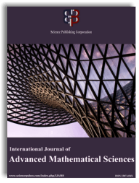
International Journal of Advanced Mathematical Sciences, 10 (2) (2024) 37-47

International Journal of Advanced Mathematical Sciences

Website: www.sciencepubco.com/index.php/IJAMS

Research paper



Roots of Tropical Polynomial from Clopen and Non-Clopen Discrete m-Topological Transformation Semigroup

M. O Francis[1*], L. F. Joseph[2], A. T. Cole[3] and B. Oshatuyi[4]

[1,2]*Department of Mathematical Sciences, Bingham University Karu, Nasarawa, Nigeria.*
[3,4]*Department of Mathematics, Federal University of Technology Minna, Niger State, Nigeria*
[*]*Corresponding author E-mail:mosesobinna1990@gmail.com*

Abstract

This paper introduces a specific subclass of m-topological transformation semigroup spaces, referred to as closed m-topological full transformation semigroup spaces. It defines both the clopen and non-clopen elements within these spaces and explores the nature of their roots. The study presents formulas for clopen and non-clopen elements, as well as for the closed m-topological full transformation semigroup spaces. Furthermore, numerical and graphical results for degrees 2 and 3 are provided to illustrate the findings.

Keywords: *m-topological closed transformation, clopen, non-clopen, tropical polynomial, coefficient, root*

1. Introduction

The term "tropical" was introduced by a French mathematician. Tropical geometry has emerged as a significant new field that connects algebraic geometry, with its techniques being employed to address problems in enumerative geometry and arithmetic geometry. This field builds on the older area of tropical mathematics, also known as max-plus algebra, which has applications in semigroup theory, computer science, and optimization. Tropical algebraic geometry is a fascinating and emerging area of mathematical research focused on the study of piecewise-linear functions that resemble algebraic varieties. In analogy to classical algebra, a tropical polynomial expression

$$f(x) = c \odot \bigoplus_{i=1}^{k} (x \odot x_i^{\odot -1}) \tag{1.1}$$

where $c = \min\{f(x) - x_i \mid i = 1, 2, \ldots, k\}$. defines a tropical polynomial function $f(x)$ in the context of tropical polynomials arising from m-topological transformation semigroups. The foundational ideas of this field have been present for some time, with early contributions from [5, 6, 11, 13, 14].

Let $X_n = \{1, 2, 3 \ldots n\}$. A partial transformation $\alpha : Dom\alpha \subseteq X_n \mapsto Im\alpha \subset X_n$ is said to be Full if $Dom\alpha = X_n$; otherwise it is called strictly partial [15]. Recently,[8], [9] introduced the concept of m-topological transformation semigroup spaces, which they defined as set of transformation semigroups that satisfy the properties of topological spaces, they studied a subclass called Regular spaces, M_{ψ_n} and also examined their workdone and power. M_{T_n} denotes m-topological full transformation semigroup spaces denoted by M_{P_n} denotes m-topological partial transformation semigroup spaces. If α and β are two elements of a transformation semigroup spaces, the following definition holds $\alpha \cap \beta = min\{\alpha x, \beta x\}, \alpha \cup \beta = max\{\alpha x, \beta x\}$, for all $\alpha x \in Im\alpha$ and $\beta x \in Im\beta$, $\alpha \subseteq M_\delta$ is open, then $\alpha^c = |n - \alpha x|$, where $n = max(X)$ and M_δ is m-topological transformation semigroup spaces. Throughout, this paper our considerations are the discrete set. The main objects of this paper are the subclasses of M_{T_n}: closed m-topological full transformation semigroup spaces denoted by M_{CT_n}, which are obtained by the complements of M_{T_n}. The notation $Cl(M_{CT_n})$ refers to the clopen elements of m-topological full transformation semigroup spaces, also denoted as M_{TCT_n}, while $NCl(M_{CT_n})$ denotes the non-clopen elements of m-topological full transformation semigroup spaces.

In the study of clopen topological spaces [2, 3, 7], some study on cubic polynomials [12, 17], recent work by [1, 10, 16] has focused on the tropicalization of elements in different classes of full transformation semigroups. The goal of this paper is to examine the nature of roots in tropical polynomial of $Cl(M_{CT_n})$ and $NCl(M_{CT_n})$. The study explore of the nature of roots in tropical polynomials within the context of $Cl(M_{CT_n})$ and $NCl(M_{CT_n})$, which is relatively new area of study. By investigating the behavior and characteristics of both clopen and non-clopen elements, the research contributes to a deeper understanding of the algebraic properties of M_{CT_n}. This analysis not only enhances knowledge in the field of tropical mathematics but also provides potential applications in areas such as optimization, semigroup theory, and computational mathematics, where understanding the interaction of these elements is crucial. It is important to note that $Cl(M_{CT_n}) \cup NCl(M_{CT_n})$ constitutes M_{CT_n}. Therefore, M_{CT_n} refers to both types of elements collectively or to either type individually.

2. Preliminaries

Definition 2.1. [9], Let δ be the chart on $X_n = \{1,2,3,...\}$. The map $\alpha : Dom(\alpha) \subseteq X_n \to Im(\alpha) \subseteq X_n$ is said to be a full transformation semigroup; denoted by T_n, if $Dom(\alpha) = X_n$, and partial transformation if $Dom(\alpha) \subseteq X_n$; denoted by P_n.

Definition 2.2. [9], A set of transformations δ is said to be an m-topological transformation semigroup (shortened as M_δ) if it satisfies the following properties:

1. $\alpha \in M_\delta$; and $\emptyset \in M_\delta$ by default.
2. α is closed under arbitrary unions in M_δ;
3. α is closed under finite intersections in M_δ.

The elements of M_{CT_n} are as follows :
When $n = 2$, we have the following transformations

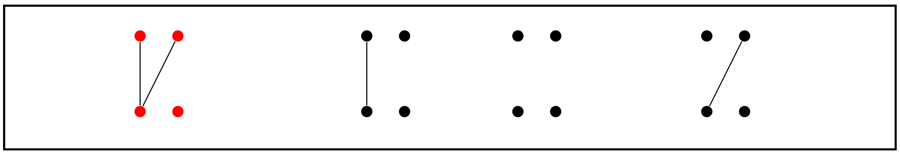


Figure 1: The Elements of M_{CT_2}

The elements of M_{CT_2} are pictured in Figure (1) above. We presented the elements of $NCl(M_{CT_2})$ with black nodes and the elements of $Cl(M_{CT_2})$ with red nodes. The diagram shows the permutation of each element. $M_{CT_2} = 4$, $Cl(M_{CT_2}) = 1$, and $NCl(M_{CT_2}) = 3$
For $n = 3$, we have the following transformations

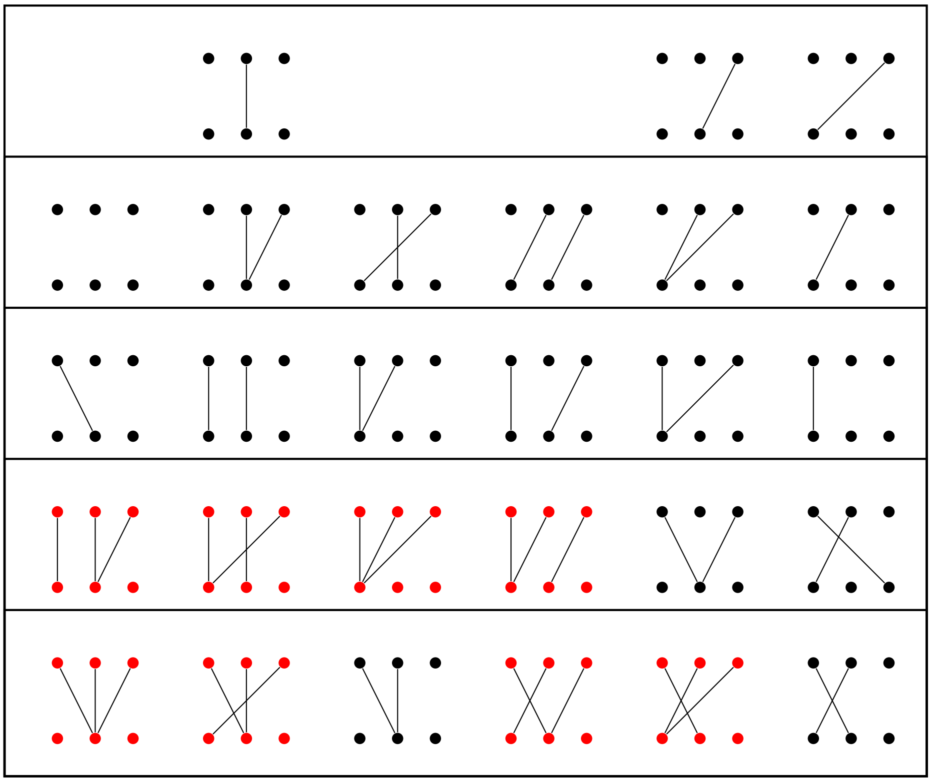


Figure 2: The Elements of M_{CT_2}

The elements of M_{CT_3} are pictured in Figure (2) above. We presented the elements of $NCl(M_{CT_3})$ with black nodes and the elements of $Cl(M_{CT_3})$ with red nodes. The diagram shows the permutation of each element.$M_{CT_3} = 27$, $Cl(M_{CT_3}) = 8$, and $NCl(M_{CT_3}) = 19$

Relationship between Discrete m-topological transformation semigroup spaces

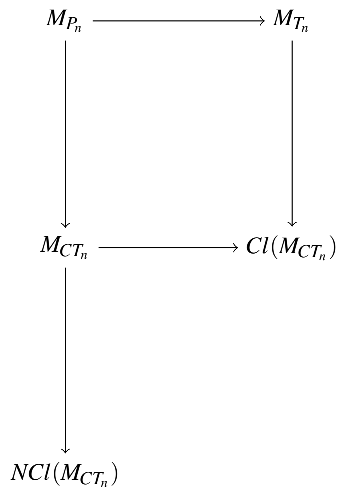


Figure 3: Relationship Diagram

Remark 2.3. It is important to note that the transformations are in this order $Cl(M_{CT_n}) \subset M_{CT_n}$ and $NCl(M_{CT_n}) \subset M_{CT_n}$, $(M_{CT_n}) \subset M_{T_n} \subset M_{P_n}$ In this paper our consideration are the elements of $Cl(M_{CT_n})$ and $NCl(M_{CT_n})$.

Lemma 2.4. *Let $M_{CT_n} \subseteq T_n$. Then for $x, \alpha x \in n$*

$M_{CT_n} = n^n$

for $n \geq 0$

Lemma 2.5. *Let $Cl(M_{CT_n}) \subseteq T_n$. Then for $x, \alpha x \in n$*

$Cl(M_{CT_n}) = (n-1)^n$

forn ≥ 0

Lemma 2.6. *Let $NCl(M_{CT_n}) \subseteq T_n$. Then for $x, \alpha x \in n$*

$NCl(M_{CT_n}) = n^n - (n-1)^n$

is always positive and grows as n increases $forn \geq 1$.

Proof. We want to prove that $n^n - (n-1)^n$ is positive and grows as n increases. We will use mathematical induction.
For $n = 1$:

$$1^1 - (1-1)^1 = 1 - 0 = 1 \tag{2.1}$$

The base case holds true. By Induction, let us assume the statement holds for some arbitrary $k \geq 1$:

$$k^k - (k-1)^k \geq 1 \tag{2.2}$$

We need to show that the statement holds for $k+1$,

$$(k+1)^{k+1} - k^{k+1} \geq 1 \tag{2.3}$$

Consider the expression:

$$(k+1)^{k+1} - k^{k+1} \tag{2.4}$$

We will use the binomial expansion for $(k+1)^{k+1}$:

$$(k+1)^{k+1} = \sum_{i=0}^{k+1} \binom{k+1}{i} k^i \tag{2.5}$$

Let us examine the difference:

$$(k+1)^{k+1} - k^{k+1} = \left(\sum_{i=0}^{k+1} \binom{k+1}{i} k^i \right) - k^{k+1} \tag{2.6}$$

Notice that all terms in the binomial expansion of $(k+1)^{k+1}$ are positive. For instance, we have:

$$\binom{k+1}{k} k^k = (k+1)k^k \tag{2.7}$$

$$\binom{k+1}{k+1} k^{k+1} = 1 \tag{2.8}$$

□

So, the terms in $(k+1)^{k+1}$ that are not k^{k+1} add to a positive amount:

$$(k+1)^{k+1} = k^{k+1} + (k+1)k^k + \text{(other positive terms)} \tag{2.9}$$

Thus,

$$(k+1)^{k+1} - k^{k+1} = (k+1)k^k + \text{(other positive terms)} \tag{2.10}$$

Since $(k+1)k^k > 0$ and all other terms are positive, we have:

$$(k+1)^{k+1} - k^{k+1} > 0 \tag{2.11}$$

Therefore, by induction, $n^n - (n-1)^n$ is positive and increases with n.
We have shown by induction that $n^n - (n-1)^n$ is always positive and grows as n increases. Thus, for all $n \geq 1$,

$$n^n - (n-1)^n \geq 1 \tag{2.12}$$

3. Main Result

3.1. Coefficient of Tropical Polynomials in (M_{CT_n})

Lemma 3.1. *For any finite set of tropical polynomials in (M_{CT_n}), $\{f_1(x), f_2(x), \ldots, f_n(x)\}$ with coefficients in $\mathbb{R}$, there exists a minimal polynomial, denoted as $f_{\min}(x)$, such that $f_{\min}(x) \leq f_i(x)$ for all $i = 1, 2, \ldots, n$.*

Proof. Consider the set of tropical polynomials $\{f_1(x), f_2(x), \ldots, f_n(x)\}$. Since these polynomials in (M_{CT_n}) are defined over the max-plus algebra, each $f_i(x)$ achieves its minimum value at certain points x_i. Let $c_i = \min\{f_i(x) \mid x \in \mathbb{R}\}$ denote the minimum value attained by $f_i(x)$. Define $f_{\min}(x) = \bigoplus_{i=1}^{n}(c_i \odot x_i^{\odot -1})$. This polynomial $f_{\min}(x)$ is constructed such that:

$$f_{\min}(x) \leq f_i(x) \quad \text{for all} \quad i = 1, 2, \ldots, n.$$

Therefore, $f_{\min}(x)$ is the minimal polynomial among $\{f_1(x), f_2(x), \ldots, f_n(x)\}$ in terms of achieving the minimum value over $\mathbb{R}$. □

Theorem 3.2. *Every tropical polynomial $f(x)$ in (M_{CT_n}) with coefficients in $\mathbb{R}$ and defined over the max-plus algebra has at least one root.*

Proof. Let $f(x) = \bigoplus_{i=0}^{n}(a_i \odot x^{\odot i})$ be a tropical polynomial form from (M_{CT_n}), where $a_i \in \mathbb{R}$ and $\oplus$, $\odot$ represent tropical addition (maximum) and multiplication (ordinary addition), respectively. Consider the minimum value $c = \min\{f(x) \mid x \in \mathbb{R}\}$ attained by $f(x)$. Since $f(x)$ is continuous and defined over a closed interval in $\mathbb{R}$, by the Extreme Value Theorem, $f(x)$ achieves its minimum value c at some point $x = t \in \mathbb{R}$.

$$c = \min\{f(x) \mid x \in \mathbb{R}\}. \tag{3.1}$$

Since $f(x)$ achieves its minimum value c at $x = \alpha$, it follows that $f(\alpha) = c$. Therefore, t is a root of $f(x)$.

$$f(\alpha) = c. \tag{3.2}$$

Thus, every tropical polynomial $f(x)$ has at least one root. □

Theorem 3.3. *Every tropical polynomial $f(x)$ in (M_{CT_n}) with coefficients in $\mathbb{R}$ and defined over the max-plus algebra can be factored into a product of linear factors.*

Proof. Let $f(x) = \bigoplus_{i=0}^{n}(a_i \odot x^{\odot i})$ be a tropical polynomial form from (M_{CT_n}), where $a_i \in \mathbb{R}$ and $\oplus$, $\odot$ represent tropical addition (maximum) and multiplication (ordinary addition), respectively.
Since $f(x)$ form of (M_{CT_n}) is a tropical polynomial, it can achieve its minimum value at specific points $x_1, x_2, \ldots, x_k$. These points are called roots of $f(x)$, where $f(x_i) = \min\{f(x) \mid x \in \mathbb{R}\}$.
Each root x_i of $f(x)$ corresponds to a linear factor $x - x_i$ in the factorization of $f(x)$. Therefore, $f(x)$ can be expressed as:

$$f(x) = c \odot \bigoplus_{i=1}^{k}(x \odot x_i^{\odot -1}) \tag{3.3}$$

where $c = \min\{f(x) - x_i \mid i = 1, 2, \ldots, k\}$. □

3.2. Roots of Tropical Polynomials in $Cl(M_{CT_n})$ and $NCl(M_{CT_n})$

Proposition 3.4. *Consider*

$$f(x) = \lambda_0 x^n + \lambda_1 x^{n-1} + \lambda_2 x^{n-2} + \ldots + \lambda_n = 0, \tag{3.4}$$

which represents either $Cl(M_{CT_n})$ or $NCl(M_{CT_n})$ of degree n and $\lambda_0 = 1$. Then, roots of the equation are either zero or negative.

Proof. By Descartes' Rule of Signs, the number of positive roots of a polynomial with real coefficients cannot exceed the number of sign changes among the non-zero coefficients. Since all the coefficients are positive, there are no sign changes, so the number of positive roots must be zero, implying that the roots are either zero or negative. □

Proposition 3.5. *Consider*

$$f(x) = \lambda_0 x^2 + \lambda_1 x + \lambda_2 = 0, \tag{3.5}$$

be a quadratic polynomial in $NCl(M_{CT_n})$, where $\lambda_2 = 0$. The polynomials has non-complex roots, and solution of the forms.

i $x_1 = 0$

ii $x_2 = -\lambda_1$

Proof. Since, $f(x)$ is of degree 2 and $NCl(M_{CT_n})$, we have $\lambda_0 = 1$.
Case I. Since, $\lambda_2 = 0$ we have $x^2 + \lambda_1 x = 0$ this implies $x(x+\lambda_1) = 0$.
Hence, $x = 0$ or $x = -\lambda_1$
Case II. Conversely, if $\lambda_2 \neq 0$. Then the roots of the polynomials has repeated nonzero root. □

Theorem 3.6. *Let*

$$f(x) = \lambda_0 x^3 + \lambda_1 x^2 + \lambda_2 x + \lambda_3 = 0 \tag{3.6}$$

be the general form of a cubic tropical polynomial in a closed m-topological transformation semigroup. The roots of this polynomial satisfy the following: at $e^{\frac{2k\pi i}{3}}$ for $k = 0, 1, 2$

$$x_1 = \frac{-\lambda_1}{3} + \sqrt[3]{\sqrt{\left(\frac{9\lambda_1\lambda_2 - 27\lambda_3 - 2\lambda_1^3}{54}\right)^2 + \left(\frac{3\lambda_2 - \lambda_1^2}{9}\right)^3} + \left(\frac{9\lambda_1\lambda_2 - 27\lambda_3 - 2\lambda_1^3}{54}\right)} + \sqrt[3]{\sqrt{\left(\frac{9\lambda_1\lambda_2 - 27\lambda_3 - 2\lambda_1^3}{54}\right)^2 + \left(\frac{3\lambda_2 - \lambda_1^2}{9}\right)^3} - \left(\frac{9\lambda_1\lambda_2 - 27\lambda_3 - 2\lambda_1^3}{54}\right)} \tag{3.7}$$

$$x_2 = \frac{-\lambda_1}{3} + \sqrt[3]{\sqrt{\left(\frac{9\lambda_1\lambda_2 - 27\lambda_3 - 2\lambda_1^3}{54}\right)^2 + \left(\frac{3\lambda_2 - \lambda_1^2}{9}\right)^3} - \left(\frac{9\lambda_1\lambda_2 - 27\lambda_3 - 2\lambda_1^3}{54}\right)}\, e^{\frac{2\pi i}{3}} + \sqrt[3]{\sqrt{\left(\frac{9\lambda_1\lambda_2 - 27\lambda_3 - 2\lambda_1^3}{54}\right)^2 - \left(\frac{3\lambda_2 - \lambda_1^2}{9}\right)^3} - \left(\frac{9\lambda_1\lambda_2 - 27\lambda_3 - 2\lambda_1^3}{54}\right)}\, e^{-\frac{2\pi i}{3}} \tag{3.8}$$

$$x_3 = \frac{-\lambda_1}{3} + \sqrt[3]{\sqrt{\left(\frac{9\lambda_1\lambda_2 - 27\lambda_3 - 2\lambda_1^3}{54}\right)^2 + \left(\frac{3\lambda_2 - \lambda_1^2}{9}\right)^3} - \left(\frac{9\lambda_1\lambda_2 - 27\lambda_3 - 2\lambda_1^3}{54}\right)}\, e^{\frac{4\pi i}{3}} + \sqrt[3]{\sqrt{\left(\frac{9\lambda_1\lambda_2 - 27\lambda_3 - 2\lambda_1^3}{54}\right)^2 - \left(\frac{3\lambda_2 - \lambda_1^2}{9}\right)^3} - \left(\frac{9\lambda_1\lambda_2 - 27\lambda_3 - 2\lambda_1^3}{54}\right)}\, e^{-\frac{4\pi i}{3}} \tag{3.9}$$

Proof. Since, equation (3.6) is the general form of tropical polynomial, we divide through by the coefficient of x^3 that is

$$f(x) = x^3 + \frac{\lambda_1 x^2}{\lambda_0} + \frac{\lambda_2 x}{\lambda_0} + \frac{\lambda_3}{\lambda_0}. \tag{3.10}$$

To express equation (3.10) in terms of Cardon's equation

$$f(x) = x^3 + \frac{\lambda_2 x}{\lambda_0} + \frac{\lambda_3}{\lambda_0} = 0. \tag{3.11}$$

We eliminate x^2 by setting $y = x + \frac{\lambda_1}{3\lambda_0}$. This implies $x = y - \frac{\lambda_1}{3\lambda_0}$ by substituting for x in equation (3.10) we have:

$$\left(y - \frac{\lambda_1}{3\lambda_0}\right)^3 + \frac{\lambda_1}{\lambda_0}\left(y - \frac{\lambda_1}{3\lambda_0}\right)^2 + \frac{\lambda_2}{\lambda_0}\left(y - \frac{\lambda_1}{3\lambda_0}\right) + \frac{\lambda_3}{\lambda_0} = 0\cdot \tag{3.12}$$

Expanding and simplifying gives

$$y^3+\left(\frac{\lambda_1}{\lambda_0}-\frac{\lambda_1}{\lambda_0}\right)y^2+\left(\frac{\lambda_1^2}{3\lambda_0}-\frac{2\lambda_1^2}{3\lambda_0}+\frac{\lambda_2}{\lambda_0}\right)y+\left(\frac{\lambda_3}{\lambda_0}-\frac{\lambda_1^3}{27\lambda_0^3}+\frac{\lambda_1^3}{9\lambda_0^2}-\frac{\lambda_1\lambda_2}{3\lambda_0^2}\right)=0. \tag{3.13}$$

From equation (3.13) $y^2=0$. Also, since our consideration are closed transformation semigroups $\lambda_0=1$, this implies

$$y^3+\left(\frac{3\lambda_2-\lambda_1^2}{2}\right)y+\left(\frac{27\lambda_3+2\lambda_1^3-9\lambda_1\lambda_2}{27}\right)=0. \tag{3.14}$$

Hence, equation (3.14) is in Cardino's form.
Therefore, let $\beta=\frac{3\lambda_2-\lambda_1^2}{2}$ and $\gamma=\frac{27\lambda_3+2\lambda_1^3-9\lambda_1\lambda_2}{27}$.
We express equation (3.14) as

$$y^3+\beta y+\gamma=0. \tag{3.15}$$

From equation (3.15) let $\tau=y+\frac{\beta}{3\tau}$ this implies $y=\tau-\frac{\beta}{3\tau}$, substituting y, gives

$$\left(\tau-\frac{\beta}{3\tau}\right)^3+\beta\left(\tau-\frac{\beta}{3\tau}\right)+\gamma=0\cdot \tag{3.16}$$

Expanding equation (3.16) gives

$$\left(\tau^3-\beta\tau+\frac{\beta^3}{3\tau}-\frac{\beta^3}{27\tau^3}\right)+\left(\beta\tau-\frac{\beta^2}{3\tau}\right)+\gamma=0. \tag{3.17}$$

By simplification we have

$$\tau^3-\frac{\beta^3}{27\tau^3}+\gamma=0\cdot \tag{3.18}$$

Multiplying through by τ^3 give

$$(\tau^3)^2+\gamma\tau^3-\frac{\beta^3}{27}+=0\cdot \tag{3.19}$$

Applying the formula method of quadratics equation gives

$$\tau^3=\frac{-\gamma\pm\sqrt{\gamma^2+\frac{4\beta^3}{27}}}{2}. \tag{3.20}$$

$$\tau^3=\frac{-\gamma\pm\sqrt{\gamma^2+\frac{4\beta^3}{27}}}{2}. \tag{3.21}$$

$$\zeta_1=\sqrt[3]{\frac{-\gamma+\sqrt{27\gamma^2+4\beta^3}}{54}},\ \zeta_2=\sqrt[3]{\frac{-\gamma-\sqrt{27\gamma^2+4\beta^3}}{54}} \tag{3.22}$$

The three roots of the depressed cubic are:

$y{=}\zeta_1e^{\frac{2k\pi i}{3}}+\zeta_2e^{-\frac{2k\pi i}{3}}$

where $k=0,1,2$. Recall that $x=y-\frac{\lambda_1}{3}$.
Hence, we have $x_k=\zeta_1e^{\frac{2k\pi i}{3}}+\zeta_2e^{-\frac{2k\pi i}{3}}-\frac{\lambda_1}{3}$.

□

3.3. Numerical Evaluations of Roots in Tropical Polynomials for $Cl(M_{CT_n}),NCl(M_{CT_n})$ and their corresponding Graphs

Example 3.7. Given:

$\lambda_0=1,\ \lambda_1=3,\ \lambda_2=5,\ \lambda_3=1$

The formula to find the root of x_1 at $k=0$ is:

$$x_1=-\frac{\lambda_1}{3}+\sqrt[3]{q+\sqrt{q^2+p^3}}+\sqrt[3]{q-\sqrt{q^2+p^3}}$$

where:

$$p=\frac{3\lambda_2-\lambda_1^2}{9}$$

$q = \frac{9\lambda_1\lambda_2 - 27\lambda_3 - 2\lambda_1^3}{54}$

First, calculate p:

$$p = \frac{3\lambda_2 - \lambda_1^2}{9} \tag{3.23}$$

$$p = \frac{3(5) - (3)^2}{9} = \frac{2}{3} \tag{3.24}$$

Next, calculate q:

$$q = \frac{9\lambda_1\lambda_2 - 27\lambda_3 - 2\lambda_1^3}{54} \tag{3.25}$$

$$q = \frac{9(3)(5) - 27(1) - 2(3)^3}{54} \tag{3.26}$$

$$q = \frac{54}{54} = 1 \tag{3.27}$$

Calculate the discriminant:

$$q^2 + p^3 = 1^2 + \left(\frac{2}{3}\right)^3 \tag{3.28}$$

$$1 + 0.2963 \approx 1.2963 \tag{3.29}$$

Find $\sqrt{q^2 + p^3}$:

$$\sqrt{1.2963} \approx 1.1388 \tag{3.30}$$

Calculate the cube roots:

$$\sqrt[3]{q + \sqrt{q^2 + p^3}} \tag{3.31}$$

$$\sqrt[3]{1 + 1.1388} = \sqrt[3]{2.1388} \approx 1.285 \tag{3.32}$$

$$\sqrt[3]{q - \sqrt{q^2 + p^3}} \tag{3.33}$$

$$\sqrt[3]{1 - 1.1388} = \sqrt[3]{-0.1388} \approx -0.5140 \tag{3.34}$$

Now sum these results and adjust for the $-\frac{\lambda_1}{3}$ term:

$$x_1 = -\frac{3}{3} + 1.285 - 0.5140 \tag{3.35}$$

$$x_1 = -1 + 0.7710 \approx -0.2291 \tag{3.36}$$

Therefore, the root x_1 is approximately:

$x_1 \approx -0.2291$

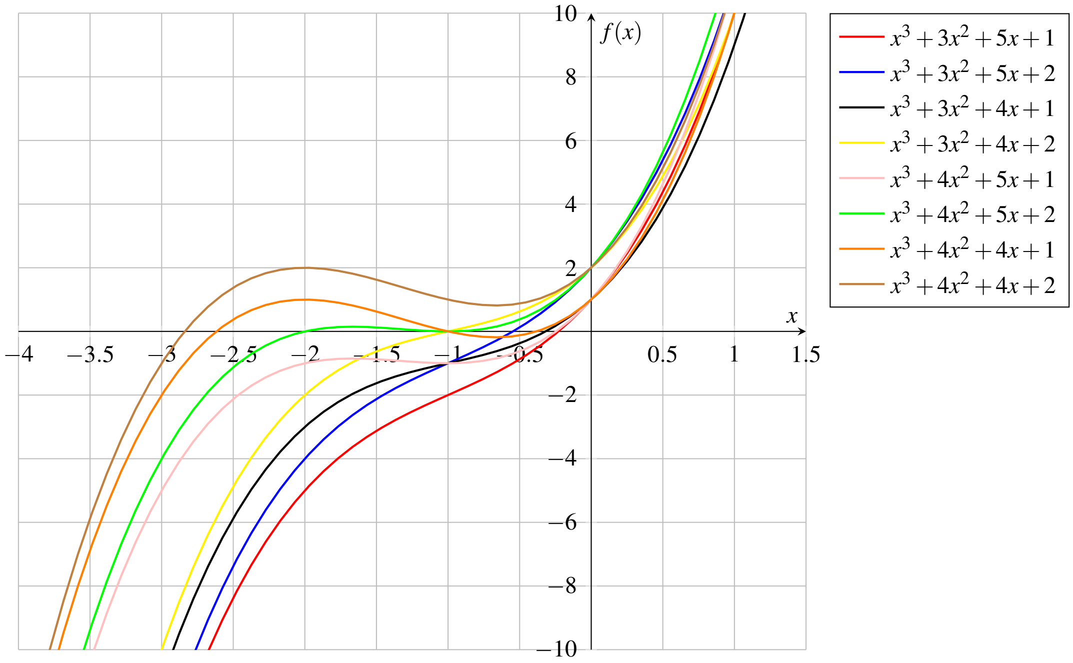


Figure 5: Graph of tropical polynomial for $Cl(M_{CT_3})$

For roots of x_2 and x_3 at $k = 1,2$. We use Python 3.9 Codes.

Polynomial	Roots
$x^2 + 3x + 1 = 0$	$x_1 \approx -0.38, \quad x_2 \approx -2.62$
$x^3 + 3x^2 + 5x + 1 = 0$	$x_1 \approx -0.2291, \quad x_2 \approx -1.3855 + 1.5639i, \quad x_3 \approx -1.3855 - 1.5639i,$
$x^3 + 3x^2 + 5x + 2 = 0$	$x_1 \approx -0.5466, \quad x_2 \approx 1.2267 + 1.4677i, \quad x_3 \approx -1.2267 - 1.4677i,$
$x^3 + 3x^2 + 4x + 1 = 0$	$x_1 \approx -0.3177, \quad x_2 \approx -1.3412 + 1.1615i, \quad x_3 \approx -1.3412 - 1.1615i$
$x^3 + 3x^2 + 4x + 2 = 0$	$x_1 \approx -1, \quad x_2 \approx -1 + 1i, \quad x_3 \approx -1 - 1i$
$x^3 + 4x^2 + 5x + 1 = 0$	$x_1 \approx -0.2451, \quad x_2 \approx -1.8774 + 0.7449i, \quad x_3 \approx -1.8774 - 0.7449i$
$x^3 + 4x^2 + 5x + 2 = 0$	$x_1 \approx -1, \quad x_2 \approx -2, \quad x_3 \approx -1$
$x^3 + 4x^2 + 4x + 1 = 0$	$x_1 \approx -1, \quad x_2 \approx -0.382, \quad x_3 \approx -2.618$
$x^3 + 4x^2 + 4x + 2 = 0$	$x_1 \approx -2.8393, \quad x_2 \approx -0.5804 + 0.6063i, \quad x_3 \approx -0.5804 - 0.6063i$

Table 1: Roots of tropical polynomials of $Cl(M_{CT_2})$ and $Cl(M_{CT_3})$

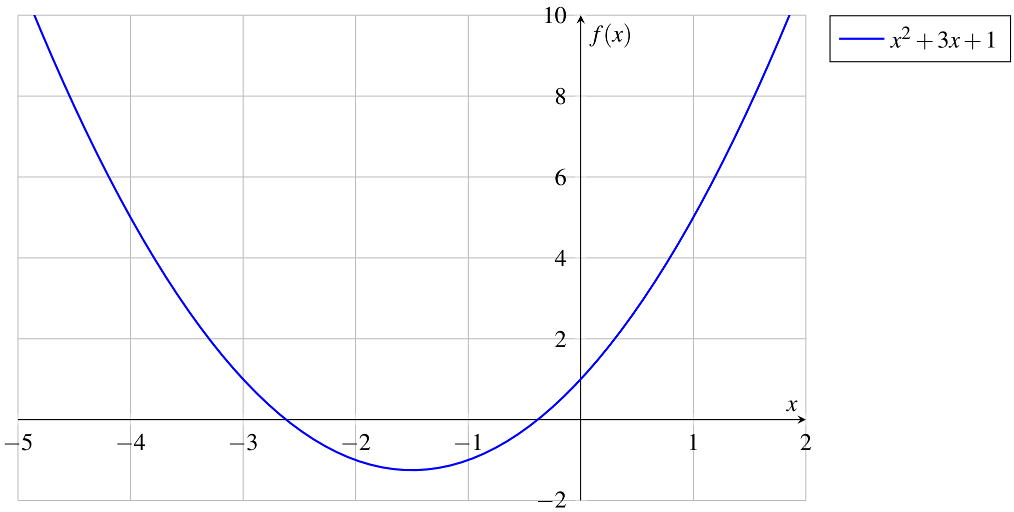


Figure 4: Graph of tropical polynomial of $Cl(M_{CT_2})$

Polynomial	**Roots**
$x^2+3x=0$	$x_1 \approx 0, \quad x_2 \approx -3.00$
$x^2+2x=0$	$x_1 \approx 0, \quad x_2 \approx -2.00$
$x^2+2x+1=0$	$x_1 \approx -1.00, \quad x_2 \approx -1.00$
$x^3+2x^2+3x=0$	$x_1 \approx 0, \quad x_2 \approx -1.00+1.41i, \quad x_3 \approx -1.00-1.41i$
$x^3+2x^2+4x=0$	$x_1 \approx 0, \quad x_2 \approx -1.00+1.73i, \quad x_3 \approx -1.00-1.73i$
$x^3+2x^2+5x=0$	$x_1 \approx 0, \quad x_2 \approx -1.00+2.00i, \quad x_3 \approx -1.00-2.00i$
$x^3+4x^2+5x=0$	$x_1 \approx 0, \quad x_2 \approx -2.00+1.00i, \quad x_3 \approx -2.00-1.00i$
$x^3+3x^2+5x=0$	$x_1 \approx 0, \quad x_2 \approx -1.50+1.66i, \quad x_3 \approx -1.50-1.66i$
$x^3+3x^2+4x=0$	$x_1 \approx 0, \quad x_2 \approx -1.50+1.32i, \quad x_3 \approx -1.50-1.32i$
$x^3+3x^2+3x=0$	$x_1 \approx 0, \quad x_2 \approx -1.50+0.86i, \quad x_3 \approx -1.50-0.86i$
$x^3+4x^2+4x=0$	$x_1 \approx 0, \quad x_2 \approx -2.00, \quad x_3 \approx -2.00$
$x^3+4x^2+3x=0$	$x_1 \approx 0, \quad x_2 \approx -1.00, \quad x_3 \approx -3.00$
$x^3+2x^2+5x+1=0$	$x_1 \approx -0.22, \quad x_2 \approx -0.89+1.95i, \quad x_3 \approx -0.89-1.95i$
$x^3+2x^2+3x+1=0$	$x_1 \approx -0.43, \quad x_2 \approx -0.78+1.31i, \quad x_3 \approx -0.78-1.31i$
$x^3+2x^2+4x+1=0$	$x_1 \approx -0.28, \quad x_2 \approx -0.86+1.67i, \quad x_3 \approx -0.86-1.67i$
$x^3+4x^2+3x+1=0$	$x_1 \approx -0.43, \quad x_2 \approx -3.15+0.37i, \quad x_3 \approx -3.15-0.37i$
$x^3+3x^2+3x+1=0$	$x_1 \approx -1.00, \quad x_2 \approx -1.00, \quad x_3 \approx -1.00$
$x^3+3x^2+3x+2=0$	$x_1 \approx -2.00, \quad x_2 \approx -0.50+0,87i, \quad x_3 \approx -0.50-0,87i$
$x^3+2x^2+3x+2=0$	$x_1 \approx -1.00, \quad x_2 \approx -0.50+1.32i, \quad x_3 \approx -0.50-1.32i$
$x^3+2x^2+5x+2=0$	$x_1 \approx -0.47, \quad x_2 \approx -0.77+1.92i, \quad x_3 \approx -0.77-1.92i$
$x^3+4x^2+3x+2=0$	$x_1 \approx -3.27, \quad x_2 \approx -0.37+0.69i, \quad x_3 \approx -0.37-0.69i$
$x^3+2x^2+4x+2=0$	$x_1 \approx -0.64, \quad x_2 \approx -0.68+1.63i, \quad x_3 \approx -0.68-1.63i$

Table 2: Roots of tropical polynomials of $NCl(M_{CT_2})$ and $NCl(M_{CT_3})$

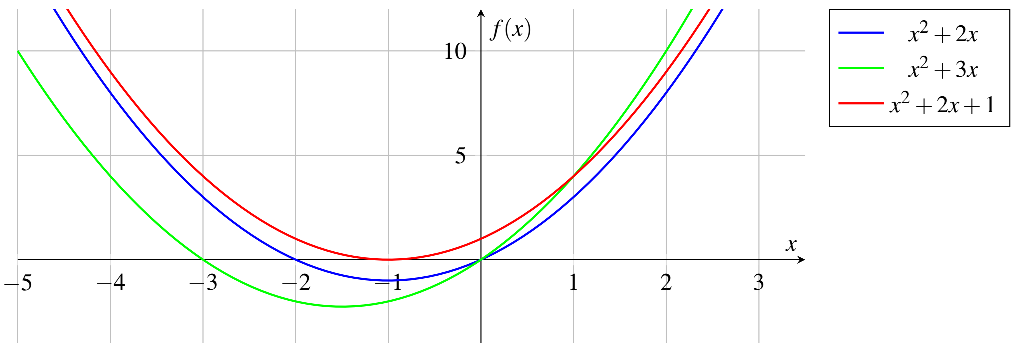


Figure 6: Graph of tropical polynomial for $NCl(M_{CT_2})$

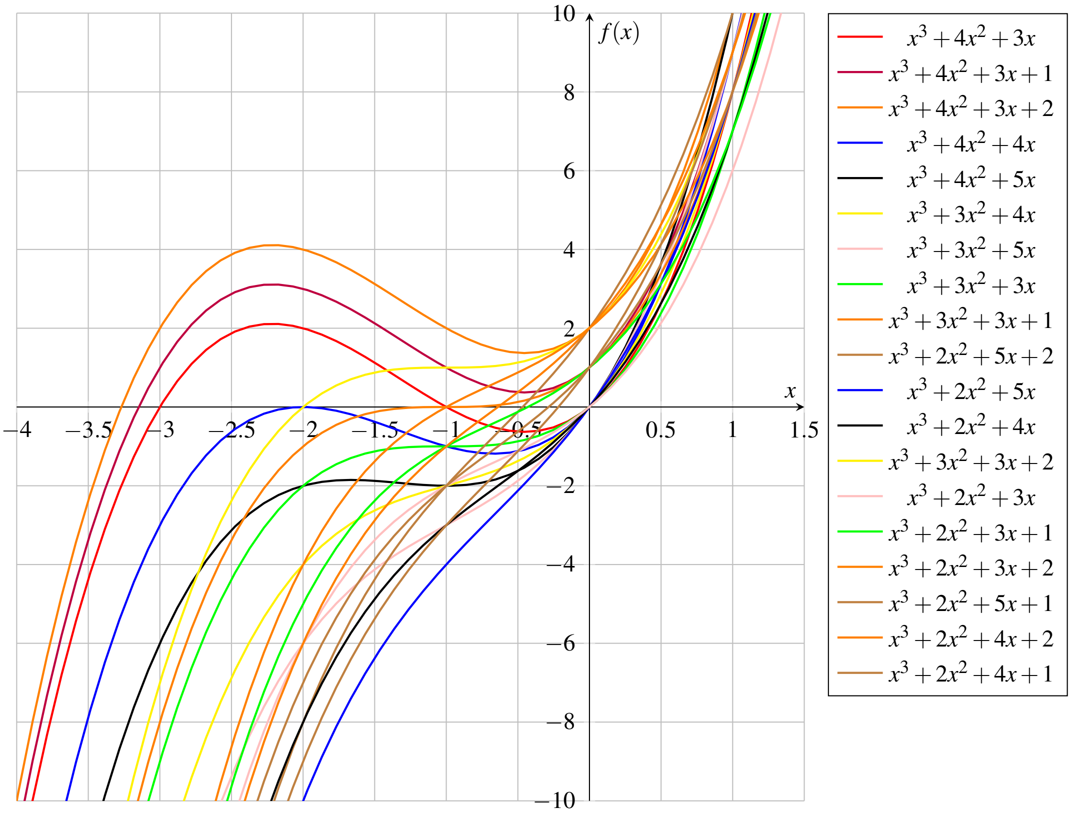


Figure 7: Graph of tropical polynomial for $NCl(M_{CT_3})$

4. Conclusion

The graphs of the roots of tropical polynomials in $Cl(M_{CT_n})$ and $NCl(M_{CT_n})$ at $n=2$ forms a parabolae with vertex $(\frac{-\lambda_0}{2}, \lambda_1 - \frac{-\lambda_0}{4})$ with have similar shapes, and this similarity also holds at $n=3$ which form cubic functions with similar shapes. As the polynomials transform, the graphs steepen slowly. The end behavior of the roots for any x shows that as $x \to \infty$, $f(x) \to \infty$ for all positive values of x, and as $x \to -\infty$, $f(x) \to -\infty$ for all negative values of x. Therefore, the roots of $Cl(M_{CT_n})$ and $NCl(M_{CT_n})$ have only negative real roots and are complex at all other points.

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