} \in \mathscr{A}_1. \tag{68}$$

For any positive integer a, (68) can be generalized as

$$N\left(\frac{T(\mu_4^a\nu_1)}{\mu_4^a} - T(\nu_1),\mathbf{C}\right) \leq \frac{1}{3\mu_4}\sum_{b=0}^{a-1} \frac{\mathscr{T}((\mu_4^b\nu_1,\mu_4^b\nu_1,\mu_4^b\nu_1),\mathbf{C})}{\mu_4^b}; \quad \text{for all } \nu_1 \in \mathscr{A}_1 \text{ and all } \mathbf{C} \in \mathscr{A}_1. \tag{69}$$

The rest of the proof is similar lines to that of Theorem 3.1. Hence the proof is complete.

From the above theorem, we have the following corollary regarding some stabilities of the functional equation (18).

Corollary 4.2 *Assume a function* $T : \mathscr{A}_1 \to \mathscr{A}_2$ *satisfies the inequality*

$$N\left(T\left(\mu_1\nu_1,\mu_2\nu_2,\mu_3\nu_3\right),\boldsymbol{C}\right) \leq \begin{cases} (C,\boldsymbol{C}); \\ C(|\nu_1,\boldsymbol{C}|^E+|\nu_2,\boldsymbol{C}|^E+|\nu_3,\boldsymbol{C}|^E); \\ C(|\nu_1,\boldsymbol{C}|^{E_1}+|\nu_2,\boldsymbol{C}|^{E_2}+|\nu_3,\boldsymbol{C}|^{E_3}); \\ C(|\nu_1,\boldsymbol{C}|^E|\nu_2,\boldsymbol{C}|^E|\nu_3,\boldsymbol{C}|^E); \\ C(|\nu_1,\boldsymbol{C}|^{E_1}|\nu_2,\boldsymbol{C}|^{E_2}|\nu_3,\boldsymbol{C}|^{E_3}); \\ C(|\nu_1,\boldsymbol{C}|^{3E}+|\nu_2,\boldsymbol{C}|^{3E}+|\nu_3,\boldsymbol{C}|^{3E}+(|\nu_1,\boldsymbol{C}|^E|\nu_2,\boldsymbol{C}|^E|\nu_3,\boldsymbol{C}|^E)); \end{cases} \quad \forall\ \nu_1,\nu_2,\nu_3 \in \mathscr{A}_1 \textit{ and all } \boldsymbol{C} \in \mathscr{A}_1 \tag{70}$$

where $C > 0$ *and* $E \neq 0$. *Then there exists a unique additive mapping* $D : \mathscr{A}_1 \to \mathscr{A}_2$ *which satisfies (18) and the inequality*

$$N(D(\nu_1)-T(\nu_1),\boldsymbol{C}) \leq \begin{cases} \left\{\frac{(C,\boldsymbol{C})}{3|\mu_4-1|}\right\}; & \\ \left\{\frac{C\ |\nu_1,\boldsymbol{C}|^E}{|\mu_4-\mu_4^E|}\right\}; & E \neq 1 \\ \left\{\frac{C\ |\nu_1,\boldsymbol{C}|^{E_1}}{3|\mu_4-\mu_4^{E_1}|} + \frac{C\ |\nu_1,\boldsymbol{C}|^{E_2}}{3|\mu_4-\mu_4^{E_2}|} + \frac{C\ |\nu_1,\boldsymbol{C}|^{E_3}}{3|\mu_4-\mu_4^{E_3}|}\right\}; & E_1,E_2,E_3 \neq 1 \\ \left\{\frac{C\ |\nu_1,\boldsymbol{C}|^{3E}}{3|\mu_4-\mu_4^{3E}|}\right\}; & 3E \neq 1 \\ \left\{\frac{C\ |\nu_1,\boldsymbol{C}|^{E_1+E_2+E_3}}{3|\mu_4-\mu_4^{E_1+E_2+E_3}|}\right\}; & E_1+E_2+E_3 \neq 1 \\ \left\{\frac{4\ C\ |\nu_1,\boldsymbol{C}|^E}{3|\mu_4-\mu_4^E|}\right\}; & 3E \neq 1 \end{cases} \quad ; \quad \textit{for all } \nu_1 \in \mathscr{A}_1 \textit{ and all } \boldsymbol{C} \in \mathscr{A}_1. \tag{71}$$

5. Stability Results : Fuzzy Banach Space

In this section, we provide the generalized Ulam – Gavruta stability of the functional equation (18) in fuzzy Banach Space. For that let us assume $(\mathscr{A}_1, F)$ be a fuzzy normed space and $(\mathscr{A}_2, F')$ be a fuzzy Banach space. Now, we present some basic definitions related to fuzzy normed spaces given in [15, 16, 19, 27, 30, 31, 39, 44].

Definition 5.1 *Let X be a real linear space. A function $F : X \times \mathbb{R} \to [0,1]$ is said to be a fuzzy norm on X if for all $x, y \in X$ and all $s, t \in \mathbb{R}$,*
$(FBS1)$ $F(x,c) = 0$ *for* $c \leq 0$;
$(FBS2)$ $x = 0$ *if and only if* $F(x,c) = 1$ *for all* $c > 0$;
$(FBS3)$ $F(cx,t) = F\left(x, \frac{t}{|c|}\right)$ *if* $c \neq 0$;
$(FBS4)$ $F(x+y, s+t) \geq min\{F(x,s), F(y,t)\}$;
$(FBS5)$ $F(x,\cdot)$ *is a non-decreasing function on* $\mathbb{R}$ *and* $\lim_{t\to\infty} F(x,t) = 1$;
$(FBS6)$ *for* $x \neq 0, F(x,\cdot)$ *is (upper semi) continuous on* $\mathbb{R}$.
The pair (X,F) is called a fuzzy normed linear space.

Example 5.2 *Let $(X, ||\cdot||)$ be a normed linear space. Then*

$$F\ (x,t) = \begin{cases} \dfrac{t}{t+\|x\|}, & t > 0,\ x \in X, \\ 0, & t \leq 0,\ x \in X \end{cases}$$

is a fuzzy norm on z.

Definition 5.3 *Let (X,F) be a fuzzy normed linear space. Let x_n be a sequence in X. Then x_n is said to be convergent if there exists $x \in X$ such that $\lim_{n\to\infty} F(x_n - x, t) = 1$ for all $t > 0$. In that case, x is called the limit of the sequence x_n and we denote it by $F\left(\lim_{n\to\infty} x_n - x, t\right) = 1$.*

Definition 5.4 *A sequence x_n in X is called Cauchy if for each $\varepsilon > 0$ and each $t > 0$ there exists n_0 such that for all $n \geq n_0$ and all $p > 0$, we have $F(x_{n+p} - x_n, t) > 1 - \varepsilon$.*

Definition 5.5 *Every convergent sequence in a fuzzy normed space is Cauchy. If each Cauchy sequence is convergent, then the fuzzy norm is said to be complete and the fuzzy normed space is called a fuzzy Banach space.*

Theorem 5.6 *Assume a function $T : \mathscr{A}_1 \to \mathscr{A}_2$ satisfies the inequality*

$$F\left(T\left(\mu_1 \nu_1, \mu_2 \nu_2, \mu_3 \nu_3\right), \boldsymbol{C}\right) \geq F'\left(\mathscr{T}(\nu_1, \nu_2, \nu_3), \boldsymbol{C}\right); \quad \text{for all } \nu_1, \nu_2, \nu_3 \in \mathscr{A}_1 \text{ and all } \boldsymbol{C} \in \mathscr{A}_1 \tag{72}$$

where $\mathscr{T} : \mathscr{A}_1^3 \to (0,\infty]$ be a function satisfying

$$F'\left(\mathscr{T}(\mu_4^{ak}\nu_1, \mu_4^{ak}\nu_2, \mu_4^{ak}\nu_3), \boldsymbol{C}\right) \geq F'\left(\boldsymbol{S}^{ak}\mathscr{T}(\nu_1, \nu_2, \nu_3), \boldsymbol{C}\right); \quad \text{for all } \nu_1, \nu_2, \nu_3 \in \mathscr{A}_1 \text{ and all } \boldsymbol{C} \in \mathscr{A}_1. \tag{73}$$

with $\boldsymbol{S} > 0$ and $0 < \frac{S}{\mu_4} < 1$ and

$$\lim_{a\to\infty} F'\left(\mathscr{T}(\mu_4^{ak}\nu_1, \mu_4^{ak}\nu_2, \mu_4^{ak}\nu_3), \mu_4^{ak}\ \boldsymbol{C}\right) = 1; \text{ with } \mu_4 = (\mu_1 + \mu_2 + \mu_3); \text{ and } k = \pm 1; \tag{74}$$

for all $\nu_1, \nu_2, \nu_3 \in \mathscr{A}_1$ and all $\boldsymbol{C} \in \mathscr{A}_1$. Then there exists a unique additive mapping $D : \mathscr{A}_1 \to \mathscr{A}_2$ which satisfies (18) and the inequality

$$F(D(\nu_1) - T(\nu_1), \boldsymbol{C}) \geq F'(\mathscr{T}(\nu_1, \nu_1, \nu_1), 3|\mu_4 - \boldsymbol{S}|\ \boldsymbol{C}); \text{where } F(D(\nu_1), \boldsymbol{C}) = F\left(\lim_{a\to\infty} \frac{T(\mu_4^{ak}\nu_1)}{\mu_4^{ak}}, \boldsymbol{C}\right); \text{ for all } \nu_1 \in \mathscr{A}_1 \text{ and all } \boldsymbol{C} \in \mathscr{A}_1. \tag{75}$$

Proof. Replacing (ν_1, ν_2, ν_3) by (ν_1, ν_1, ν_1) in (72) and using oddness of T, we get

$$F(3T(\mu_4\nu_1) - 3\mu_4 T(\nu_1), \mathbf{C}) \geq \mathscr{T}((\nu_1, \nu_1, \nu_1), \mathbf{C}); \quad \text{for all } \nu_1 \in \mathscr{A}_1 \text{ and all } \mathbf{C} \in \mathscr{A}_1. \tag{76}$$

It follows from (76) and *(FBS3)* that

$$F\left(\frac{T(\mu_4\nu_1)}{\mu_4} - T(\nu_1), \frac{1}{3\mu_4}\ \mathbf{C}\right) \geq F'\left(\mathscr{T}(\nu_1, \nu_1, \nu_1),\ \mathbf{C}\right); \quad \text{for all } \nu_1 \in \mathscr{A}_1 \text{ and all } \mathbf{C} \in \mathscr{A}_1. \tag{77}$$

Setting ν_1 by $\mu_4^a \nu_1$ in (76) using (73) and *(FBS3)*, we obtain

$$\begin{aligned} F\left(\frac{T(\mu_4^a\ \mu_4\nu_1)}{\mu_4^a\ \mu_4} - \frac{T(\mu_4^a\nu_1)}{\mu_4^a}, \frac{1}{3\mu_4\ \mu_4^a}\ \mathbf{C}\right) &\geq F'\left(\mathscr{T}(\mu_4^a\nu_1, \mu_4^a\nu_1, \mu_4^a\nu_1),\ \mathbf{C}\right) \\ &\geq F'\left(\mathbf{S}^a\mathscr{T}(\nu_1, \nu_1, \nu_1),\ \mathbf{C}\right) \\ &= F'\left(\mathscr{T}(\nu_1, \nu_1, \nu_1), \frac{1}{\mathbf{S}^a}\mathbf{C}\right); \quad \text{for all } \nu_1 \in \mathscr{A}_1 \text{ and all } \mathbf{C} \in \mathscr{A}_1. \end{aligned} \tag{78}$$

Putting $\mathbf{C}$ by $\mathbf{S}^a\ \mathbf{C}$ in (78), we have

$$F\left(\frac{T(\mu_4^{a+1}\nu_1)}{\mu_4^{a+1}} - \frac{T(\mu_4^a\nu_1)}{\mu_4^a}, \frac{1}{3\mu_4} \cdot \frac{\mathbf{S}^a}{\mu_4^a}\ \mathbf{C}\right) \geq F'\left(\mathscr{T}(\nu_1, \nu_1, \nu_1),\ \mathbf{C}\right); \quad \text{for all } \nu_1 \in \mathscr{A}_1 \text{ and all } \mathbf{C} \in \mathscr{A}_1. \tag{79}$$

It follows form (79) and *(FBS4)* that

$$F\left(\frac{T(\mu_4^a\nu_1)}{\mu_4^a}-T(\nu_1),\frac{1}{3\mu_4}\cdot\sum_{b=0}^{a-1}\frac{\mathbf{S}^b}{\mu_4^b}\mathbf{C}\right)=F\left(\sum_{b=0}^{a-1}\frac{T(\mu_4^{b+1}\nu_1)}{\mu_4^{b+1}}-\frac{T(\mu_4^b\nu_1)}{\mu_4^b},\frac{1}{3\mu_4}\cdot\sum_{b=0}^{a-1}\frac{\mathbf{S}^b}{\mu_4^b}\mathbf{C}\right)$$
$$\geq \min\bigcup_{b=0}^{a-1}\left\{F\left(\frac{T(\mu_4^{b+1}\nu_1)}{\mu_4^{b+1}}-\frac{T(\mu_4^b\nu_1)}{\mu_4^b},\frac{1}{3\mu_4}\cdot\frac{\mathbf{S}^b}{\mu_4^b}\mathbf{C}\right)\right\}$$
$$\geq F'\left(\mathscr{T}(\nu_1,\nu_1,\nu_1),\ \mathbf{C}\right);\quad \text{for all } \nu_1\in\mathscr{A}_1 \text{ and all } \mathbf{C}\in\mathscr{A}_1. \tag{80}$$

Setting ν_1 by $\mu_4^c\nu_1$ in (80) and using (73) and *(FBS3)*, we arrive

$$F\left(\frac{T(\mu_4^a\mu_4^c\nu_1)}{\mu_4^a\mu_4^c}-\frac{T(\mu_4^c\nu_1)}{\mu_4^c},\frac{1}{3\mu_4}\cdot\sum_{b=0}^{a-1}\frac{\mathbf{S}^b}{\mu_4^b\mu_4^c}\mathbf{C}\right)\geq F'\left(\mathscr{T}(\mu_4^c\nu_1,\mu_4^c\nu_1,\mu_4^c\nu_1),\ \mathbf{C}\right)$$
$$\geq F'\left(\mathbf{S}^c\mathscr{T}(\nu_1,\nu_1,\nu_1),\ \mathbf{C}\right)$$
$$=F'\left(\mathscr{T}(\nu_1,\nu_1,\nu_1),\frac{1}{\mathbf{S}^c}\mathbf{C}\right);\quad \text{for all } \nu_1\in\mathscr{A}_1 \text{ and all } \mathbf{C}\in\mathscr{A}_1. \tag{81}$$

Putting $\mathbf{C}$ by $\mathbf{S}^c$ $\mathbf{C}$ in (81), we achieve

$$F\left(\frac{T(\mu_4^a\mu_4^c\nu_1)}{\mu_4^a\mu_4^c}-\frac{T(\mu_4^c\nu_1)}{\mu_4^c},\frac{1}{3\mu_4}\cdot\sum_{b=0}^{a-1}\frac{\mathbf{S}^b\mathbf{S}^c}{\mu_4^b\mu_4^c}\mathbf{C}\right)\geq F'\left(\mathscr{T}(\nu_1,\nu_1,\nu_1),\mathbf{C}\right);\quad \text{for all } \nu_1\in\mathscr{A}_1 \text{ and all } \mathbf{C}\in\mathscr{A}_1. \tag{82}$$

Changing $\mathbf{C}$ by $\dfrac{\mathbf{C}}{\frac{1}{3\mu_4}\cdot\sum_{b=0}^{a-1}\frac{\mathbf{S}^b\mathbf{S}^c}{\mu_4^b\mu_4^c}}$ in (82), we reach

$$F\left(\frac{T(\mu_4^a\mu_4^c\nu_1)}{\mu_4^a\mu_4^c}-\frac{T(\mu_4^c\nu_1)}{\mu_4^c},\ \mathbf{C}\right)\geq F'\left(\mathscr{T}(\nu_1,\nu_1,\nu_1),\frac{\mathbf{C}}{\frac{1}{\frac{1}{3\mu_4}\cdot\sum_{b=0}^{a-1}\frac{\mathbf{S}^b\mathbf{S}^c}{\mu_4^b\mu_4^c}}}\right);\quad \text{for all } \nu_1\in\mathscr{A}_1 \text{ and all } \mathbf{C}\in\mathscr{A}_1. \tag{83}$$

By data and the Cauchy criterion for convergence, *(FBS5)*, implies that the sequence $\left\{\dfrac{T(\mu_4^a\nu_1)}{\mu_4^a}\right\}$ is a Cauchy sequence in the fuzzy Banach space $(\mathscr{A}_2,F')$ and this sequence converges to $D(\nu_1)$. So, by notation, we write

$$F\left(D(\nu_1)-\lim_{a\to\infty}\frac{T(\mu_4^a\nu_1)}{\mu_4^a},\mathbf{C}\right)=1;\quad i.e., F(D(\nu_1),\mathbf{C})=F\left(\lim_{a\to\infty}\frac{T(\mu_4^a\nu_1)}{\mu_4^a},\mathbf{C}\right);\text{ for all } \nu_1\in\mathscr{A}_1 \text{ and all } \mathbf{C}\in\mathscr{A}_1. \tag{84}$$

Letting $c=0$ and $a\to\infty$ in (83) and using (84), we land

$$F\left(D(\nu_1)-T(\nu_1),\ \mathbf{C}\right)\geq F'\left(\mathscr{T}(\nu_1,\nu_1,\nu_1),3\ (\mu_4-\mathbf{S})\mathbf{C}\right);\quad \text{for all } \nu_1\in\mathscr{A}_1 \text{ and all } \mathbf{C}\in\mathscr{A}_1. \tag{85}$$

Therefore (85) holds for $k=1$. In order to show the existence of $D(\nu_1)$ satisfies (18), changing (ν_1,ν_2,ν_3) by $(\mu_4^a\nu_1,\mu_4^a\nu_2,\mu_4^a\nu_3)$ in (72), we attain

$$F\Big(\frac{1}{\mu_4^a}\Big(T(\mu_4^a(\mu_1\nu_1+\mu_2\nu_2+\mu_3\nu_3))+T(\mu_4^a(\mu_1\nu_1+\mu_2\nu_2-\mu_3\nu_3))+T(\mu_4^a(\mu_1\nu_1-\mu_2\nu_2+\mu_3\nu_3))+T(\mu_4^a(-\mu_1\nu_1+\mu_2\nu_2+\mu_3\nu_3))$$
$$+T(\mu_4^a(\mu_1\nu_1-\mu_2\nu_2-\mu_3\nu_3))+T(\mu_4^a(-\mu_1\nu_1+\mu_2\nu_2-\mu_3\nu_3))+T(\mu_4^a(-\mu_1\nu_1-\mu_2\nu_2+\mu_3\nu_3))+T(\mu_4^a(\mu_2\nu_1+\mu_3\nu_2+\mu_1\nu_3))$$
$$+T(\mu_4^a(\mu_2\nu_1-\mu_3\nu_2+\mu_1\nu_3))+T(\mu_4^a(-\mu_2\nu_1+\mu_3\nu_2+\mu_1\nu_3))+T(\mu_4^a(\mu_2\nu_1+\mu_3\nu_2-\mu_1\nu_3))+T(\mu_4^a(-\mu_2\nu_1-\mu_3\nu_2+\mu_1\nu_3))$$
$$+T(\mu_4^a(\mu_2\nu_1-\mu_3\nu_2-\mu_1\nu_3))+T(\mu_4^a(-\mu_2\nu_1+\mu_3\nu_2-\mu_1\nu_3))+T(\mu_4^a(\mu_3\nu_1+\mu_1\nu_2+\mu_2\nu_3))+T(\mu_4^a(-\mu_3\nu_1+\mu_1\nu_2+\mu_2\nu_3))$$
$$+T(\mu_4^a(\mu_3\nu_1+\mu_1\nu_2-\mu_2\nu_3))+T(\mu_4^a(\mu_3\nu_1-\mu_1\nu_2+\mu_2\nu_3))+T(\mu_4^a(-\mu_3\nu_1+\mu_1\nu_2-\mu_2\nu_3))+T(\mu_4^a(-\mu_3\nu_1-\mu_1\nu_2+\mu_2\nu_3))$$
$$+T(\mu_4^a(\mu_3\nu_1-\mu_1\nu_2-\mu_2\nu_3))-(\mu_1+\mu_2+\mu_3)\{T(\mu_4^a\nu_1)+T(\mu_4^a\nu_2)+T(\mu_4^a\nu_3)\}\Big),\mathbf{C}\Big)$$
$$\geq F'\left(\mathscr{T}(\mu_4^a\nu_1,\mu_4^a\nu_2,\mu_4^a\nu_3),\mu_4^a\mathbf{C}\right);\quad \text{for all } \nu_1,\nu_2,\nu_3\in\mathscr{A}_1 \text{ and all } \mathbf{C}\in\mathscr{A}_1. \tag{86}$$

Now,

$$F\Big(D(\mu_1\nu_1+\mu_2\nu_2+\mu_3\nu_3)+D(\mu_1\nu_1+\mu_2\nu_2-\mu_3\nu_3)+D(\mu_1\nu_1-\mu_2\nu_2+\mu_3\nu_3)+D(-\mu_1\nu_1+\mu_2\nu_2+\mu_3\nu_3)+D(\mu_1\nu_1-\mu_2\nu_2-\mu_3\nu_3)$$
$$+D(-\mu_1\nu_1+\mu_2\nu_2-\mu_3\nu_3)+D(-\mu_1\nu_1-\mu_2\nu_2+\mu_3\nu_3)+D(\mu_2\nu_1+\mu_3\nu_2+\mu_1\nu_3)+D(\mu_2\nu_1-\mu_3\nu_2+\mu_1\nu_3)+D(-\mu_2\nu_1+\mu_3\nu_2+\mu_1\nu_3)$$
$$+D(\mu_2\nu_1+\mu_3\nu_2-\mu_1\nu_3)+D(-\mu_2\nu_1-\mu_3\nu_2+\mu_1\nu_3)+D(\mu_2\nu_1-\mu_3\nu_2-\mu_1\nu_3)+D(-\mu_2\nu_1+\mu_3\nu_2-\mu_1\nu_3)+D(\mu_3\nu_1+\mu_1\nu_2+\mu_2\nu_3)$$
$$+D(-\mu_3\nu_1+\mu_1\nu_2+\mu_2\nu_3)+D(\mu_3\nu_1+\mu_1\nu_2-\mu_2\nu_3)+D(\mu_3\nu_1-\mu_1\nu_2+\mu_2\nu_3)+D(-\mu_3\nu_1+\mu_1\nu_2-\mu_2\nu_3)+D(-\mu_3\nu_1-\mu_1\nu_2+\mu_2\nu_3)$$
$$+D(\mu_3\nu_1-\mu_1\nu_2-\mu_2\nu_3)-(\mu_1+\mu_2+\mu_3)\{D(\nu_1)+D(\nu_2)+D(\nu_3)\},\mathbf{C}\Big)$$
$$\geq \min\ F\Big\{\Big(D(\mu_1\nu_1+\mu_2\nu_2+\mu_3\nu_3)-\frac{1}{\mu_4^a}\,T(\mu_4^a(\mu_1\nu_1+\mu_2\nu_2+\mu_3\nu_3)),\frac{\mathbf{C}}{23}\Big),\Big(D(\mu_1\nu_1+\mu_2\nu_2-\mu_3\nu_3)-\frac{1}{\mu_4^a}\,T(\mu_4^a(\mu_1\nu_1+\mu_2\nu_2-\mu_3\nu_3)),\frac{\mathbf{C}}{23}\Big),$$
$$\Big(D(\mu_1\nu_1-\mu_2\nu_2+\mu_3\nu_3)-\frac{1}{\mu_4^a}\,T(\mu_4^a(\mu_1\nu_1-\mu_2\nu_2+\mu_3\nu_3)),\frac{\mathbf{C}}{23}\Big),\Big(D(-\mu_1\nu_1+\mu_2\nu_2+\mu_3\nu_3)-\frac{1}{\mu_4^a}\,T(\mu_4^a(-\mu_1\nu_1+\mu_2\nu_2+\mu_3\nu_3)),\frac{\mathbf{C}}{23}\Big),$$
$$\Big(D(\mu_1\nu_1-\mu_2\nu_2-\mu_3\nu_3)-\frac{1}{\mu_4^a}\,T(\mu_4^a(\mu_1\nu_1-\mu_2\nu_2-\mu_3\nu_3)),\frac{\mathbf{C}}{23}\Big),\Big(D(-\mu_1\nu_1+\mu_2\nu_2-\mu_3\nu_3)-\frac{1}{\mu_4^a}\,T(\mu_4^a(-\mu_1\nu_1+\mu_2\nu_2-\mu_3\nu_3)),\frac{\mathbf{C}}{23}\Big),$$
$$\Big(D(-\mu_1\nu_1-\mu_2\nu_2+\mu_3\nu_3)-\frac{1}{\mu_4^a}\,T(\mu_4^a(-\mu_1\nu_1-\mu_2\nu_2+\mu_3\nu_3)),\frac{\mathbf{C}}{23}\Big),\Big(D(\mu_2\nu_1+\mu_3\nu_2+\mu_1\nu_3)-\frac{1}{\mu_4^a}\,T(\mu_4^a(\mu_2\nu_1+\mu_3\nu_2+\mu_1\nu_3)),\frac{\mathbf{C}}{23}\Big),$$
$$\Big(D(\mu_2\nu_1-\mu_3\nu_2+\mu_1\nu_3)-\frac{1}{\mu_4^a}\,T(\mu_4^a(\mu_2\nu_1-\mu_3\nu_2+\mu_1\nu_3)),\frac{\mathbf{C}}{23}\Big),\Big(D(-\mu_2\nu_1+\mu_3\nu_2+\mu_1\nu_3)-\frac{1}{\mu_4^a}\,T(\mu_4^a(-\mu_2\nu_1+\mu_3\nu_2+\mu_1\nu_3)),\frac{\mathbf{C}}{23}\Big)$$
$$\Big(D(\mu_2\nu_1+\mu_3\nu_2-\mu_1\nu_3)-\frac{1}{\mu_4^a}\,T(\mu_4^a(\mu_2\nu_1+\mu_3\nu_2-\mu_1\nu_3)),\frac{\mathbf{C}}{23}\Big),\Big(D(-\mu_2\nu_1-\mu_3\nu_2+\mu_1\nu_3)-\frac{1}{\mu_4^a}\,T(\mu_4^a(-\mu_2\nu_1-\mu_3\nu_2+\mu_1\nu_3)),\frac{\mathbf{C}}{23}\Big),$$
$$\Big(D(\mu_2\nu_1-\mu_3\nu_2-\mu_1\nu_3)-\frac{1}{\mu_4^a}\,T(\mu_4^a(\mu_2\nu_1-\mu_3\nu_2-\mu_1\nu_3)),\frac{\mathbf{C}}{23}\Big),\Big(D(-\mu_2\nu_1+\mu_3\nu_2-\mu_1\nu_3)-\frac{1}{\mu_4^a}\,T(\mu_4^a(-\mu_2\nu_1+\mu_3\nu_2-\mu_1\nu_3)),\frac{\mathbf{C}}{23}\Big),$$
$$\Big(D(\mu_3\nu_1+\mu_1\nu_2+\mu_2\nu_3)-\frac{1}{\mu_4^a}\,T(\mu_4^a(\mu_3\nu_1+\mu_1\nu_2+\mu_2\nu_3)),\frac{\mathbf{C}}{23}\Big),\Big(D(-\mu_3\nu_1+\mu_1\nu_2+\mu_2\nu_3)-\frac{1}{\mu_4^a}\,T(\mu_4^a(-\mu_3\nu_1+\mu_1\nu_2+\mu_2\nu_3)),\frac{\mathbf{C}}{23}\Big)$$
$$\Big(D(\mu_3\nu_1+\mu_1\nu_2-\mu_2\nu_3)-\frac{1}{\mu_4^a}\,T(\mu_4^a(\mu_3\nu_1+\mu_1\nu_2-\mu_2\nu_3)),\frac{\mathbf{C}}{23}\Big),\Big(D(\mu_3\nu_1-\mu_1\nu_2+\mu_2\nu_3)-\frac{1}{\mu_4^a}\,T(\mu_4^a(\mu_3\nu_1-\mu_1\nu_2+\mu_2\nu_3)),\frac{\mathbf{C}}{23}\Big)$$
$$\Big(D(-\mu_3\nu_1+\mu_1\nu_2-\mu_2\nu_3)-\frac{1}{\mu_4^a}\,T(\mu_4^a(-\mu_3\nu_1+\mu_1\nu_2-\mu_2\nu_3)),\frac{\mathbf{C}}{23}\Big),$$
$$\Big(D(-\mu_3\nu_1-\mu_1\nu_2+\mu_2\nu_3)-\frac{1}{\mu_4^a}\,T(\mu_4^a(-\mu_3\nu_1+\mu_1\nu_2-\mu_2\nu_3)),\frac{\mathbf{C}}{23}\Big),\Big(D(\mu_3\nu_1-\mu_1\nu_2-\mu_2\nu_3)-\frac{1}{\mu_4^a}\,T(\mu_4^a(\mu_3\nu_1-\mu_1\nu_2-\mu_2\nu_3)),\frac{\mathbf{C}}{23}\Big),$$
$$\Big(-(\mu_1+\mu_2+\mu_3)\{D(\nu_1)+D(\nu_2)+D(\nu_3)\}-\frac{(\mu_1+\mu_2+\mu_3)}{\mu_4^a}\{T(\mu_4^a\nu_1)+T(\mu_4^a\nu_2)+T(\mu_4^a\nu_3)\},\frac{\mathbf{C}}{23}\Big),$$
$$F\Big(\frac{1}{\mu_4^a}\Big(T(\mu_4^a(\mu_1\nu_1+\mu_2\nu_2+\mu_3\nu_3))+T(\mu_4^a(\mu_1\nu_1+\mu_2\nu_2-\mu_3\nu_3))+T(\mu_4^a(\mu_1\nu_1-\mu_2\nu_2+\mu_3\nu_3))+T(\mu_4^a(-\mu_1\nu_1+\mu_2\nu_2+\mu_3\nu_3))$$
$$+T(\mu_4^a(\mu_1\nu_1-\mu_2\nu_2-\mu_3\nu_3))+T(\mu_4^a(-\mu_1\nu_1+\mu_2\nu_2-\mu_3\nu_3))+T(\mu_4^a(-\mu_1\nu_1-\mu_2\nu_2+\mu_3\nu_3))+T(\mu_4^a(\mu_2\nu_1+\mu_3\nu_2+\mu_1\nu_3))$$
$$+T(\mu_4^a(\mu_2\nu_1-\mu_3\nu_2+\mu_1\nu_3))+T(\mu_4^a(-\mu_2\nu_1+\mu_3\nu_2+\mu_1\nu_3))+T(\mu_4^a(\mu_2\nu_1+\mu_3\nu_2-\mu_1\nu_3))+T(\mu_4^a(-\mu_2\nu_1-\mu_3\nu_2+\mu_1\nu_3))$$
$$+T(\mu_4^a(\mu_2\nu_1-\mu_3\nu_2-\mu_1\nu_3))+T(\mu_4^a(-\mu_2\nu_1+\mu_3\nu_2-\mu_1\nu_3))+T(\mu_4^a(\mu_3\nu_1+\mu_1\nu_2+\mu_2\nu_3))+T(\mu_4^a(-\mu_3\nu_1+\mu_1\nu_2+\mu_2\nu_3))$$
$$+T(\mu_4^a(\mu_3\nu_1+\mu_1\nu_2-\mu_2\nu_3))+T(\mu_4^a(\mu_3\nu_1-\mu_1\nu_2+\mu_2\nu_3))+T(\mu_4^a(-\mu_3\nu_1+\mu_1\nu_2-\mu_2\nu_3))+T(\mu_4^a(-\mu_3\nu_1-\mu_1\nu_2+\mu_2\nu_3))$$
$$+T(\mu_4^a(\mu_3\nu_1-\mu_1\nu_2-\mu_2\nu_3))-(\mu_1+\mu_2+\mu_3)\{T(\mu_4^a\nu_1)+T(\mu_4^a\nu_2)+T(\mu_4^a\nu_3)\}\Big),\frac{\mathbf{C}}{23}\Big\};\text{for all } \nu_1,\nu_2,\nu_3\in\mathscr{A}_1 \text{ and all } \mathbf{C}\in\mathscr{A}_1. \tag{87}$$

Letting $a\to\infty$ in (87) and using (84), (86), (74), we obtain that

$$F\Big(D(\mu_1\nu_1+\mu_2\nu_2+\mu_3\nu_3)+D(\mu_1\nu_1+\mu_2\nu_2-\mu_3\nu_3)+D(\mu_1\nu_1-\mu_2\nu_2+\mu_3\nu_3)+D(-\mu_1\nu_1+\mu_2\nu_2+\mu_3\nu_3)+D(\mu_1\nu_1-\mu_2\nu_2-\mu_3\nu_3)$$
$$+D(-\mu_1\nu_1+\mu_2\nu_2-\mu_3\nu_3)+D(-\mu_1\nu_1-\mu_2\nu_2+\mu_3\nu_3)+D(\mu_2\nu_1+\mu_3\nu_2+\mu_1\nu_3)+D(\mu_2\nu_1-\mu_3\nu_2+\mu_1\nu_3)+D(-\mu_2\nu_1+\mu_3\nu_2+\mu_1\nu_3)$$
$$+D(\mu_2\nu_1+\mu_3\nu_2-\mu_1\nu_3)+D(-\mu_2\nu_1-\mu_3\nu_2+\mu_1\nu_3)+D(\mu_2\nu_1-\mu_3\nu_2-\mu_1\nu_3)+D(-\mu_2\nu_1+\mu_3\nu_2-\mu_1\nu_3)+D(\mu_3\nu_1+\mu_1\nu_2+\mu_2\nu_3)$$
$$+D(-\mu_3\nu_1+\mu_1\nu_2+\mu_2\nu_3)+D(\mu_3\nu_1+\mu_1\nu_2-\mu_2\nu_3)+D(\mu_3\nu_1-\mu_1\nu_2+\mu_2\nu_3)+D(-\mu_3\nu_1+\mu_1\nu_2-\mu_2\nu_3)+D(-\mu_3\nu_1-\mu_1\nu_2+\mu_2\nu_3)$$
$$+D(\mu_3\nu_1-\mu_1\nu_2-\mu_2\nu_3)-(\mu_1+\mu_2+\mu_3)\{D(\nu_1)+D(\nu_2)+D(\nu_3)\},\mathbf{C}\Big)$$
$$\geq \min\ F\Big\{1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1\Big\}=1;\text{for all } \nu_1,\nu_2,\nu_3\in\mathscr{A}_1 \text{ and all } \mathbf{C}\in\mathscr{A}_1. \tag{88}$$

By *(FBS2)*, we see that $D(\nu_1)$ satisfies (18) for all $\nu_1,\nu_2,\nu_3\in\mathscr{A}_1$. To prove the existence of $D(\nu_1)$ is unique, assume $D'(\nu_1)$ be a another mapping satisfies (18), (84), (85). Now, using (73), $(FBS3)$, $(FBS5)$ and $(FBS2)$, we arrive that

$$\begin{aligned} F\Big(D(\nu_1)-D'(\nu_1),\mathbf{C}\Big) &\geq F\Big(D(\mu_4^c\nu_1)-D(\mu_4^c\nu_1)+T(\mu_4^c\nu_1)-D'(\mu_4^c\nu_1),\mu_4^c\mathbf{C}\Big) \\ &\geq \min\left\{F\Big(D(\mu_4^c\nu_1)-D(\mu_4^c\nu_1),\frac{\mu_4^c\,\mathbf{C}}{2}\Big),F\Big(T(\mu_4^c\nu_1)-D'(\mu_4^c\nu_1),\frac{\mu_4^c\,\mathbf{C}}{2}\Big)\right\} \\ &\geq F'\left(\mathscr{T}(\mu_4^c\nu_1,\mu_4^c\nu_1,\mu_4^c\nu_1),\frac{3(\mu_4-\mathbf{S})\mu_4^c\mathbf{C}}{2}\right)\geq F'\left(\mathscr{T}(\nu_1,\nu_1,\nu_1),\frac{3(\mu_4-\mathbf{S})\mu_4^c\mathbf{C}}{2\mathbf{S}^c}\right) \\ &\Rightarrow F\Big(D(\nu_1)-D'(\nu_1),\mathbf{C}\Big)=1 \end{aligned}$$

Thus $D(\nu_1)$ is unique. Therefore the theorem holds for $k=1$.
Replacing ν_1 by $\frac{\nu_1}{\mu_4}$ in (76) , we get

$$F\left(3T(\nu_1)-3\mu_4T\Big(\frac{\nu_1}{\mu_4}\Big),\mathbf{C}\right)\geq\mathscr{T}\left(\Big(\frac{\nu_1}{\mu_4},\frac{\nu_1}{\mu_4},\frac{\nu_1}{\mu_4}\Big),\mathbf{C}\right);\quad\text{for all }\nu_1\in\mathscr{A}_1\text{ and all }\mathbf{C}\in\mathscr{A}_1. \tag{89}$$

It follows from (89) and *(FBS3)* that

$$F\left(T(\nu_1)-\mu_4T\Big(\frac{\nu_1}{\mu_4}\Big),\frac{1}{3}\,\mathbf{C}\right)\geq F'\left(\mathscr{T}\Big((\frac{\nu_1}{\mu_4},\frac{\nu_1}{\mu_4},\frac{\nu_1}{\mu_4}),\mathbf{C}\Big),\ \mathbf{C}\right);\quad\text{for all }\nu_1\in\mathscr{A}_1\text{ and all }\mathbf{C}\in\mathscr{A}_1. \tag{90}$$

Setting ν_1 by $\frac{\nu_1}{\mu_4^a}$ in (90) using (73) and *(FBS3)*, we obtain

$$\begin{aligned} F\left(\mu_4^aT\Big(\frac{\nu_1}{\mu_4^a}\Big)-\mu_4^a\mu_4T\Big(\frac{\nu_1}{\mu_4\mu_4^a}\Big),\frac{\mu_4^a}{3\mu_4}\,\mathbf{C}\right) &\geq F'\left(\mathscr{T}\Big(\frac{\nu_1}{\mu_4^a},\frac{\nu_1}{\mu_4^a},\frac{\nu_1}{\mu_4^a}\Big),\ \mathbf{C}\right) \\ &\geq F'\left(\frac{1}{\mathbf{S}^a}\mathscr{T}(\nu_1,\nu_1,\nu_1),\ \mathbf{C}\right) \\ &= F'\left(\mathscr{T}(\nu_1,\nu_1,\nu_1),\mathbf{S}^a\,\mathbf{C}\right);\quad\text{for all }\nu_1\in\mathscr{A}_1\text{ and all }\mathbf{C}\in\mathscr{A}_1. \end{aligned} \tag{91}$$

Putting $\mathbf{C}$ by $\frac{\mathbf{C}}{\mathbf{S}^a}$ in (91), we have

$$F\left(\mu_4^aT\Big(\frac{\nu_1}{\mu_4^a}\Big)-\mu_4^{a+1}T\Big(\frac{\nu_1}{\mu_4^{a+1}}\Big),\frac{1}{3}\cdot\frac{\mu_4^a}{\mathbf{S}^a}\mathbf{C}\right)\geq F'\left(\mathscr{T}(\nu_1,\nu_1,\nu_1),\ \mathbf{C}\right);\quad\text{for all }\nu_1\in\mathscr{A}_1\text{ and all }\mathbf{C}\in\mathscr{A}_1. \tag{92}$$

The rest of the proof is similar lines to that of $k=1$. So the theorem is true for $k=-1$. Hence the proof is complete.
From the above theorem, we have the following corollary regarding some stabilities of the functional equation (18) .
Corollary 5.7 *Assume a function $T:\mathscr{A}_1\to\mathscr{A}_2$ satisfies the inequality*

$$F\Big(T\Big(\mu_1\nu_1,\mu_2\nu_2,\mu_3\nu_3\Big),\boldsymbol{C}\Big)\geq\begin{cases} F'\left(C,\boldsymbol{C}\right); \\ F'\left(C(|\nu_1|^E+|\nu_2|^E+|\nu_3|^E),\boldsymbol{C}\right); \\ F'\left(C(|\nu_1|^{E_1}+|\nu_2|^{E_2}+|\nu_3|^{E_3}),\boldsymbol{C}\right); \\ F'\left(C(|\nu_1|^E|\nu_2|^E|\nu_3|^E),\boldsymbol{C}\right); \\ F'\left(C(|\nu_1|^{E_1}|\nu_2|^{E_2}|\nu_3|^{E_3}),\boldsymbol{C}\right); \\ F'\left(C(|\nu_1|^{3E}+|\nu_2|^{3E}+|\nu_3|^{3E}+(|\nu_1|^E|\nu_2|^E|\nu_3|^E)),\boldsymbol{C}\right); \end{cases} \tag{93}$$

for all $\nu_1,\nu_2,\nu_3\in\mathscr{A}_1$ and all $\boldsymbol{C}\in\mathscr{A}_1$. Then there exists a unique additive mapping $D:\mathscr{A}_1\to\mathscr{A}_2$ which satisfies (18) and the inequality

$$F\left(D(\nu_1)-T(\nu_1),\boldsymbol{C}\right)\geq\begin{cases} F'\left(C,3|\mu_4-1|\,\boldsymbol{C}\right); & \\ F'\left(C\,|\nu_1|^E,3|\mu_4-\mu_4^E|\,\boldsymbol{C}\right); & E\neq1 \\ F'\left(C\,|\nu_1|^{E_1}+C\,|\nu_1|^{E_2}+C\,|\nu_1|^{E_3},3\{|\mu_4-\mu_4^{E_1}|+|\mu_4-\mu_4^{E_2}|+|\mu_4-\mu_4^{E_3}|\}\,\boldsymbol{C}\right); & E_1,E_2,E_3\neq1 \\ F'\left(C\,|\nu_1|^{3E},3|\mu_4-\mu_4^{3E}|\,\boldsymbol{C}\right); & 3E\neq1 \\ F'\left(C\,|\nu_1|^{E_1+E_2+E_3},3|\mu_4-\mu_4^{E_1+E_2+E_3}|\,\boldsymbol{C}\right); & E_1+E_2+E_3\neq1 \\ F'\left(C\,|\nu_1|^{3E},3|\mu_4-\mu_4^{3E}|\,\boldsymbol{C}\right); & 3E\neq1 \end{cases} \tag{94}$$

for all $\nu_1\in\mathscr{A}_1$ and all $\boldsymbol{C}\in\mathscr{A}_1$.

6. Stability Results : Random Banach Space

In this section, we provide the generalized Ulam – Gavruta stability of the functional equation (18) in Ramdom Banach Space. For that let us assume $\mathscr{A}_1$ be a linear space and $(\mathscr{A}_2,\mathbf{R},\mathbf{T})$ be a Random Banach space. Now, we present some basic definitions related to Random normed spaces given in [18, 23, 24, 40, 41].
Hereafter, in this section, Δ^+ is the space of distribution functions, that is, the space of all mappings $F:R\cup\{-\infty,+\infty\}\to[0,1]$ such that F is leftcontinuous and nondecreasing on $R, F(0)=0$ and $F(+\infty)=1$ D^+ is a subset of Δ^+ consisting of all functions $F\in\Delta^+$ for which $l^-\ \ F(+\infty)=1$ where $l^-f(x)$ denotes the left limit of the function f at the point x, $l^-f(x)=\lim_{t\to x}f(t)$. The space Δ^+ is partially ordered by the usual point-wise ordering of functions, i.e., $F\leq G$ if and only if $F(t)\leq G(t)$ for all $t\in R$. The maximal element for Δ^+ in this order is the d.f. given by: $\varepsilon_0(t)=\begin{cases}0, & if\ \ t\leq0,\\ 1, & if\ \ t>0.\end{cases}$
Definition 6.1 *A mapping $T:[0,1]\times[0,1]\to[0,1]$ is called a continuous triangular norm (briefly, a continuous $t-$norm) if $\boldsymbol{T}$ satisfies the following conditions:*

(a) $\boldsymbol{T}$ *is commutative and associative;*
(b) $\boldsymbol{T}$ *is continuous;*
(c) $\boldsymbol{T}(a,1)=a$ *for all* $a\in[0,1]$*;*
(d) $\boldsymbol{T}(a,b)\leq \boldsymbol{T}(c,d)$ *whenever* $a\leq c$ *and* $b\leq d$ *for all* $a,b,c,d\in[0,1]$.

Definition 6.2 *A random normed space (briefly, RN-space) is a triple* $(X,\boldsymbol{R},\boldsymbol{T})$*, where X is a vector space,* $\boldsymbol{T}$ *is a continuous* $t-$*norm and* $\boldsymbol{R}$ *is a mapping from X into* D^+ *satisfying the following conditions:*

(RBN1) $\boldsymbol{R}_x(t)=\varepsilon(t)$ *for all* $t>0$ *if and only if* $x=0$*;*
(RBN2) $\boldsymbol{R}_{\alpha\, x}(t)=\boldsymbol{R}_x(t/|\alpha|)$ *for all* $x\in X$*, and* $\alpha\in\mathbb{R}$ *with* $\alpha\neq 0$*;*
(RBN3) $\boldsymbol{R}_{x+y}(t+s)\geq \boldsymbol{T}\left(\boldsymbol{R}_x(t),\boldsymbol{R}_y(s)\right)$ *for all* $x,y\in X$ *and* $t,s\geq 0$.

Example 6.3 *Every normed spaces* $(X,||\cdot||)$ *defines a random normed space* $(X,\boldsymbol{R},\boldsymbol{T}_M)$*, where* $\boldsymbol{R}_x(t)=\frac{t}{t+||x||}$ *and* $\boldsymbol{T}_M$ *is the minimum* $t-$*norm. This space is called the induced random normed space.*

Definition 6.4 *Let* $(X,\boldsymbol{R},\boldsymbol{T})$ *be a RN-space. A sequence* $\{x_n\}$ *in X is said to be convergent to a point* $x\in X$ *if, for any* $\varepsilon>0$ *and* $\lambda>0$*, there exists a positive integer N such that* $\boldsymbol{R}_{x_n-x}(\varepsilon)>1-\lambda$ *for all* $n\geq N$.

Definition 6.5 *Let* $(X,\boldsymbol{R},\boldsymbol{T})$ *be a RN-space. A sequence* $\{x_n\}$ *in X is called a Cauchy sequence if, for any* $\varepsilon>0$ *and* $\lambda>0$*, there exists a positive integer N such that* $\boldsymbol{R}_{x_n-x_m}(\varepsilon)>1-\lambda$ *for all* $n\geq m\geq N$.

Definition 6.6 *A RN-space* $(X,\boldsymbol{R},\boldsymbol{T})$ *is said to be complete if every Cauchy sequence in X is convergent to a point in X.*

Theorem 6.7 *Assume a function* $T:\mathscr{A}_1\to\mathscr{A}_2$ *satisfies the inequality*

$$\boldsymbol{R}_{T(\mu_1\nu_1,\mu_2\nu_2,\mu_3\nu_3)}(\boldsymbol{C})\geq \boldsymbol{R}'_{\nu_1,\nu_2,\nu_3}(\boldsymbol{C});\quad \text{for all } \nu_1,\nu_2,\nu_3\in\mathscr{A}_1 \text{ and all } \boldsymbol{C}\in\mathscr{A}_1 \tag{95}$$

where $\mathscr{T}:\mathscr{A}_1^3\to(0,\infty]$ *be a function satisfying*

$$\lim_{a\to\infty}\boldsymbol{T}_{b=0}^{\infty}\boldsymbol{R}'_{\mu_4^{akb}\nu_1,\mu_4^{akb}\nu_1,\mu_4^{akb}\nu_1}\left(\mu_4^{akb}\,\boldsymbol{C}\right)=1=\lim_{a\to\infty}\boldsymbol{R}'_{\mu_4^{ak}\nu_1,\mu_4^{ak}\nu_2,\mu_4^{ak}\nu_3}\left(\mu_4^{ak}\,\boldsymbol{C}\right);\ \text{with } \mu_4=(\mu_1+\mu_2+\mu_3);\ \text{and } k=\pm1; \tag{96}$$

for all $\nu_1\in\mathscr{A}_1$ *and all* $\boldsymbol{C}\in\mathscr{A}_1$*. Then there exists a unique additive mapping* $D:\mathscr{A}_1\to\mathscr{A}_2$ *which satisfies (18) and the inequality*

$$\boldsymbol{R}_{D(\nu_1)-T(\nu_1)}(\boldsymbol{C})\geq \boldsymbol{T}_{b=0}^{\infty}\boldsymbol{R}'_{\mu_4^{akb}\nu_1,\mu_4^{akb}\nu_1,\mu_4^{akb}\nu_1}\left(3\mu_4\cdot\mu_4^{b}\,\boldsymbol{C}\right);\ \boldsymbol{R}_{D(\nu_1)}(\boldsymbol{C})=\lim_{a\to\infty}\boldsymbol{R}_{\frac{T(\mu_4^{ak}\nu_1)}{\mu_4^{ak}}}(\boldsymbol{C});\ \text{for all } \nu_1\in\mathscr{A}_1 \text{ and all } \boldsymbol{C}\in\mathscr{A}_1. \tag{97}$$

Proof. Replacing (ν_1,ν_2,ν_3) by (ν_1,ν_1,ν_1) in (95) and using oddness of T, we get

$$\mathbf{R}_{3T(\mu_4\nu_1)-3\mu_4T(\nu_1)}(\mathbf{C})\geq \mathbf{R}'_{\nu_1,\nu_1,\nu_1}(\mathbf{C});\quad \text{for all } \nu_1\in\mathscr{A}_1 \text{ and all } \mathbf{C}\in\mathscr{A}_1. \tag{98}$$

It follows from (98) and *(RBS2)* that

$$\mathbf{R}_{\frac{T(\mu_4\nu_1)}{\mu_4}-T(\nu_1)}\left(\frac{\mathbf{C}}{3\mu_4}\right)\geq \mathbf{R}'_{\nu_1,\nu_1,\nu_1}(\mathbf{C});\quad \text{for all } \nu_1\in\mathscr{A}_1 \text{ and all } \mathbf{C}\in\mathscr{A}_1. \tag{99}$$

Setting ν_1 by $\mu_4^a\nu_1$ in (99) using *(RBS2)*, we obtain

$$\mathbf{R}_{\frac{T(\mu_4^a\,\mu_4\nu_1)}{\mu_4^a\,\mu_4}-\frac{T(\mu_4^a\nu_1)}{\mu_4^a}}\left(\frac{\mathbf{C}}{3\mu_4\,\mu_4^a}\right)\geq \mathbf{R}'_{\mu_4^a\nu_1,\mu_4^a\nu_1,\mu_4^a\nu_1}(\mathbf{C});\quad \text{for all } \nu_1\in\mathscr{A}_1 \text{ and all } \mathbf{C}\in\mathscr{A}_1. \tag{100}$$

It follows from (100) and *(RBS3)* that

$$\begin{aligned}\mathbf{R}_{\frac{T(\mu_4^a\nu_1)}{\mu_4^a}-T(\nu_1)},\left(\frac{\mathbf{C}}{3\mu_4}\cdot\sum_{b=0}^{a-1}\frac{1}{\mu_4^b}\right)&=\mathbf{R}_{\sum_{b=0}^{a-1}\frac{T(\mu_4^{b+1}\nu_1)}{\mu_4^{b+1}}-\frac{T(\mu_4^b\nu_1)}{\mu_4^b}}\left(\frac{\mathbf{C}}{3\mu_4}\cdot\sum_{b=0}^{a-1}\frac{1}{\mu_4^b}\right)\\&\geq \mathbf{T}_{b=0}^{a-1}\mathbf{R}_{\frac{T(\mu_4^{b+1}\nu_1)}{\mu_4^{b+1}}-\frac{T(\mu_4^b\nu_1)}{\mu_4^b}}\left(\frac{\mathbf{C}}{3\mu_4}\cdot\frac{1}{\mu_4^b}\right)\\&\geq \mathbf{T}_{b=0}^{a-1}\mathbf{R}'_{\mu_4^b\nu_1,\mu_4^b\nu_1,\mu_4^b\nu_1}(\mathbf{C});\quad \text{for all } \nu_1\in\mathscr{A}_1 \text{ and all } \mathbf{C}\in\mathscr{A}_1.\end{aligned} \tag{101}$$

In order to prove the convergence of the sequence $\left\{\dfrac{T(\mu_4^a\nu_1)}{\mu_4^a}\right\}$, replace ν_1 by $\mu_4^c\nu_1$ in (101) for any $c>a>0$, we arrive

$$\mathbf{R}_{\frac{T(\mu_4^{a+c}\nu_1)}{\mu_4^{a+v}}-\frac{T(\mu_4^c\nu_1)}{\mu_4^c}},\left(\frac{\mathbf{C}}{3\mu_4}\cdot\sum_{b=0}^{a-1}\frac{1}{\mu_4^{b+c}}\right)\geq \mathbf{T}_{b=0}^{a-1}\mathbf{R}'_{\mu_4^{b+c}\nu_1,\mu_4^{b+c}\nu_1,\mu_4^{b+c}\nu_1}(\mathbf{C});\quad \text{for all } \nu_1\in\mathscr{A}_1 \text{ and all } \mathbf{C}\in\mathscr{A}_1. \tag{102}$$

It follows from (102), *(RBS2)*, *(RBS3)* and (96) that

$$\mathbf{R}_{\frac{T(\mu_4^{a+c}\nu_1)}{\mu_4^{a+c}}-\frac{T(\mu_4^c\nu_1)}{\mu_4^c}},(\mathbf{C})\geq \mathbf{T}_{b=0}^{a-1}\mathbf{R}'_{\mu_4^{b+c}\nu_1,\mu_4^{b+c}\nu_1,\mu_4^{b+c}\nu_1}\left(3\mu_4\cdot\mu_4^{b+c}\,\mathbf{C}\right)\quad\to\quad 1\quad \text{as}\quad a\quad to\quad \infty;\quad \text{for all } \nu_1\in\mathscr{A}_1 \text{ and all } \mathbf{C}\in\mathscr{A}_1. \tag{103}$$

This implies that the sequence $\left\{\dfrac{T(\mu_4^a\nu_1)}{\mu_4^a}\right\}$ is a Cauchy sequence in the Random Banach space $(\mathscr{A}_2,\mathbf{R},T)$ and this sequence converges to $D(\nu_1)$. So, by notation, we write

$$\mathbf{R}_{D(\nu_1)}(\mathbf{C})=\lim_{a\to\infty}\mathbf{R}_{\frac{T(\mu_4^a\nu_1)}{\mu_4^a}}(\mathbf{C});\ \text{for all } \nu_1\in\mathscr{A}_1 \text{ and all } \mathbf{C}\in\mathscr{A}_1. \tag{104}$$

Letting $c=0$ and $a\to\infty$ in (103) and using (104), we land

$$\mathbf{R}_{D(\nu_1)-T(\nu_1)}(\mathbf{C}) \geq \mathbf{T}_{b=0}^{\infty}\mathbf{R}'_{\mu_4^b\nu_1,\mu_4^b\nu_1,\mu_4^b\nu_1}\left(3\mu_4\cdot\mu_4^b\,\mathbf{C}\right); \quad \text{for all } \nu_1\in\mathscr{A}_1 \text{ and all } \mathbf{C}\in\mathscr{A}_1. \tag{105}$$

Therefore (105) holds for $k=1$. In order to show the existence of $D(\nu_1)$ satisfies (18), changing (ν_1,ν_2,ν_3) by $(\mu_4^a\nu_1,\mu_4^a\nu_2,\mu_4^a\nu_3)$ in (95), we attain

$$\mathbf{R}_{T(\mu_1\mu_4^a\nu_1,\mu_2\mu_4^a\nu_2,\mu_3\mu_4^a\nu_3)}(\mathbf{C}) \geq \mathbf{R}'_{\mu_4^a\nu_1,\mu_4^a\nu_2,\mu_4^a\nu_3}(\mathbf{C}); \quad \text{for all } \nu_1,\nu_2,\nu_3\in\mathscr{A}_1 \text{ and all } \mathbf{C}\in\mathscr{A}_1 \tag{106}$$

Letting $a\to\infty$ in (106) and using (104), (96), we see that $D(\nu_1)$ satisfies (18) for all $\nu_1,\nu_2,\nu_3\in\mathscr{A}_1$. To prove the existence of $D(\nu_1)$ is unique, assume $D'(\nu_1)$ be a another mapping satisfies (18), (104), (105). Now, using (96), $(RBS3)$, $(RBS2)$, $(RBS1)$, we arrive that

$$\begin{aligned}
\mathbf{R}_{D(\nu_1)-D'(\nu_1)}\left(2\mathbf{C}\right) &= \mathbf{R}_{D(\mu_4^c\nu_1)-D(\mu_4^c\nu_1)+T(\mu_4^c\nu_1)-D'(\mu_4^c\nu_1),\mu_4^c}\left(2\mathbf{C}\right)\\
&\geq \mathbf{T}\left\{\mathbf{R}_{D(\mu_4^c\nu_1)-D(\mu_4^c\nu_1)}\left(\mu_4^c\,\mathbf{C}\right),\mathbf{R}_{T(\mu_4^c\nu_1)-D'(\mu_4^c\nu_1)}\left(\mu_4^c\,\mathbf{C}\right)\right\}\\
&\geq \mathbf{T}\left\{\mathbf{T}_{b=0}^{\infty}\mathbf{R}'_{\mu_4^{b+c}\nu_1,\mu_4^{b+c}\nu_1,\mu_4^{b+c}\nu_1}\left(3\mu_4\cdot\mu_4^{b+c}\,\mathbf{C}\right),\mathbf{T}_{b=0}^{\infty}\mathbf{R}'_{\mu_4^{b+c}\nu_1,\mu_4^{b+c}\nu_1,\mu_4^{b+c}\nu_1}\left(3\mu_4\cdot\mu_4^{b+c}\,\mathbf{C}\right)\right\}\\
&\to \quad 1 \quad as \quad c \quad \to \quad \infty\\
&\Rightarrow \mathbf{R}_{D(\nu_1)-D'(\nu_1)}\left(2\mathbf{C}\right)=1
\end{aligned}$$

Thus $D(\nu_1)$ is unique. Therefore the theorem holds for $k=1$. Replacing ν_1 by $\frac{\nu_1}{\mu_4}$ in (98), we get

$$\mathbf{R}_{3T(\nu_1)-3\mu_4T\left(\frac{\nu_1}{\mu_4}\right)}(\mathbf{C}) \geq \mathbf{R}'_{\frac{\nu_1}{\mu_4},\frac{\nu_1}{\mu_4},\frac{\nu_1}{\mu_4}}(\mathbf{C}); \quad \text{for all } \nu_1\in\mathscr{A}_1 \text{ and all } \mathbf{C}\in\mathscr{A}_1. \tag{107}$$

It follows from (107) and *(RBS2)* that

$$\mathbf{R}_{T(\nu_1)-\mu_4T\left(\frac{\nu_1}{\mu_4}\right)}\left(\frac{1}{3}\,\mathbf{C}\right) \geq \mathbf{R}'_{\frac{\nu_1}{\mu_4},\frac{\nu_1}{\mu_4},\frac{\nu_1}{\mu_4}}(\mathbf{C}); \quad \text{for all } \nu_1\in\mathscr{A}_1 \text{ and all } \mathbf{C}\in\mathscr{A}_1. \tag{108}$$

Setting ν_1 by $\frac{\nu_1}{\mu_4^a}$ in (108) using *(RBS2)*, we obtain

$$\mathbf{R}_{\mu_4^aT\left(\frac{\nu_1}{\mu_4^a}\right)-\mu_4^{a+1}T\left(\frac{\nu_1}{\mu_4^{a+1}}\right)}\left(\frac{\mu_4^a}{3}\,\mathbf{C}\right) \geq \mathbf{R}'_{\frac{\nu_1}{\mu_4^{a+1}},\frac{\nu_1}{\mu_4^{a+1}},\frac{\nu_1}{\mu_4^{a+1}}}(\mathbf{C}); \quad \text{for all } \nu_1\in\mathscr{A}_1 \text{ and all } \mathbf{C}\in\mathscr{A}_1. \tag{109}$$

The rest of the proof is similar lines to that of $k=1$. So the theorem is true for $k=-1$. Hence the proof is complete. From the above theorem, we have the following corollary regarding some stabilities of the functional equation (18) .

Corollary 6.8 *Assume a function $T:\mathscr{A}_1\to\mathscr{A}_2$ satisfies the inequality*

$$\boldsymbol{R}_{T\left(\mu_1\nu_1,\mu_2\nu_2,\mu_3\nu_3\right)}(\boldsymbol{C}) \geq \begin{cases}
\boldsymbol{R}'_C(\boldsymbol{C});\\
\boldsymbol{R}'_{C(|\nu_1|^E+|\nu_2|^E+|\nu_3|^E)}(\boldsymbol{C});\\
\boldsymbol{R}'_{C(|\nu_1|^{E_1}+|\nu_2|^{E_2}+|\nu_3|^{E_3})}(\boldsymbol{C});\\
\boldsymbol{R}'_{C(|\nu_1|^E|\nu_2|^E|\nu_3|^E)}(\boldsymbol{C});\\
\boldsymbol{R}'_{C(|\nu_1|^{E_1}|\nu_2|^{E_2}|\nu_3|^{E_3})}(\boldsymbol{C});\\
\boldsymbol{R}'_{C(|\nu_1|^{3E}+|\nu_2|^{3E}+|\nu_3|^{3E}+(|\nu_1|^E|\nu_2|^E|\nu_3|^E))}(\boldsymbol{C});
\end{cases} \tag{110}$$

for all $\nu_1,\nu_2,\nu_3\in\mathscr{A}_1$ and all $\boldsymbol{C}\in\mathscr{A}_1$. Then there exists a unique additive mapping $D:\mathscr{A}_1\to\mathscr{A}_2$ which satisfies (18) and the inequality

$$\boldsymbol{R}_{D(\nu_1)-T(\nu_1)}(\boldsymbol{C}) \geq \begin{cases}
\boldsymbol{R}'_C\left(3|\mu_4-1|\,\boldsymbol{C}\right); & \\
\boldsymbol{R}'_{C\,|\nu_1|^E}\left(3|\mu_4-\mu_4^E|\,\boldsymbol{C}\right); & E\neq 1\\
\boldsymbol{R}'_{C\,|\nu_1|^{E_1}+C\,|\nu_1|^{E_2}+C\,|\nu_1|^{E_3}}\left(3\{|\mu_4-\mu_4^{E_1}|+|\mu_4-\mu_4^{E_2}|+|\mu_4-\mu_4^{E_3}|\}\,\boldsymbol{C}\right); & E_1,E_2,E_3\neq 1\\
\boldsymbol{R}'_{C\,|\nu_1|^{3E}}\left(3|\mu_4-\mu_4^{3E}|\,\boldsymbol{C}\right); & 3E\neq 1\\
\boldsymbol{R}'_{C\,|\nu_1|^{E_1+E_2+E_3}}\left(3|\mu_4-\mu_4^{E_1+E_2+E_3}|\,\boldsymbol{C}\right); & E_1+E_2+E_3\neq 1\\
\boldsymbol{R}'_{C\,|\nu_1|^{3E}}\left(3|\mu_4-\mu_4^{3E}|\,\boldsymbol{C}\right); & 3E\neq 1
\end{cases} \tag{111}$$

for all $\nu_1\in\mathscr{A}_1$ and all $\boldsymbol{C}\in\mathscr{A}_1$.

7. Conclusion

This article gives a new type of additive functional equation which yet now not introduced. Also, the generalized Ulam – Gavruta stability theorems of this additive functional equation in Banach space, Generalized 2 Banach space, Fuzzy Banach Space and Random Banach space using Hyers method are proved.

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