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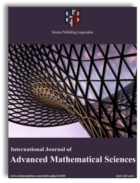

Matrix Representation of Generalized Quadranacci Number Sequences

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Abstract

In this paper, we present a matrix representation for the generalized Quadranacci number sequence, defined by the 4^{th} -order generalized recurrence relation $V_n\left(a_j, p_j\right) = \sum_{j=1}^{4} p_j V_{n-j}$, $p_4 \neq 0, n > 5$, initial terms $V_j = a_j$, and constant coefficients $p_j,\ j = 1,2,3,4$. The parameters a_j, and p_j are arbitrarily chosen real numbers. Fundamental results based on this definition are established in general symbolic form. A generalized 4×4 companion matrix M, associated with the recurrence relation, is introduced to analyze the properties of Quadranacci numbers. Subsequently, M^n is derived in a generalized form, enabling the application of matrix techniques to study the properties of Quadranacci number sequences. By appropriately specifying the initial values a_j, and the constant coefficients $p_j,\ j = 1,2,3,4$, several existing results are shown to be special cases of the derived results. Moreover, all the results obtained are implicitly applicable to the generalized Tribonacci and Fibonacci sequences, which are governed by lower-order recurrence relations.

Keywords: *Quadranacci number sequences, 4^{th} order linear recurrence relations, companion matrix,Tribonacci and Fibonacci sequences*

1. Introduction

In recurrence relations, sequences are generated by a process where a succeeding term is the sum of preceding terms. Recurrence relations play a significant role in solving various problems in mathematical and physical sciences and serve as a fundamental mathematical technique for calculations. Generally, recurrence sequences are a consequence of recursive relations. The Fibonacci sequence and many other well-known sequences are outcomes of such relations.

Sequences generated by 4^{th} -order linear recurrence relations are referred to as Tetranacci in Greek and Quadranacci in Latin. In this article, we focus on the Quadranacci number sequence. Initially, the Tetranacci sequence was described by [2], and subsequent studies by [3, 8] have considered the Tetranacci sequence in detail. If the four initial terms of the Quadranacci sequence are arbitrary, it is referred to as a Tetranacci-like sequence. However, if both the initial terms and the constant coefficients of the fourth-order linear recurrence relations are arbitrary, the resulting sequences are characterized as generalized Quadranacci sequences.

The matrix representation of generalized Quadranacci sequences is an effective technique for studying these sequences and establishing new results and properties. Authors such as [1, 5] have employed matrix techniques to develop identities for these sequences. Other researchers, including [6, 9, 10, 12, 13, 14, 15], have studied similar sequences, their generalizations, and derived numerous interesting properties and results. For further reference, one can consult Koshy's book [7], which discusses lower-order sequences and their generalizations along with practical applications. Researchers [4, 11] have also explored the properties of Quadranacci sequences and their applications.

In the present paper, we consider the most generalized 4^{th}-order recurrence relation. Initially, based on its definition, some preliminary results are obtained. Subsequently, a 4×4 companion matrix M, corresponding to the characteristic monic polynomial of the linear recurrence relation, is constructed. The n^{th} power of this generating matrix M is derived and applied to establish the properties of Quadranacci sequences. Using the generating matrix M^n , the terms of the Quadranacci sequence, corresponding to a linear system of matrix equations, are characterized.The implications of the obtained generalized Quadranacci matrices are analyzed, with closed-form formulas for their entries established. These formulas are related to the combinatorial representation of the real coefficients $p_i,\ i = 1,2,3,4$. Lower-order sequences are discussed as special cases of the generalized results obtained. Finally, some applications of this generalized matrix representation are presented.

1.1. General Definition

Definition We define the eneralized sequences $\{V_n\}$ by a fourth order recursive relations:

$$V_n = p_1V_{n-1} + p_2V_{n-2} + p_3V_{n-3} + p_4V_{n-4}, p_4 \neq 0, n \geq 5 \quad (1)$$

with initial conditions, $V_j = a_j,\ j = 1,2,3,4$. $a_j$and p_j. are any non–zero real numbers. Sequences $\{V_n\}$ designated as the generalized "Quadranacci" (Latin) number sequence.

2. Basic results based on definition

Theorem 2.1. *For $n > 5$, $V_n = \sum_{j=1}^{4} p_jV_{n-j}$, then*

$$V_n = (1+p_1)V_{n-1} + \sum_{j=2}^{4} \left(p_j - p_{j-1}\right)V_{n-j} - p_4V_{n-5}.$$

Proof. Using the definition (1), we have

$$\begin{aligned} V_n &= \sum_{j=1}^{4} p_jV_{n-j} \\ &= p_1V_{n-1} + p_2V_{n-2} + p_3V_{n-3} + p_4V_{n-4} \\ &= p_1V_{n-1} + p_2V_{n-2} + p_3V_{n-3} + p_4V_{n-4} + (V_{n-1} - V_{n-1}) \\ &= p_1V_{n-1} + p_2V_{n-2} + p_3V_{n-3} + p_4V_{n-4} + [V_{n-1} - (p_1V_{n-2} + p_2V_{n-3} + p_3V_{n-4} + p_4V_{n-5})] \\ &= (1+p_1)V_{n-1} + (p_2-p_1)V_{n-2} + (p_3-p_2)V_{n-3} + (p_4-p_3)V_{n-4} - p_4V_{n-5} \\ &= (1+p_1)V_{n-1} + \sum_{j=2}^{4} \left(p_j - p_{j-1}\right)V_{n-j} - p_4V_{n-5} \end{aligned}$$

Hence the theorem □

Corollary 2.2. *If we substitute $p_1 = 1, p_2 = 1, p_3 = 1, p_4 = 1$, then above results become*

$$V_n = 2V_{n-1} - V_{n-5}.$$

Theorem 2.3. *For $n > 7$,*

$$\begin{aligned} V_n &= \left[(1+p_1)^3 + 2(1+p_1)(p_2-p_1) + (p_3-p_2)\right]V_{n-3} \\ &+ \left((1+p_1)^2(p_2-p_1) + (1+p_1)(p_3-p_2) + (p_4-p_3) + (p_2-p_1)^2\right)V_{n-4} \\ &+ \left[(1+p_1)^2(p_3-p_2) + (1+p_1)(p_3-p_2) + (p_2-p_1)(p_3-p_2) - p_4\right]V_{n-5} \\ &+ \left((1+p_1)^2(p_4-p_3) - (1+p_1)p_4 + (p_2-p_1)(p_4-p_3)\right)V_{n-6} - \left[(1+p_1)^2p_4 + (p_2-p_1)\right]V_{n-7} \end{aligned}$$

Proof. Apply above Theorem (2.1), three times and simplify, we obtain the result. □

Corollary 2.4. *If we substitute $p_1 = 1, p_2 = 1, p_3 = 1, p_4 = 1$, then above results become*

$$V_n = 2^3V_{n-3} - V_{n-5} - 2V_{n-6} - 2^4V_{n-7}.$$

Definition. The n^{th} term of the third order sequence $\{V_n\}$ defined in (1) can be put in the form of formula

$$V_n = T_{n-1}V_4 + (p_2T_{n-2} + p_3T_{n-3} + p_4T_{n-4})V_3 + (p_3T_{n-2} + p_4T_{n-3})V_2 + p_4T_{n-2}V_1,\ n \geq 5, \quad (2)$$

where

$$\begin{pmatrix} T_n \\ T_{n-1} \\ T_{n-2} \\ T_{n-3} \end{pmatrix} = \begin{pmatrix} p_1 & p_2 & p_3 & p_4 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}^{n-4} \begin{pmatrix} p_1^2 + p_2 \\ p_1 \\ 1 \\ 0 \end{pmatrix} n \geq 5, \quad (3)$$

$T_1 = 0,\ T_2 = 1,\ T_3 = p_1, T_4 = p_1^2 + p_2.$

Theorem 2.5.

$$V_n = \begin{cases} a_i \text{ if } 1 \le n \le 4 \\ p_1V_{n-1} + p_2V_{n-2} + p_3V_{n-3} + p_4V_{n-4} \text{ if } n \ge 5 \end{cases}$$

a_j and p_j are any non–zero real numbers. Then

$$V_n = T_{n-1}V_4 + (p_2T_{n-2} + p_3T_{n-3} + p_4T_{n-4})V_3 + (p_3T_{n-2} + p_4T_{n-3})V_2 + p_4T_{n-2}V_1,\ n \ge 5, \tag{4}$$

where

$$\begin{pmatrix} T_n \\ T_{n-1} \\ T_{n-2} \\ T_{n-3} \end{pmatrix} = \begin{pmatrix} p_1 & p_2 & p_3 & p_4 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}^{n-4} \begin{pmatrix} p_1^2 + p_2 \\ p_1 \\ 1 \\ 0 \end{pmatrix} n \ge 5,$$

$T_0 = T_1 = T_2 = 0, T_3 = 1,\ T_4 = p_1,$

Proof. We will prove the theorem using induction for $n \ge 5$.
The theorem true for $n = 5, 6$ etc. which is easy to verify.
For $n = 5$, we have

$$V_5 = p_1V_4 + p_2V_3 + p_3V_2 + p_4V_1 = p_1a_4 + p_2a_3 + p_3a_2 + p_4a_1.$$

Using definition of the sequence.
Now RHS of (4), we have

$$\begin{aligned} V_5 &= T_4V_4 + (p_2T_3 + p_3T_2 + p_4T_1)V_2 + (p_3T_3 + p_4T_2)V_2 + p_4T_3V_1 \\ &= p_1a_4 + (p_21 + p_30 + p_40)a_3 + (p_31 + p_40)a_2 + p_41a_1 \\ &= p_1a_4 + p_2a_3 + p_3a_2 + p_4a_1 \end{aligned}.$$

Using

$$V_n = T_{n-1}V_4 + (p_2T_{n-2} + p_3T_{n-3} + p_4T_{n-4})V_3 + (p_3T_{n-2} + p_4T_{n-3})V_2 + p_4T_{n-2}V_1,\ .$$

Thus the theorem is valid for $n = 5$.
Now For $n = 6$, LHS of (4) is

$$\begin{aligned} V_6 &= p_1V_5 + p_2V_4 + p_3V_3 + p_4V_2 \\ &= p_1(p_1a_4 + p_2a_3 + p_3a_2 + p_4a_1) + p_2a_4 + p_3a_3 + p_4a_2p_1 \\ &= \left(p_1^2 + p_2\right)a_4 + (p_1p_2 + p_3)a_3 + (p_1p_3 + p_4)a_2 + p_1p_4a_1 \end{aligned}.$$

(by definition) RHS

$$\begin{aligned} V_6 &= T_5V_4 + (p_2T_4 + p_3T_3 + p_4T_2)V_3 + (p_3T_4 + p_4T_3)V_2 + p_4T_4V_1, \\ &= \left(p_1^2 + p_2\right)V_4 + (p_2p_1 + p_31 + p_40)V_3 + (p_3p_1 + p_41)V_2 + p_4p_1V_1, \\ &= \left(p_1^2 + p_2\right)a_4 + (p_2p_1 + p_3)a_3 + (p_3p_1 + p_4)a_2 + p_4p_1a_1 \end{aligned}.$$

Employing the second definition. Thus the theorem is valid for $n = 6$. Assuming that the theorem is valid for $n = k$ (is greater than 5). For $k > 5$,

$$V_k = T_{k-1}V_4 + (p_2T_{k-2} + p_3T_{k-3} + p_4T_{k-4})V_3 + (p_3T_{k-2} + p_4T_{k-3})V_2 + p_4T_{k-2}V_1, \tag{5}$$

Now we shall show that theorem is also valid for $n = k+1$ when it is valid for $n = k$. For this we shall show that

$$V_{k+1} = T_{(k+1)-1}V_4 + \left(p_2T_{(k+1)-2} + p_3T_{(k+1)-3} + p_4T_{(k+1)-4}\right)V_3 + \left(p_3T_{(k+1)-2} + p_4T_{(k+1)-3}\right)V_2 + p_4T_{(k+1)-2}V_1.$$

Now multiply the assumption (5) by p_1 and adding terms p_2V_{k-1}, p_3V_{k-2} and p_4V_{k-2}, we obtain

$$\begin{aligned} & p_1V_k + p_2V_{k-1} + p_3V_{k-2} + p_4V_{k-3} \\ &= p_1(T_{k-1}V_4 + (p_2T_{k-2} + p_3T_{k-3} + p_4T_{k-4})V_3 + (p_3T_{k-2} + p_4T_{k-3})V_2 + p_4T_{k-2}V_1,) \\ &+ p_2(T_{k-2}V_4 + (p_2T_{k-3} + p_3T_{k-4} + p_4T_{k-5})V_3 + (p_3T_{k-3} + p_4T_{k-4})V_2 + p_4T_{k-3}V_1) \\ &+ p_3(T_{k-3}V_4 + (p_2T_{k-4} + p_3T_{k-5} + p_4T_{k-6})V_3 + (p_3T_{k-4} + p_4T_{k-5})V_2 + p_4T_{k-4}V_1) \\ &+ p_4(T_{k-4}V_4 + (p_2T_{k-5} + p_3T_{k-6} + p_4T_{k-7})V_3 + (p_3T_{k-5} + p_4T_{k-6})V_2 + p_4T_{k-5}V_1) \end{aligned}.$$

Left hand side is $p_1V_k + p_2V_{k-1} + p_3V_{k-2} + p_4V_{k-3} = V_{k+1}$ by definition and the right hand side after simplification become

$$V_{k+1} = (p_1T_{k-1} + p_2T_{k-2} + p_3T_{k-3} + p_4T_{k-4})V_4$$

$$+\begin{bmatrix} p_2\left(p_1T_{k-2}+p_2T_{k-3}+p_3T_{k-4}+p_4T_{k-5}\right) \\ +p_3\left(p_1T_{k-3}+p_2T_{k-4}+p_3T_{k-5}+p_4T_{k-6}\right) \\ +p_4\left(p_1T_{k-4}+p_2T_{k-5}+p_3T_{k-6}+p_4T_{k-7}\right) \end{bmatrix} V_3$$

$$+\begin{bmatrix} p_3\left(p_1T_{k-2}+p_2T_{k-3}+p_3T_{k-4}+p_4T_{k-5}\right) \\ +p_4\left(p_1T_{k-3}+p_2T_{k-4}+p_3T_{k-5}+p_4T_{k-6}\right) \end{bmatrix} V_2$$

$$+p_4\left(p_1T_{k-2}+p_2T_{k-3}+p_3T_{k-4}+p_4T_{k-5}\right)V_1.$$

We obtain

$$V_{k+1}=T_kV_4+\left(p_2T_{k-1}+p_3T_{k-2}+p_4T_{k-3}\right)V_3+\left(p_3T_{k-1}+p_4T_{k-2}\right)V_2+p_4T_{k-1}V_1.$$

For simplication, the equations used are

$$\begin{array}{l} p_1T_{k-1}+p_2T_{k-2}+p_3T_{k-3}+p_4T_{k-4}=T_k \\ p_1T_{k-2}+p_2T_{k-3}+p_3T_{k-4}+p_4T_{k-5}=T_{k-1} \\ p_1T_{k-3}+p_2T_{k-4}+p_3T_{k-5}+p_4T_{k-6}=T_{k-2} \\ p_1T_{k-4}+p_2T_{k-5}+p_3T_{k-6}+p_4T_{k-7}=T_{k-3} \end{array}.$$

Hence, result follows by mathematical induction.

□

3. Matrix representation of 4rd order recurrence relation

Definition. We define a square matrix M of order 4 corresponding to the fourth order recurrence relations as:

$$M=\begin{pmatrix} p_1 & p_2 & p_3 & p_4 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \tag{6}$$

then

$$\begin{pmatrix} V_n \\ V_{n-1} \\ V_{n-2} \\ V_{n-3} \end{pmatrix}=M^{n-4}\begin{pmatrix} V_4 \\ V_3 \\ V_2 \\ V_1 \end{pmatrix}, n>4. \tag{7}$$

Theorem 3.1. *For $n>4$,*

$$M=\begin{pmatrix} p_1 & p_2 & p_3 & p_4 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

$$M^n=\begin{pmatrix} T_{n+3} & p_2T_{n+2}+p_3T_{n+1}+p_4T_n & p_3T_{n+2}+p_4T_{n+1} & p_4T_{n+2} \\ T_{n+2} & p_2T_{n+1}+p_3T_n+p_4T_{n-1} & p_3T_{n+1}+p_4T_n & p_4T_{n+1} \\ T_{n+1} & p_2T_n+p_3T_{n-1}+p_4T_{n-2} & p_3T_n+p_4T_{n-1} & p_4T_n \\ T_n & p_2T_{n-1}+p_3T_{n-2}+p_4T_{n-3} & p_3T_{n-1}+p_4T_{n-2} & p_4T_{n-1} \end{pmatrix}$$

where

$$\begin{pmatrix} T_n \\ T_{n-1} \\ T_{n-2} \\ T_{n-3} \end{pmatrix}=\begin{pmatrix} p_1 & p_2 & p_3 & p_4 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}^{n-4}\begin{pmatrix} p_1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, n>4.$$

Proof. To prove the theorem we shall use useinduction on n Let $n=1$, then we have

$$M^1=\begin{pmatrix} T_4 & p_2T_3+p_3T_2+p_4T_1 & p_3T_3+p_4T_2 & p_4T_3 \\ T_3 & p_2T_2+p_3T_1+p_4T_{-1} & p_3T_2+p_4T_1 & p_4T_2 \\ T_2 & p_2T_1+p_3T_0+p_4T_{-2} & p_3T_1+p_4T_0 & p_4T_1 \\ T_1 & p_2T_0+p_3T_{-1}+p_4T_{-3} & p_3T_0+p_4T_{-1} & p_4T_0 \end{pmatrix},$$

The result is true for $n=1$. Assume that the result be true for $n=m$. Then

$$M^m=\begin{pmatrix} T_{m+3} & p_2T_{m+2}+p_3T_{m+1}+p_4T_m & p_3T_{m+2}+p_4T_{m+1} & p_4T_{m+2} \\ T_{m+2} & p_2T_{m+1}+p_3T_m+p_4T_{m-1} & p_3T_{m+1}+p_4T_m & p_4T_{m+1} \\ T_{m+1} & p_2T_m+p_3T_{m-1}+p_4T_{m-2} & p_3T_m+p_4T_{m-1} & p_4T_m \\ T_m & p_2T_{m-1}+p_3T_{m-2}+p_4T_{m-3} & p_3T_{m-1}+p_4T_{m-2} & p_4T_{m-1} \end{pmatrix}. \tag{8}$$

Now

$M^{m+1} = M^m M$

$$= \begin{pmatrix} T_{m+3} & p_2T_{m+2}+p_3T_{m+1}+p_4T_m & p_3T_{m+2}+p_4T_{m+1} & p_4T_{m+2} \\ T_{m+2} & p_2T_{m+1}+p_3T_m+p_4T_{m-1} & p_3T_{m+1}+p_4T_m & p_4T_{m+1} \\ T_{m+1} & p_2T_m+p_3T_{m-1}+p_4T_{m-2} & p_3T_m+p_4T_{m-1} & p_4T_m \\ T_m & p_2T_{m-1}+p_3T_{m-2}+p_4T_{m-3} & p_3T_{m-1}+p_4T_{m-2} & p_4T_{m-1} \end{pmatrix} \begin{pmatrix} p_1 & p_2 & p_3 & p_4 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} T_{m+4} & p_2T_{m+3}+p_3T_{m+2}+p_4T_{m+1} & p_3T_{m+3}+p_4T_{m+2} & p_4T_{m+3} \\ T_{m+3} & p_2T_{m+2}+p_3T_{m+1}+p_4T_m & p_3T_{m+2}+p_4T_{m+1} & p_4T_{m+2} \\ T_{m+2} & p_2T_{m+1}+p_3T_m+p_4T_{m-1} & p_3T_{m+1}+p_4T_m & p_4T_{m+1} \\ T_{m+1} & p_2T_m+p_3T_{m-1}+p_4T_{m-2} & p_3T_m+p_4T_{m-1} & p_4T_m \end{pmatrix}$$

$T_{m+j+1} = p_1T_{m+j} + p_2T_{m+j-1} + p_3T_{m+j-2} + p_4T_{m+j-3},\ j = 0,1,2,3,4.$

Hence the theorem. □

Theorem 3.2. *If*

$$M^n = \left(m_{i,j}^{(n)}\right)_{4\times 4},$$

where

$$\left[m_{i,j}^{(n)} = m_{i,j}^{(n-1)}p_1 + m_{i,j}^{(n-2)}p_2 + m_{i,j}^{(n-3)}p_3 + m_{i,j}^{(n-4)}p_4,\ 1 \le i,j \le 4\right],$$

then

$$M^n = p_1M^{n-1} + p_2M^{n-2} + p_3M^{n-3} + p_4M^{n-4}.$$

Proof.

$$M^n = \begin{pmatrix} T_{n+3} & p_2T_{n+2}+p_3T_{n+1}+p_4T_n & p_3T_{n+2}+p_4T_{n+1} & p_4T_{n+2} \\ T_{n+2} & p_2T_{n+1}+p_3T_n+p_4T_{n-1} & p_3T_{n+1}+p_4T_n & p_4T_{n+1} \\ T_{n+1} & p_2T_n+p_3T_{n-1}+p_4T_{n-2} & p_3T_n+p_4T_{n-1} & p_4T_n \\ T_n & p_2T_{n-1}+p_3T_{n-2}+p_4T_{n-3} & p_3T_{n-1}+p_4T_{n-2} & p_4T_{n-1} \end{pmatrix},$$

M^n can be expressed as

$$M^n = \begin{pmatrix} \begin{matrix} T_{n+2}+T_{n+1} \\ +T_n+T_{n-1} \end{matrix} & \begin{matrix} p_2\begin{pmatrix} T_{n+1}+T_n \\ +T_{n-1}+T_{n-2} \end{pmatrix} \\ +p_3\begin{pmatrix} T_n+T_{n-1} \\ +T_{n-2}+T_{n-3} \end{pmatrix} \\ +p_4\begin{pmatrix} T_{n-1}+T_{n-2} \\ +T_{n-3}+T_{n-4} \end{pmatrix} \end{matrix} & \begin{matrix} p_3\begin{pmatrix} T_{n+1}+T_n \\ +T_{n-1}+T_{n-2} \end{pmatrix} \\ +p_4\begin{pmatrix} T_n+T_{n-1} \\ +T_{n-2}+T_{n-3} \end{pmatrix} \end{matrix} & p_4\begin{pmatrix} T_{n+1}+T_n \\ +T_{n-1}+T_{n-2} \end{pmatrix} \\ \begin{matrix} T_{n+1}+T_n \\ +T_{n-1}+T_{n-2} \end{matrix} & \begin{matrix} p_2\begin{pmatrix} T_n+T_{n-1} \\ +T_{n-2}+T_{n-3} \end{pmatrix} \\ +p_3\begin{pmatrix} T_{n-1}+T_{n-2} \\ +T_{n-3}+T_{n-4} \end{pmatrix} \\ +p_4\begin{pmatrix} T_{n-2}+T_{n-3} \\ +T_{n-4}+T_{n-5} \end{pmatrix} \end{matrix} & \begin{matrix} p_3\begin{pmatrix} T_n+T_{n-1} \\ +T_{n-2}+T_{n-3} \end{pmatrix} \\ +p_4\begin{pmatrix} T_{n-1}+T_{n-2} \\ +T_{n-3}+T_{n-4} \end{pmatrix} \end{matrix} & p_4\begin{pmatrix} T_n+T_{n-1} \\ +T_{n-2}+T_{n-3} \end{pmatrix} \\ \begin{matrix} T_n+T_{n-1} \\ +T_{n-2}+T_{n-3} \end{matrix} & \begin{matrix} p_2\begin{pmatrix} T_{n-1}+T_{n-2} \\ +T_{n-3}+T_{n-4} \end{pmatrix} \\ +p_3\begin{pmatrix} T_{n-2}+T_{n-3} \\ +T_{n-4}+T_{n-5} \end{pmatrix} \\ +p_4\begin{pmatrix} T_{n-3}+T_{n-4} \\ +T_{n-5}+T_{n-6} \end{pmatrix} \end{matrix} & \begin{matrix} p_3\begin{pmatrix} T_{n-1}+T_{n-2} \\ +T_{n-3}+T_{n-4} \end{pmatrix} \\ +p_4\begin{pmatrix} T_{n-2}+T_{n-3} \\ +T_{n-4}+T_{n-5} \end{pmatrix} \end{matrix} & p_4\begin{pmatrix} T_{n-1}+T_{n-2} \\ +T_{n-3}+T_{n-4} \end{pmatrix} \\ \begin{matrix} T_{n-1}+T_{n-2} \\ +T_{n-3}+T_{n-4} \end{matrix} & \begin{matrix} p_2\begin{pmatrix} T_{n-2}+T_{n-3} \\ +T_{n-4}+T_{n-5} \end{pmatrix} \\ +p_3\begin{pmatrix} T_{n-3}+T_{n-4} \\ +T_{n-5}+T_{n-6} \end{pmatrix} \\ +p_4\begin{pmatrix} T_{n-4}+T_{n-5} \\ +T_{n-6}+T_{n-7} \end{pmatrix} \end{matrix} & \begin{matrix} p_3\begin{pmatrix} T_{n-2}+T_{n-3} \\ +T_{n-4}+T_{n-5} \end{pmatrix} \\ +p_4\begin{pmatrix} T_{n-3}+T_{n-4} \\ +T_{n-5}+T_{n-6} \end{pmatrix} \end{matrix} & p_4\begin{pmatrix} T_{n-2}+T_{n-3} \\ +T_{n-4}+T_{n-5} \end{pmatrix} \end{pmatrix}.$$

On expanding, employing matrix properties and simplification, we obtain

$$M^n = \begin{pmatrix} T_{n+2} & p_2T_{n+1}+p_3T_n+p_4T_{n-1} & p_3T_{n+1}+p_4T_n & p_4T_{n+1} \\ T_{n+1} & p_2T_n+p_3T_{n-1}+p_4T_{n-2} & p_3T_n+p_4T_{n-1} & p_4T_n \\ T_n & p_2T_{n-1}+p_3T_{n-2}+p_4T_{n-3} & p_3T_{n-1}+p_4T_{n-2} & p_4T_{n-1} \\ T_{n-1} & p_2T_{n-2}+p_3T_{n-3}+p_4T_{n-4} & p_3T_{n-2}+p_4T_{n-3} & p_4T_{n-2} \end{pmatrix}$$

$$+\begin{pmatrix} T_{n+1} & p_2T_n+p_3T_{n-1}+p_4T_{n-2} & p_3T_n+p_4T_{n-1} & p_4T_n \\ T_n & p_2T_{n-1}+p_3T_{n-2}+p_4T_{n-3} & p_3T_{n-1}+p_4T_{n-2} & p_4T_{n-1} \\ T_{n-1} & p_2T_{n-2}+p_3T_{n-3}+p_4T_{n-4} & p_3T_{n-2}+p_4T_{n-3} & p_4T_{n-2} \\ T_{n-2} & p_2T_{n-3}+p_3T_{n-4}+p_4T_{n-5} & p_3T_{n-3}+p_4T_{n-4} & p_4T_{n-3} \end{pmatrix}$$
$$+\begin{pmatrix} T_n & p_2T_{n-1}+p_3T_{n-2}+p_4T_{n-3} & p_3T_{n-1}+p_4T_{n-2} & p_4T_{n-1} \\ T_{n-1} & p_2T_{n-2}+p_3T_{n-3}+p_4T_{n-4} & p_3T_{n-2}+p_4T_{n-3} & p_4T_{n-2} \\ T_{n-2} & p_2T_{n-3}+p_3T_{n-4}+p_4T_{n-5} & p_3T_{n-3}+p_4T_{n-4} & p_4T_{n-3} \\ T_{n-3} & p_2T_{n-4}+p_3T_{n-5}+p_4T_{n-6} & p_3T_{n-4}+p_4T_{n-5} & p_4T_{n-4} \end{pmatrix}$$
$$+\begin{pmatrix} T_{n-1} & p_2T_{n-2}+p_3T_{n-3}+p_4T_{n-4} & p_3T_{n-2}+p_4T_{n-3} & p_4T_{n-2} \\ T_{n-2} & p_2T_{n-3}+p_3T_{n-4}+p_4T_{n-5} & p_3T_{n-3}+p_4T_{n-4} & p_4T_{n-3} \\ T_{n-3} & p_2T_{n-4}+p_3T_{n-5}+p_4T_{n-6} & p_3T_{n-4}+p_4T_{n-5} & p_4T_{n-4} \\ T_{n-4} & p_2T_{n-5}+p_3T_{n-6}+p_4T_{n-7} & p_3T_{n-5}+p_4T_{n-6} & p_4T_{n-5} \end{pmatrix},$$

which can be written as

$$M^n = p_1M^{n-1}+p_2M^{n-2}+p_3M^{n-3}+p_4M^{n-4}.$$

Hence

$$M^n = p_1M^{n-1}+p_2M^{n-2}+p_3M^{n-3}+p_4M^{n-4},$$

and

$$\begin{pmatrix} T_n \\ T_{n-1} \\ T_{n-2} \\ T_{n-3} \end{pmatrix} = \begin{pmatrix} p_1 & p_2 & p_3 & p_4 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}^{n-4} \begin{pmatrix} p_1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, n > 4.$$

Hence the result. □

4. Some Properties of M^n Matrix

- $Trace\,(M^n) = \sum_{i=1}^{4} m_{i,i}^{(n-i)} p_i$.
- $\det M^n = (-p_4)^n$.
- $\frac{1}{-p_4}\begin{pmatrix} 0 & -p_4 & 0 & 0 \\ 0 & 0 & -p_4 & 0 \\ 0 & 0 & 0 & -p_4 \\ -1 & p_1 & p_2 & p_3 \end{pmatrix}.$
- $M^{n_1}M^{n_2} = M^{n_2}M^{n_1} = M^{n_1+n_2}$, $n_1, n_2 \geq 0$.
- Characteristic equation of matrix M is $f(\lambda) = \lambda^4 - p_1\lambda^3 - p_2\lambda^2 - p_3\lambda - p_4 = 0$.

5. Generalized Quadranacci sequences and Ciphering

In this section, generalized Tetranacci sequence in terms arbitrary coefficients $p_j, j = 1,2,3,4$ of fourth order recurrence relation and initial terms $V_j = a_j$ is defined. Consider any 16-digits number, which is taken as 16-digits initial code in the form of 4×4 square matrix. Let the original message in the form of a square matrix of order 4 is

$$N = \begin{pmatrix} n_1 & n_2 & n_3 & n_4 \\ n_5 & n_6 & n_7 & n_8 \\ n_9 & n_{10} & n_{11} & n_{12} \\ n_{13} & n_{14} & n_{15} & n_{16} \end{pmatrix}.$$

Now for coding, consider the generalized tetranacci matrix M^n, choose any value of n, so taking $n = 5$, then matrix reprsentation of M^5 is

$$M^5 = \begin{pmatrix} T_8 & p_2T_7+p_3T_6+p_4T_5 & p_3T_7+p_4T_6 & p_4T_7 \\ T_7 & p_2T_6+p_3T_5+p_4T_4 & p_3T_6+p_4T_5 & p_4T_6 \\ T_6 & p_2T_5+p_3T_4+p_4T_3 & p_3T_5+p_4T_4 & p_4T_5 \\ T_5 & p_2T_4+p_3T_3+p_4T_2 & p_3T_4+p_4T_3 & p_4T_4 \end{pmatrix},$$

where

$T_0 = 0,$

$T_{13} = 1,$

$T_2 = p_1,$

$T_3 = p_1^2+p_2,$

$T_4 = p_1^3+2p_1p_2+p_3$

$T_5 = p_1^4+3p_1^2p_2+2p_1p_3+p_2^2+p_4$

$T_6 = p_1^5+4p_1^3p_2+3p_1^2p_3+3p_1p_2^2+2p_2p_3+2p_1p_4$

$T_7 = p_1^6+5p_1^4p_2+6p_1^2p_2^2+4p_3p_1^3+6p_1p_2p_3+p_2^3+p_3^2+2p_2p_4+3p_1^2p_4$

....

Now the inverse of the above matrix M^{-n} for $n=5$ is obtained as

$$M^{-5}=\begin{pmatrix}\left(A-B.D^{-1}C\right)^{-1} & -\left(A-B.D^{-1}C\right)^{-1}BD^{-1}\\ -D^{-1}C\left(A-B.D^{-1}C\right)^{-1} & D^{-1}+D^{-1}C\left(A-B.D^{-1}C\right)^{-1}BD^{-1}\end{pmatrix}$$

$$A=\begin{pmatrix}T_8 & p_2T_7+p_3T_6+p_4T_5\\ T_7 & p_2T_6+p_3T_5+p_4T_4\end{pmatrix},B=\begin{pmatrix}p_3T_7+p_4T_6 & p_4T_7\\ p_3T_6+p_4T_5 & p_4T_6\end{pmatrix}$$

$$C=\begin{pmatrix}T_6 & p_2T_5+p_3T_4+p_4T_3\\ T_5 & p_2T_4+p_3T_3+p_4T_2\end{pmatrix},D=\begin{pmatrix}p_3T_5+p_4T_4 & p_4T_5\\ p_3T_4+p_4T_3 & p_4T_4\end{pmatrix}.$$

$$D^{-1}=\frac{1}{p_4{}^2\left(-(T_4)^2+T_5T_3\right)}\begin{pmatrix}-p_4T_4 & p_4T_5\\ p_3T_4+p_4T_3 & p_3T_n+p_4T_4\end{pmatrix}$$

The matrix NM^5 is the Generalized Quadranacci coding of the message,can be written as

$$NM^5=\begin{pmatrix}r_1 & r_2 & r_3 & r_4\\ r_5 & r_6 & r_7 & r_8\\ r_9 & r_{10} & r_{11} & r_{12}\\ r_{13} & r_{14} & r_{15} & r_{16}\end{pmatrix}=R.$$

The code matrix R is sent, decoding is done using the matrices R and M^{-5}, the message obtained as

$$RM^{-5}=\begin{pmatrix}n_1 & n_2 & n_3 & n_4\\ n_5 & n_6 & n_7 & n_8\\ n_9 & n_{10} & n_{11} & n_{12}\\ n_{13} & n_{14} & n_{15} & n_{16}\end{pmatrix}=N.$$

5.1. Determinant of the code matrix R

$\det(NM^n)=\det(N)\det(M^n)=p_3^n\det(N).$

Remark 5.1. If $p_4=1$, then

$\det(NM^n)=\det(N)\det(M^n)=(-1)^n\det(N)=\det(N).$

Remark 5.2. If $p_1=1,p_2=1,p_3=1,p_4=1,$, then

$\det(NM^n)=\det(N)\det(M^n)=\det(N).$

5.2. Illustration

Example 5.3. First of all choose any $p_j, j=1,2,3,4$ for the generalized Quadranacci sequence and the original message. Let us take $p_1=3,p_2=1,p_3=3,p_4=2$, and the original message ,say, ”2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47, 53” (first 16 prime numbers) for the companion matrix M, we have

$$N=\begin{pmatrix}2 & 3 & 5 & 7\\ 11 & 13 & 17 & 19\\ 23 & 29 & 31 & 37\\ 41 & 43 & 47 & 53\end{pmatrix}.$$

Now using the Generalized Quadranacci matrix M^n , taking power $n=5$ of the matrix, we have a coding matrix,

$$M^5=\begin{pmatrix}6738 & 3269 & 5820 & 73268\\ 1634 & 918 & 1635 & 918\\ 459 & 257 & 459 & 258\\ 129 & 72 & 128 & 72\end{pmatrix}.$$

Now, using the above data, the message forwarded in the form of a following matrix,

$$R=NM^5=\begin{pmatrix}2 & 3 & 5 & 7\\ 11 & 13 & 17 & 19\\ 23 & 29 & 31 & 37\\ 41 & 43 & 47 & 53\end{pmatrix}\begin{pmatrix}6738 & 3269 & 5820 & 73268\\ 1634 & 918 & 1635 & 918\\ 459 & 257 & 459 & 258\\ 129 & 72 & 128 & 72\end{pmatrix}=\begin{pmatrix}21576 & 11081 & 19736 & 11084\\ 105614 & 53630 & 95510 & 53636\\ 221362 & 112440 & 200240 & 112448\\ 370340 & 186828 & 332692 & 186824\end{pmatrix}.$$

Message received and on decoding in the form of a matrix is

$$RM^{-5}=\begin{pmatrix}0.001 & -0.004 & 0.001 & -0.004\\ -0.002 & 0.119 & -1.002 & 2.176\\ 0.004 & 0.32 & 0.004 & -4.266\\ -0.007 & -0.68 & 0.993 & 5.43\end{pmatrix}\begin{pmatrix}21576 & 11081 & 19736 & 11084\\ 105614 & 53630 & 95510 & 53636\\ 221362 & 112440 & 200240 & 112448\\ 370340 & 186828 & 332692 & 186824\end{pmatrix}.$$

$$=\begin{pmatrix}2 & 3 & 5 & 7\\ 11 & 13 & 17 & 19\\ 23 & 29 & 31 & 37\\ 41 & 43 & 47 & 53\end{pmatrix}=N$$

6. Conclusion

This work investigates fourth-order recursive relations in their most general form, defining the relationship between a term and its four preceding terms in a sequence. A general matrix M is derived to encapsulate the coefficients of these relations, offering a concise and structured framework for their analysis and solution. The n^{th} power of matrix M is obtained in its generalized form, enabling the study of Quadranacci sequences. The derived results demonstrate that second-order, third-order, and numerous fourth-order recurrence relations emerge as special cases of this generalized approach. Future research could explore higher-order generalized sequences for broader applications and delve deeper into number theory to uncover further identities and results using alternative approaches.

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