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Research paper

Common fixed-point theorems on complex partial b- metric space

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Abstract

In this paper, we establish some unique common fixed-point theorems for two pairs of weakly compatible mappings in complex partial b-metric space which generalize and improves several well- known common fixed point results in partial b-metric space.

Keywords: *Complex Valued B-Metric Space; Complex Partial B-Metric Space; Weakly Compatible Maps; Common Fixed-Point Semicolon.*

1. Introduction

The fixed point theory is very useful and powerful tools in several branches of sciences, engineering and the development of non-linear analysis. In 1989, Backhtin developed the concept of b-metric spaces. After that many fixed point theorems have been proved on b-metric space by different authors. In 2011, Azam et.al introduced the concept of complex valued metric spaces. Rao et.al.[8] devolved a common fixed point theorem in complex b-metric spaces. After that M. Gunaseelan [5] introduced the notion of complex valued partial b-metric space and proved existence and uniqueness of fixed point theorem. Afterward, Dhivya and Marudai [6] extended all the preceding results in the setting of complex partial metric spaces making use of a rational type contraction. Recently, Maheswari et.al [4] introduced to complex partial b-metric space and proved the existence of coupled fixed point result under contractive conditions in this space .There are many researchers extended the concept of Partial metric space such as Partial b-metric space, Complex Valued partial metric space and proved the existence of fixed point theorem via contraction mappings.

The aim of this research paper to prove a common fixed point theorem for two pairs of weakly compatible mappings in Complex valued partial b-metric spaces. The present research work will attempt to extends, generalize and improve several results from the existing literature in this field.

2. Preliminaries

Let ∉ be the set of complex numbers and $\omega_1, \omega_2 \in$ ∉. Define a partial order $\leq$ on ∉ as follows:
$\omega_1 \leq \omega_2$ iff $R_e(\omega_1) \leq R_e(\omega_2)$ and $I_m(\omega_1) \leq I_m(\omega_2)$.
Consequently $\omega_1 \leq \omega_2$ if one of the following condition is satisfied:

(a) $R_e(\omega_1) = R_e(\omega_2), \quad I_m(\omega_1) < I_m(\omega_2),$
(b) $R_e(\omega_1) < R_e(\omega_2), \quad I_m(\omega_1) = I_m(\omega_2),$
(c) $R_e(\omega_1) < R_e(\omega_2), \quad I_m(\omega_1) < I_m(\omega_2),$
(d) $R_e(\omega_1) = R_e(\omega_2), \quad I_m(\omega_1) = I_m(\omega_2),$

Specifically, we write $\omega_1 \lneqq \omega_2$ if $\omega_1 \neq \omega_2$ and one of (a),(b) and (c) is satisfied.
We will write $\omega_1 < \omega_2$ if only condition (c) is satisfied. Observe that

(i) If $0 \leq \omega_1 \lneqq \omega_2$, then $|\omega_1| < |\omega_2|$,
(ii) If $\omega_1 \leq \omega_2$ and $\omega_2 < \omega_3$, then $\omega_1 < \omega_3$,
(iii) If $m, n \in \mathbb{R}$ and $m \leq n$, then $mh \leq nh$, for all $h \in$ ∉

Definition 2.1:(see [6]). Let X be a non empty set. A mapping $\mathbb{P}_c: \mathbb{X} \times \mathbb{X} \to \mathbb{C}^+$ is said to be complex partial matrix on $\mathbb{X}$, if the following conditions are satisfied :

$(\mathbb{P}_{c_1})$: $0 \le \mathbb{P}_c(x,x) \le \mathbb{P}_c(x,y)$, $\forall\, x, y \in \mathbb{X}$.
$(\mathbb{P}_{c_2})$: $\mathbb{P}_c(x,y) = \mathbb{P}_c(y,x)$
$(\mathbb{P}_{c_3})$: $\mathbb{P}_c(x,x) = \mathbb{P}_c(x,y) = \mathbb{P}_c(y,y)$ if and only if $x = y$
$(\mathbb{P}_{c_4})$: $\mathbb{P}_c(x,y) \le \mathbb{P}_c(x,z) + \mathbb{P}_c(z,y) - \mathbb{P}_c(z,z)$, $\forall\, x, y, z \in \mathbb{X}$.
A complex partial metric space is a pair $(\mathbb{X}, \mathbb{P}_c)$ such that $\mathbb{X}$ is a non empty set and $\mathbb{P}_c$ is a complex partial metric on $\mathbb{X}$.

Definition 2.2:(see[8]). Let $\mathbb{X}$ be a non empty set and let $k \ge 1$ be a given real number. A mapping $d_{cb}: \mathbb{X} \times \mathbb{X} \to \mathbb{C}^+$ is said to be a complex valued b-metric space if the following conditions are satisfied:

(d_{cb_1}) : $0 \le d_{cb}(x,y)$ and $d_{cb}(x,y) = 0 \Leftrightarrow x = y$, $\forall\, x, y \in \mathbb{X}$
(d_{cb_2}) : $d_{cb}(x,y) = d_{cb}(y,x)$, $\forall\, x, y \in \mathbb{X}$
(d_{cb_3}) : $d_{cb}(x,y) \le k[d_{cb}(x,z) + d_{cb}(z,y)]$, $\forall\, x, y, z \in \mathbb{X}$.
The pair $(\mathbb{X}, d_{cb})$ is called a complex valued b-metric space.

Definition 2.3:(see[5]). Let $\mathbb{X}$ be a non empty set and let $k \ge 1$ be a given real number. A mapping $\mathbb{P}_{cb}: \mathbb{X} \times \mathbb{X} \to \mathbb{C}^+$ is said to be a complex partial b-metric space on $\mathbb{X}$, if the following conditions are satisfied:

$(\mathbb{P}_{cb_1})$: $0 \le \mathbb{P}_{cb}(x,x) \le \mathbb{P}_{cb}(x,y)$, $\forall\, x, y \in \mathbb{X}$
$(\mathbb{P}_{cb_2})$: $\mathbb{P}_{cb}(x,y) = \mathbb{P}_{cb}(y,x)$, $\forall\, x, y \in \mathbb{X}$
$(\mathbb{P}_{cb_3})$: $\mathbb{P}_{cb}(x,x) = \mathbb{P}_{cb}(x,y) = \mathbb{P}_{cb}(y,y)$, iff $x = y$
$(\mathbb{P}_{cb_4})$: $\mathbb{P}_{cb}(x,y) \le k[\mathbb{P}_{cb}(x,z) + \mathbb{P}_{cb}(z,y) - \mathbb{P}_{cb}(z,z)]$, $\forall\, x, y, z \in \mathbb{X}$.
The pair $(\mathbb{X}, \mathbb{P}_{cb})$ is called a complex partial b-metric space. The number k is called the coefficient of $(\mathbb{X}, \mathbb{P}_{cb})$.

Remark 2.1:(see[5]). In a complex partial b-metric space $(\mathbb{X}, \mathbb{P}_{cb})$ if $x, y \in \mathbb{X}$ and $\mathbb{P}_{cb}(x,y) = 0$, then $x = y$,but converse may not be true.

Remark 2.1:(see[5]). It is clear that every complex partial metric space is a complex partial b-metric space with Coefficient $k = 1$ and every complex valued b-metric space is a complex partial b-metric space with the same Coefficient and zero self- distance. However, the converse of this fact need not be proved.

Definition 2.4:*(*see[5]). Let $(\mathbb{X}, \mathbb{P}_{cb})$ is a complex partial b-metric space with coefficient k. Let $\{t_n\}$ be any sequence in $\mathbb{X}$ and $t \in \mathbb{R}$, then

i) The sequence $\{t_n\}$ is said to be convergent w.r.t. $\mathbb{P}_{cb}$ and converges to t,if $\lim_{n\to\infty} \mathbb{P}_{cb}(t_n, t) = \mathbb{P}_{cb}(t,t)$
ii) The sequence $\{t_n\}$ is said to be Cauchy sequence in $(\mathbb{X}, \mathbb{P}_{cb})$, if $\lim_{n,m\to\infty} \mathbb{P}_{cb}(t_n, t_m)$ exists and finite.
iii) $(\mathbb{X}, \mathbb{P}_{cb})$ is said to be complete complex partial b-metric space if for every Cauchy sequence $\{t_n\}$ in $\mathbb{X}$ there exists $t \in \mathbb{X}$ such that $\lim_{n,m\to\infty} \mathbb{P}_{cb}(t_n, t_m) = \lim_{n\to\infty} \mathbb{P}_{cb}(t_n, t) = \mathbb{P}_{cb}(t,t)$.
iv) A mapping $\mu: \mathbb{X} \to \mathbb{X}$ is said to be continuous at $t_0 \in \mathbb{X}$ if for every $\varepsilon > 0$, there exists $\delta > 0$, such that $\mu\big(B\mathbb{P}_{cb}(t_0, \delta)\big) \subset (B\mathbb{P}_{cb}(t_0, \varepsilon))$.

Definition 2.5:(see[12]). Let $\mathbb{X}$ be a non empty set and $A, B : \mathbb{X} \to \mathbb{X}$. If $\omega = Ax = Bx$, for some $x \in \mathbb{X}$, then x is called a coincidence point of A and B , and ω is called a point of coincidence of A and B.

If $\omega = x$, then x is a common fixed point of A and B.

***Definition 2.6*:**(see[12]). Let P and Q be two self- maps defined on a non empty set $\mathbb{X}$. Then P and Q are said to be weakly compatible if they commute at their coincident point i.e. if $Pt = Qt$ for some $t \in (\mathbb{X}, \mathbb{P}_{cb})$, then $PQt = QPt$.

3. Main Result

Theorem 3.1: Let $(\mathbb{X}, \mathbb{P}_{cb})$is a complete complex partial b-metric space with coefficient $k \ge 1$ and let P, Q, S and T are four self maps of $\mathbb{X}$ such that $T(\mathbb{X}) \subseteq P(\mathbb{X})$ and $S(\mathbb{X}) \subseteq Q(\mathbb{X})$ Satisfying

i) $\mathbb{P}_{cb}(Sx, Ty) \le \frac{\mu}{k^2}\left[max\left\{\mathbb{P}_{cb}(Px,Qy), \mathbb{P}_{cb}(Px,Sx), \mathbb{P}_{cb}(Qy,Sx), \mathbb{P}_{cb}(Qy,Ty), \frac{\mathbb{P}_{cb}(Qy,Ty)(1+\mathbb{P}_{cb}(Px,Sx))}{1+\mathbb{P}_{cb}(Px,Qy)}\right\}\right]$,

where $k \ge 1$ and $\mu \in (0,1)$, $\forall\, x, y \in \mathbb{X}$. …....................... (3.1.1)

ii) The pairs (P,S) and (Q,T) are weakly compatible, then P, Q, S and T have a unique common fixed point.

Proof. We have $T(\mathbb{X}) \subseteq P(\mathbb{X})$ and $S(\mathbb{X}) \subseteq Q(\mathbb{X})$. Let $x_0 \in \mathbb{X}$ be arbitrary, there exists $x_1, x_2 \in \mathbb{X}$ such that $y_0 = Qx_1 = Sx_0$ and $y_1 = Px_2 = Tx_1$.

We construct the sequences $\{y_n\}$ in $\mathbb{X}$ such that $y_{2n} = Qx_{2n+1} = Sx_{2n}$ and $y_{2n+1} = Px_{2n+2} = Tx_{2n+1}$, $n = 1,2,3, \ldots$..
From (3.1.1), we have

$$\mathbb{P}_{cb}(y_{2n}, y_{2n+1}) = \mathbb{P}_{cb}(Sx_{2n}, Tx_{2n+1})$$
$$\leq \frac{\mu}{k^2} \left[max \left\{ \mathbb{P}_{cb}(Px_{2n}, Qx_{2n+1}), \mathbb{P}_{cb}(Px_{2n}, Sx_{2n}), \mathbb{P}_{cb}(Qx_{2n+1}, Sx_{2n}), \mathbb{P}_{cb}(Qx_{2n+1}, Tx_{2n+1}), \frac{\mathbb{P}_{cb}(Qx_{2n+1}, Tx_{2n+1})(1 + \mathbb{P}_{cb}(Px_{2n}, Sx_{2n}))}{1 + \mathbb{P}_{cb}(Px_{2n}, Qx_{2n+1})} \right\} \right]$$
$$= \frac{\mu}{k^2} \left[max \left\{ \mathbb{P}_{cb}(y_{2n-1}, y_{2n}), \mathbb{P}_{cb}(y_{2n-1}, y_{2n+1}), \mathbb{P}_{cb}(y_{2n}, y_{2n}), \mathbb{P}_{cb}(y_{2n}, y_{2n+1}), \frac{\mathbb{P}_{cb}(y_{2n}, y_{2n+1})(1 + \mathbb{P}_{cb}(y_{2n-1}, y_{2n}))}{1 + \mathbb{P}_{cb}(y_{2n-1}, y_{2n})} \right\} \right]$$
$$= \frac{\mu}{k^2} \left[max\{\mathbb{P}_{cb}(y_{2n-1}, y_{2n}), \mathbb{P}_{cb}(y_{2n-1}, y_{2n}), 0, \mathbb{P}_{cb}(y_{2n}, y_{2n+1}), \mathbb{P}_{cb}(y_{2n}, y_{2n+1})\} \right]$$
$$= \frac{\mu}{k^2} \left[\mathbb{P}_{cb}(y_{2n}, y_{2n+1}) \right]$$

$\Rightarrow \left(1 - \frac{\mu}{k^2}\right)[\mathbb{P}_{cb}(y_{2n}, y_{2n+1}) \leq 0.$
Which is a contradiction, since $k \geq 1$ and $\mu \in (0,1)$.
Therefore $\mathbb{P}_{cb}(y_{2n}, y_{2n+1}) \leq \frac{\mu}{k^2} [\mathbb{P}_{cb}(y_{2n-1}, y_{2n})]$.
Similarly, $\mathbb{P}_{cb}(y_{2n+1}, y_{2n+2}) \leq \frac{\mu}{k^2} [\mathbb{P}_{cb}(y_{2n}, y_{2n+1})]$.
It follows that
$\mathbb{P}_{cb}(y_{2n}, y_{2n+1}) \leq \frac{\mu}{k^2} [\mathbb{P}_{cb}(y_{2n-1}, y_{2n})] \leq \ldots\ldots\ldots \leq (\frac{\mu}{k^2})^n[\mathbb{P}_{cb}(y_0, y_1)]$.
For any $m, n \in N$ with $m > n$, it follows that

$$|\mathbb{P}_{cb}(y_n, y_m)| \leq (\tfrac{\mu}{k^2})k[\mathbb{P}_{cb}(y_n, y_{n+1}) + \mathbb{P}_{cb}(y_{n+1}, y_m) - \mathbb{P}_{cb}(y_{n+1}, y_{n+1})]$$
$$\leq k(\tfrac{\mu}{k^2}) [\mathbb{P}_{cb}(y_n, y_{n+1}) + \mathbb{P}_{cb}(y_{n+1}, y_m)]$$
$$\leq k(\tfrac{\mu}{k^2})[\mathbb{P}_{cb}(y_n, y_{n+1})] + k(\tfrac{\mu}{k^2})[\mathbb{P}_{cb}(y_{n+1}, y_m)]$$
$$\leq k(\tfrac{\mu}{k^2})[\mathbb{P}_{cb}(y_n, y_{n+1})] + k^2(\tfrac{\mu}{k^2})[\mathbb{P}_{cb}(y_{n+1}, y_{n+2}) + \mathbb{P}_{cb}(y_{n+2}, y_m) - \mathbb{P}_{cb}(y_{n+2}, y_{n+2})]$$
$$\leq k(\tfrac{\mu}{k^2})[\mathbb{P}_{cb}(y_n, y_{n+1})] + k^2(\tfrac{\mu}{k^2})[\mathbb{P}_{cb}(y_{n+1}, y_{n+2}) + \mathbb{P}_{cb}(y_{n+2}, y_m)]$$
$$\leq k(\tfrac{\mu}{k^2})[\mathbb{P}_{cb}(y_n, y_{n+1})] + k^2(\tfrac{\mu}{k^2})[\mathbb{P}_{cb}(y_{n+1}, y_{n+2})] + k^2(\tfrac{\mu}{k^2}) \mathbb{P}_{cb}(y_{n+2}, y_m)]$$
$$\leq k(\tfrac{\mu}{k^2})^n|\mathbb{P}_{cb}(y_0, y_1)| + k^2(\tfrac{\mu}{k^2})^{n+1}|\mathbb{P}_{cb}(y_0, y_1)| + \cdots + k^{m-n}(\tfrac{\mu}{k^2})^{m-n}|\mathbb{P}_{cb}(y_0, y_1)|$$
$$\leq k((\tfrac{\mu}{k^2})^n[1 + k(\tfrac{\mu}{k^2}) + k^2(\tfrac{\mu}{k^2})^2 + \cdots + k^{m-n-1}(\tfrac{\mu}{k^2})^{m-n-1}]|\mathbb{P}_{cb}(y_0, y_1)|$$
$$= k((\tfrac{\mu}{k^2})^n \left[1 + \left(\tfrac{\mu}{k}\right) + \left(\tfrac{\mu}{k^2}\right)^2 + \ldots + \left(\tfrac{\mu}{k^2}\right)^{m-n-1}\right] |\mathbb{P}_{cb}(y_0, y_1)|$$
$$= k((\tfrac{\mu}{k^2})^n \left[\frac{1}{1 - \frac{\mu}{k}}\right] |\mathbb{P}_{cb}(y_0, y_1)|$$

Hence $|\mathbb{P}_{cb}(y_n, y_{n+1})| \leq \frac{(\mu/_k)^n}{1 - \mu/_k} |\mathbb{P}_{cb}(y_0, y_1)|$
As $n \to \infty$, we get $\mathbb{P}_{cb}(y_n, y_{n+1}) \to 0$.
Therefore $\{y_n\}$ is a Cauchy sequence in complete complex partial b-metric space $(\mathbb{X}, \mathbb{P}_{cb})$.
So there exist a point $r \in \mathbb{X}$, such that $\lim_{n\to\infty} y_n = r$ and $\mathbb{P}_{cb}(r, r) = \lim_{n\to\infty} \mathbb{P}_{cb}(r, y_n) = \mathbb{P}_{cb}(y_n, y_n) = 0$.
Therefore $\lim_{n\to\infty} Qx_{2n+1} = r$, $\lim_{n\to\infty} Sx_{2n} = r$, $\lim_{n\to\infty} Px_{2n+1} = r$ and $\lim_{n\to\infty} Tx_{2n} = r$.
Since $T(\mathbb{X}) \subseteq P(\mathbb{X})$, there exists a point $u \in \mathbb{X}$ such that $Pu = r$.
Now we have to prove that $Su = r$.
Suppose that $\mathbb{P}_{cb}(Su, r) > 0$, then using (3.1.1), we have

$$\mathbb{P}_{cb}(Su, r) \leq k[\mathbb{P}_{cb}(Su, Tx_{2n}) + \mathbb{P}_{cb}(Tx_{2n}, r) - \mathbb{P}_{cb}(Tx_{2n} , Tx_{2n})]$$
$$\leq k \, \mathbb{P}_{cb}(Su, Tx_{2n}) + k \, \mathbb{P}_{cb}(Tx_{2n}, r)$$
$$\leq k \frac{\mu}{k^2} \left[max \left\{ \mathbb{P}_{cb}(Pu, Qx_{2n}), \mathbb{P}_{cb}(Pu, Su), \mathbb{P}_{cb}(Qx_{2n}, Su), \mathbb{P}_{cb}(Qx_{2n}, Tx_{2n}), \frac{\mathbb{P}_{cb}(Qx_{2n}, Tx_{2n})(1 + \mathbb{P}_{cb}(Pu, Su))}{1 + \mathbb{P}_{cb}(Pu, Qx_{2n})} \right\} \right] + k \, \mathbb{P}_{cb}(Tx_{2n}, r)$$

Taking limit as $n \to \infty$, we get

$$\mathbb{P}_{cb}(Su,r) \leq k\frac{\mu}{k^2}\ [\ max\ \{\ \mathbb{P}_{cb}(Pu,r), \mathbb{P}_{cb}(Pu,Su), \mathbb{P}_{cb}(r,Su), \mathbb{P}_{cb}(r,r),$$

$$\frac{\mathbb{P}_{cb}(r,r)(1+\mathbb{P}_{cb}(Pu,Su)\ )}{1+\mathbb{P}_{cb}(Pu,r)}\}\ ]\ +\ k\ \mathbb{P}_{cb}(r,r)$$

$$\leq k\frac{\mu}{k^2}\ [\ max\ \{\ \mathbb{P}_{cb}(r,r), \mathbb{P}_{cb}(r,Su), \mathbb{P}_{cb}(r,Su), \mathbb{P}_{cb}(r,r),$$

$$\frac{\mathbb{P}_{cb}(r,r)(1+\mathbb{P}_{cb}(r,Su)\ )}{1+\mathbb{P}_{cb}(r,r)}\}\ ]\ +\ k\ \mathbb{P}_{cb}(r,r)$$

$$\Rightarrow \mathbb{P}_{cb}(Su,r) \leq\ k\frac{\mu}{k^2}\ [\ max\ \{\ 0, \mathbb{P}_{cb}(r,Su), \mathbb{P}_{cb}(r,Su), 0,0]$$

$$\leq\ k\frac{\mu}{k^2}\mathbb{P}_{cb}(r,Su)$$

$$\leq\ \frac{\mu}{k}\mathbb{P}_{cb}(r,Su)$$

$$\Rightarrow |\mathbb{P}_{cb}(Su,r)|\ \leq \frac{\mu}{k}\mathbb{P}_{cb}(r,Su)$$

$\Rightarrow |\mathbb{P}_{cb}(Su,r)|\left(1-\frac{\mu}{k}\right) \leq 0$, which is a contradiction.

Hence $Pu = Su = r$.

Again, since $S(\mathbb{X}) \subseteq Q(\mathbb{X})$, there exists a point $v \in \mathbb{X}$ such that $Qv = r$.

Now we have to prove that $Tv = r$.

Suppose that $\mathbb{P}_{cb}(\ r, Tv) > 0$, then using (3.1.1), we have

$$\mathbb{P}_{cb}(r,Tv)\ \leq\ \mathbb{P}_{cb}(Su,Tv)$$

$$\leq \frac{\mu}{k^2}\ [\ max\ \{\ \mathbb{P}_{cb}(Pu,Qv), \mathbb{P}_{cb}(Pu,Su), \mathbb{P}_{cb}(Qv,Su), \mathbb{P}_{cb}(Qv,Tv),$$

$$\frac{\mathbb{P}_{cb}(Qv,Tv)(1+\mathbb{P}_{cb}(Pu,Su)\ )}{1+\mathbb{P}_{cb}(Pu,Qv)}\}\ ]$$

$$\leq\ \frac{\mu}{k^2}\ [\ max\ \{\ \mathbb{P}_{cb}(r,r), \mathbb{P}_{cb}(r,r), \mathbb{P}_{cb}(r,r), \mathbb{P}_{cb}(r,Tv), \frac{\mathbb{P}_{cb}(r,Tv)(1+\mathbb{P}_{cb}(r,r)\ )}{1+\mathbb{P}_{cb}(r,r)}\}]$$

$$\leq \frac{\mu}{k^2}\ [\ max\ \{\ 0,0,0, \mathbb{P}_{cb}(r,Tv), \mathbb{P}_{cb}(r,Tv)\ \}]$$

$$\leq \frac{\mu}{k^2}\ [\mathbb{P}_{cb}(r,Tv)]$$

$$\Rightarrow (1-\frac{\mu}{k^2})\ \mathbb{P}_{cb}(r,Tv) \leq 0$$

$\Rightarrow\ \mathbb{P}_{cb}(r,Tv)\ \leq\ 0$, which is a contradiction.

Therefore $Tv = Qv = r$.

Hence $Pu = Su = Tv = Qv = r$.

Since P and S are weakly compatible maps, then $SPr\ = PSr$. Therefore $Sr = Pr$.

Now we have to prove that r is a fixed point of$S$.

Suppose that $\mathbb{P}_{cb}(Sr,r) > 0$, then we have

$$\mathbb{P}_{cb}(Sr,r)\ \leq\ \mathbb{P}_{cb}(Sr,Tv)$$

$$\leq \frac{\mu}{k^2}\ [\ max\ \{\ \mathbb{P}_{cb}(Pr,Qv), \mathbb{P}_{cb}(Pr,Sr), \mathbb{P}_{cb}(Qv,Sr), \mathbb{P}_{cb}(Qv,Tv)\ , \frac{\mathbb{P}_{cb}(Qv,Tv)(1+\mathbb{P}_{cb}(Pr,Sr)\ )}{1+\mathbb{P}_{cb}(Pr,Qv)}\}\ ]$$

$$\leq\ \frac{\mu}{k^2}\ [\ max\ \{\ \mathbb{P}_{cb}(Sr,r), \mathbb{P}_{cb}(Sr,Sr), \mathbb{P}_{cb}(r,Sr), \mathbb{P}_{cb}(r,r), \frac{\mathbb{P}_{cb}(r,r)(1+\mathbb{P}_{cb}(Sr,Sr)\ )}{1+\mathbb{P}_{cb}(Sr,r)}\}]$$

$$\leq\ \frac{\mu}{k^2}\ [\ max\ \{\ \mathbb{P}_{cb}(Sr,r), 0, \mathbb{P}_{cb}(r,Sr), 0,\ 0\}]$$

$$\leq\ \frac{\mu}{k^2}\ [\ \mathbb{P}_{cb}(Sr,r)]$$

$$\Rightarrow (1-\frac{\mu}{k^2})\ \mathbb{P}_{cb}(Sr,r) \leq 0$$

$\Rightarrow\ \mathbb{P}_{cb}(Sr,r) \leq 0$, which is a contradiction.

Therefore $Sr = r$. Hence $Sr = Pr =\ r$.

Similarly, $Q$and T are weakly compatible, then$TQr = QTr$. Therefore $Qr = Tr$.

We have to prove that$r$ is a fixed point of T.

Suppose that $\mathbb{P}_{cb}(Tr,r) > 0$, then we have

$$\mathbb{P}_{cb}(r,Tr) \leq \mathbb{P}_{cb}(Sr,Tr)$$

$$\leq \frac{\mu}{k^2} [\, max \{ \mathbb{P}_{cb}(Pr,Qr), \mathbb{P}_{cb}(Pr,Sr), \mathbb{P}_{cb}(Qr,Sr), \mathbb{P}_{cb}(Qr,Tr), \frac{\mathbb{P}_{cb}(Qr,Tr)(1+\mathbb{P}_{cb}(Pr,Sr))}{1+\mathbb{P}_{cb}(Pr,Qr)} \}]$$

$$\leq \frac{\mu}{k^2} [\, max \{ \mathbb{P}_{cb}(r,Tr), \mathbb{P}_{cb}(r,r), \mathbb{P}_{cb}(Tr,r), \mathbb{P}_{cb}(Tr,Tr), \frac{\mathbb{P}_{cb}(Tr,Tr)(1+\mathbb{P}_{cb}(r,r))}{1+\mathbb{P}_{cb}(r,Tr)} \}]$$

$$\leq \frac{\mu}{k^2} [max\{\mathbb{P}_{cb}(r,Tr), 0, \mathbb{P}_{cb}(Tr,r), 0, 0\}]$$

$$\leq \frac{\mu}{k^2} [\, \mathbb{P}_{cb}(r,Tr)]$$

$$\Rightarrow (1-\frac{\mu}{k^2})\, \mathbb{P}_{cb}(r,Tr) \leq 0$$

$\Rightarrow \mathbb{P}_{cb}(r,Tr) \leq 0$, which is a contradiction.
Hence $Qr = Tr = r$.
Therefore $Pr = Sr = Qr = Tr = r$.
It follows that r is a common fixed point of P, Q, S and T.

Uniqueness: To prove that uniqueness of r , let r and ω are distinct common fixed point of P, Q, S and T.
We have to prove that $\omega = r$.
Suppose that $\mathbb{P}_{cb}(\omega, r) > 0$, then by using (3.1.1), we have

$$\mathbb{P}_{cb}(r,\omega) \leq \mathbb{P}_{cb}(Sr,T\omega)$$

$$\leq \frac{\mu}{k^2} [\, max \{ \mathbb{P}_{cb}(Pr,Q\omega), \mathbb{P}_{cb}(Pr,Sr), \mathbb{P}_{cb}(Q\omega,Sr), \mathbb{P}_{cb}(Q\omega,T\omega), \frac{\mathbb{P}_{cb}(Q\omega,T\omega)(1+\mathbb{P}_{cb}(Pr,Sr))}{1+\mathbb{P}_{cb}(Pr,Qw)} \}]$$

$$\leq \frac{\mu}{k^2} [\, max \{ \mathbb{P}_{cb}(r,\omega), \mathbb{P}_{cb}(r,r), \mathbb{P}_{cb}(\omega,r), \mathbb{P}_{cb}(\omega,\omega), \frac{\mathbb{P}_{cb}(\omega,\omega)(1+\mathbb{P}_{cb}(r,r))}{1+\mathbb{P}_{cb}(r,\omega)} \}]$$

$$\leq \frac{\mu}{k^2} [max\{\mathbb{P}_{cb}(r,\omega), \mathbb{P}_{cb}(\omega,r)\}]$$

$$\leq \frac{\mu}{k^2} [\mathbb{P}_{cb}(r,\omega)]$$

$$\Rightarrow (1-\frac{\mu}{k^2})\, \mathbb{P}_{cb}(r,\omega) \leq 0$$

$\Rightarrow \mathbb{P}_{cb}(r,\omega) \leq 0$, which is a contradiction.
Therefore $r = \omega$. Hence r is a unique common fixed point of P, Q, $S$and T.

Corollary 3.2: Let $(\mathbb{X}, \mathbb{P}_{cb})$is a complete complex partial b-metric space with coefficient $k \geq 1$ and let P, Q, $S$and T are four self maps of $\mathbb{X}$ such that $T(\mathbb{X}) \subseteq P(\mathbb{X})$ and$S(\mathbb{X}) \subseteq Q(\mathbb{X})$ Satisfying

i) $\mathbb{P}_{cb}(Sx,Ty) \leq \frac{\mu}{k^2}[max \{ \mathbb{P}_{cb}(Px,Qy), \mathbb{P}_{cb}(Px,Sx), \mathbb{P}_{cb}(Qy,Sx), \mathbb{P}_{cb}(Qy,Ty),$

$$\frac{1}{2}[\mathbb{P}_{cb}(Qy,Sx)+\mathbb{P}_{cb}(Ty,Px)], \frac{\mathbb{P}_{cb}(Qy,Ty)(1+\mathbb{P}_{cb}(Px,Sx))}{1+\mathbb{P}_{cb}(Px,Qy)}\}],$$

Where $k \geq 1$ and $\mu \in (0,1), \forall\, x, y \in \mathbb{X}$. ………………… (3.2.1)

ii) The pairs (P,S) and (Q,T) are weakly compatible, then P, Q, $S$and T have a unique common fixed point.

Corollary 3.3: Let $(\mathbb{X}, \mathbb{P}_{cb})$is a complete complex partial b-metric space with coefficient $k \geq 1$ and let $P$and S are two self maps of $\mathbb{X}$ such that $S(\mathbb{X}) \subseteq P(\mathbb{X})$ Satisfying

i) $\mathbb{P}_{cb}(Sx,Sy) \leq \frac{\mu}{k^2}[max \{ \mathbb{P}_{cb}(Px,Py), \mathbb{P}_{cb}(Px,Sx), \mathbb{P}_{cb}(Py,Sx), \mathbb{P}_{cb}(Py,Sy),$

$$\frac{1}{2}[\mathbb{P}_{cb}(Py,Sx)+\mathbb{P}_{cb}(Sx,Px)], \frac{\mathbb{P}_{cb}(Py,Sy)(1+\mathbb{P}_{cb}(Px,Sx))}{1+\mathbb{P}_{cb}(Px,Py)}\}]$$

Where $k \geq 1$and $\mu \in (0,1)$,$\forall\, x, y \in \mathbb{X}$. ……………..(3.3.1)

ii) The pair (P,S) is weakly compatible, then $P$and S have a unique common fixed point.

4. Conclusion

In this paper, we obtained some common fixed point theorems for two pairs of weakly compatible mappings in Complex valued partial b-metric spaces. The results presented in this paper extend, generalize and improve many results from the existing literature regarding complex partial b-metric spaces.

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